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WEAKLY $\mathcal{I}_{g\delta}$-CLOSED SETS

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Abstract – In this paper, the notion of weakly $\mathcal{I}_{g\delta}$-closed sets in ideal topological spaces is introduced and studied. The relationships of weakly $\mathcal{I}_{g\delta}$-closed sets and various properties of weakly $\mathcal{I}_{g\delta}$-closed sets are investigated.

Keywords – *generalized class of* τ^**, weakly* $\mathcal{I}_{g\delta}$*-closed set, ideal topological space, generalized closed set,* $\mathcal{I}_{g\delta}$*-closed set, pre*$^*_{\mathcal{I}}$*-closed set, pre*$^*_{\mathcal{I}}$*-open set.*

1 Introduction and Preliminaries

In this paper, (X, τ) represents topological space on which no separation axioms are assumed unless explicitly stated. The closure and the interior of a subset G of a space X will be denoted by cl(G) and int(G), respectively.

In 1937, Stone [16] introduced and studied the notion of regular open sets in topological spaces. A subset G of X is said to be regular open [16] if int(cl(G))=G. The complement of regular open set is regular closed. In 1968, Veličko [19] introduced the notion of δ-open sets, which are stronger than open sets in order to investigate the characterization of H-closed spaces. A point x $\in$ X is called a δ-cluster point of G if G $\cap$ U $\neq \emptyset$ for every regular open set U containing x. The set of all δ-cluster points of G is called the δ-closure of G and is denoted by cl_δ(G). If cl_δ(G)=G, then G is called δ-closed. The complement of a δ-closed set is δ-open. In 1968, Zaitsav [20] introduced and studied the notion of π-open sets. A finite union of regular open sets is said to be π-open [20]. The complement of a π-open set is π-closed.

In 1999, Dontchev et al. studied the notion of generalized closed sets in ideal topological spaces called $\mathcal{I}_g$-closed sets [2]. In 2008, Navaneethakrishnan and Paulraj Joseph have studied some characterizations of normal spaces via $\mathcal{I}_g$-open sets [10].

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In 2013, Ekici and Ozen [6] introduced a generalized class of τ^*. Ravi et. al [14, 15] introduced another generalized classes of τ^* called weakly $\mathcal{I}_g$-closed sets and weakly $\mathcal{I}_{\pi g}$-closed sets respectively.

The main aim of this paper is to study the notion of weakly $\mathcal{I}_{g\delta}$-closed sets in ideal topological spaces. Moreover, this generalized class of τ^* generalize $\mathcal{I}_{g\delta}$-open sets and weakly $\mathcal{I}_{g\delta}$-open sets. The relationships of weakly $\mathcal{I}_{g\delta}$-closed sets and various properties of weakly $\mathcal{I}_{g\delta}$-closed sets are discussed.

Definition 1.1. A subset G of a topological space (X, τ) is said to be

1. g-closed [9] if cl(G) $\subseteq$ H whenever G $\subseteq$ H and H is open in X;
2. g-open [9] if X\G is g-closed;
3. weakly g-closed [17] if cl(int(G)) $\subseteq$ H whenever G $\subseteq$ H and H is open in X.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

1. A $\in \mathcal{I}$ and B $\subseteq$ A imply B $\in \mathcal{I}$ and
2. A $\in \mathcal{I}$ and B $\in \mathcal{I}$ imply A$\cup$B $\in \mathcal{I}$ [8].

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X if $\mathcal{P}$(X) is the set of all subsets of X, a set operator $(\bullet)^\star : \mathcal{P}(X) \to \mathcal{P}(X)$, called a local function [8] of A with respect to τ and $\mathcal{I}$, is defined as follows: for A $\subseteq$ X, $A^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $cl^\star(\bullet)$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called the $\star$-topology and finer than τ, is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$ [18]. We will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. On the other hand, (A, τ_A, $\mathcal{I}_A$) where τ_A is the relative topology on A and $\mathcal{I}_A = \{A \cap J : J \in \mathcal{I}\}$ is an ideal topological space for an ideal topological space (X, τ, $\mathcal{I}$) and A $\subseteq$ X [7]. For a subset A $\subseteq$ X, cl*(A) and int*(A) will, respectively, denote the closure and the interior of A in (X, τ^*).

Definition 1.2. A subset G of an ideal topological space (X, τ, $\mathcal{I}$) is said to be

1. $\mathcal{I}_g$-closed [2] if G* $\subseteq$ H whenever G $\subseteq$ H and H is open in (X, τ, $\mathcal{I}$).
2. $\mathcal{I}_{rg}$-closed [11] if G* $\subseteq$ H whenever G $\subseteq$ H and H is regular open in (X, τ, $\mathcal{I}$).
3. $\mathcal{I}_{\pi g}$-closed [13] if G* $\subseteq$ H whenever G $\subseteq$ H and H is π-open in (X, τ, $\mathcal{I}$).
4. pre*$_{\mathcal{I}}$-open [5] if G $\subseteq$ int*(cl(G)).
5. pre*$_{\mathcal{I}}$-closed [5] if X\G is pre*$_{\mathcal{I}}$-open.
6. $\mathcal{I}$-R closed [1] if G = cl*(int(G)).
7. $*$-closed [7] if G = cl*(G) or G* $\subseteq$ G.

Remark 1.3. [6] In any ideal topological space, every $\mathcal{I}$-R closed set is $*$-closed but not conversely.

Definition 1.4. Let (X, τ, $\mathcal{I}$) be an ideal topological space. A subset G of X is said to be

1. a weakly $\mathcal{I}_g$-closed set [14] if (int(G))* $\subseteq$ H whenever G $\subseteq$ H and H is an open set in X.
2. a weakly $\mathcal{I}_{\pi g}$-closed set [15] if (int(G))* $\subseteq$ H whenever G $\subseteq$ H and H is a π-open set in X.
3. a weakly $\mathcal{I}_{rg}$-closed set [6] if (int(G))* $\subseteq$ H whenever G $\subseteq$ H and H is a regular open set in X.

Remark 1.5. [3] The following holds in any topological space:
regular open set $\Rightarrow$ π-open set $\Rightarrow$ δ-open set $\Rightarrow$ open set.

These implications are not reversible.

2 Properties of Weakly $\mathcal{I}_{g\delta}$-closed Sets

Definition 2.1. A subset G of an ideal topological space (X, τ, $\mathcal{I}$) is said to be

1. $\mathcal{I}_{g\delta}$-closed if G* $\subseteq$ H whenever G $\subseteq$ H and H is δ-open in (X, τ, $\mathcal{I}$).
2. weakly $\mathcal{I}_{g\delta}$-closed if (int(G))* $\subseteq$ H whenever G $\subseteq$ H and H is δ-open in (X, τ, $\mathcal{I}$).

Theorem 2.2. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. The following properties are equivalent:

1. G is a weakly $\mathcal{I}_{g\delta}$-closed set,
2. cl*(int(G)) $\subseteq$ H whenever G $\subseteq$ H and H is a δ-open set in X.

Proof. (1) $\Rightarrow$ (2) : Let G be a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$). Suppose that G $\subseteq$ H and H is a δ-open set in X. We have (int(G))* $\subseteq$ H. Since int(G) $\subseteq$ G $\subseteq$ H, then (int(G))* $\cup$ int(G) $\subseteq$ H. This implies that cl*(int(G)) $\subseteq$ H.

(2) $\Rightarrow$ (1) : Let cl*(int(G)) $\subseteq$ H whenever G $\subseteq$ H and H is a δ-open in X. Since (int(G))* $\cup$ int(G) $\subseteq$ H, then (int(G))* $\subseteq$ H whenever G $\subseteq$ H and H is a δ-open set in X. Therefore G is a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$).

Theorem 2.3. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is δ-open and weakly $\mathcal{I}_{g\delta}$-closed, then G is $*$-closed.

Proof. Let G be a δ-open and weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$). Since G is δ-open and weakly $\mathcal{I}_{g\delta}$-closed, cl*(G) = cl*(int(G)) $\subseteq$ G. Thus, G is a $*$-closed set in (X, τ, $\mathcal{I}$).

Theorem 2.4. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_{g\delta}$-closed set, then (int(G))* \G contains no any nonempty δ-closed set.

Proof. Let G be a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$). Suppose that H is a δ-closed set such that H $\subseteq$ (int(G))*\G. Since G is a weakly $\mathcal{I}_{g\delta}$-closed set, X\H is δ-open and G $\subseteq$ X\H, then (int(G))* $\subseteq$ X\H. We have H $\subseteq$ X\(int(G))*. Hence, H $\subseteq$ (Int(G))* $\cap$ (X\(int(G))*) = $\emptyset$. Thus, (int(G))*\G contains no any nonempty δ-closed set.

Theorem 2.5. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_{g\delta}$-closed set, then cl*(int(G))\G contains no any nonempty δ-closed set.

Proof. Suppose that H is a δ-closed set such that H $\subseteq$ cl*(int(G))\G. By Theorem 2.4, it follows from the fact that cl*(int(G))\G = ((int(G))* $\cup$ int(G))\G.

Theorem 2.6. Let (X, τ, $\mathcal{I}$) be an ideal topological space. The following properties are equivalent:

1. G is pre*$_{\mathcal{I}}$-closed for each weakly $\mathcal{I}_{g\delta}$-closed set G in (X, τ, $\mathcal{I}$),
2. Each singleton {x} of X is a δ-closed set or {x} is pre*$_{\mathcal{I}}$-open.

Proof. (1) $\Rightarrow$ (2) : Let G be pre*$_{\mathcal{I}}$-closed for each weakly $\mathcal{I}_{g\delta}$-closed set G in (X, τ, $\mathcal{I}$) and x $\in$ X. We have cl*(int(G)) $\subseteq$ G for each weakly $\mathcal{I}_{g\delta}$-closed set G in (X, τ, $\mathcal{I}$). Assume that {x} is not a δ-closed set. It follows that X is the only δ-open set containing X\{x}. Then, X\{x} is a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$). Thus, cl*(int(X\{x})) $\subseteq$ X\{x} and hence {x} $\subseteq$ int*(cl({x})). Consequently, {x} is pre*$_{\mathcal{I}}$-open.

(2) $\Rightarrow$ (1) : Let G be a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$). Let x $\in$ cl*(int(G)).

Suppose that {x} is pre*$_{\mathcal{I}}$-open. We have {x} $\subseteq$ int*(cl({x})). Since x $\in$ cl*(int(G)), then int*(cl({x})) $\cap$ int(G) $\neq \emptyset$. It follows that cl({x}) $\cap$ int(G) $\neq \emptyset$. We have cl({x} $\cap$ int(G)) $\neq \emptyset$ and then {x} $\cap$ int(G) $\neq \emptyset$. Hence, x $\in$ int(G). Thus, we have x $\in$ G.

Suppose that {x} is a δ-closed set. By Theorem 2.5, cl*(int(G))\G does not contain {x}. Since x $\in$ cl*(int(G)), then we have x $\in$ G. Consequently, we have x $\in$ G.

Thus, cl*(int(G)) $\subseteq$ G and hence G is pre*$_{\mathcal{I}}$-closed.

Theorem 2.7. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If cl*(int(G))\G contains no any nonempty $*$-closed set, then G is a weakly $\mathcal{I}_{g\delta}$-closed set.

Proof. Suppose that cl*(int(G))\G contains no any nonempty $*$-closed set in (X, τ, $\mathcal{I}$). Let G $\subseteq$ H and H be a δ-open set. Assume that cl*(int(G)) is not contained in H. It follows that cl*(int(G))$\cap$(X\H) is a nonempty $*$-closed subset of cl*(int(G))\G. This is a contradiction. Hence G is a weakly $\mathcal{I}_{g\delta}$-closed set.

Theorem 2.8. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_{g\delta}$-closed set, then int(G) = H\K where H is $\mathcal{I}$-R closed and K contains no any nonempty δ-closed set.

Proof. Let G be a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$). Take K = (int(G))*\G. Then, by Theorem 2.4, K contains no any nonempty δ-closed set. Take H = cl*(int(G)). Then H = cl*(int(H)). Moreover, we have H\K = ((int(G))* $\cup$ int(G))\((int(G))*\G) = ((int(G))* $\cup$ int(G)) $\cap$ (X\(int(G))* $\cup$ G) = int(G).

Theorem 2.9. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. Assume that G is a weakly $\mathcal{I}_{g\delta}$-closed set. The following properties are equivalent:

1. G is pre*$_{\mathcal{I}}$-closed,
2. cl*(int(G))\G is a δ-closed set,
3. (int(G))* \G is a δ-closed set.

Proof. (1) $\Rightarrow$ (2) : Let G be pre*$_{\mathcal{I}}$-closed. We have cl*(int(G)) $\subseteq$ G. Then, cl*(int(G))\G = $\emptyset$. Thus, cl*(int(G))\G is a δ-closed set.

(2) $\Rightarrow$ (1) : Let cl*(int(G))\G be a δ-closed set. Since G is a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$), then by Theorem 2.5, cl*(int(G))\G = $\emptyset$. Hence, we have cl*(int(G)) $\subseteq$ G. Thus, G is pre*$_{\mathcal{I}}$-closed.

(2) $\Leftrightarrow$ (3) : It follows easily from that cl*(int(G))\G = (int(G))*\G.

Theorem 2.10. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X be a weakly $\mathcal{I}_{g\delta}$-closed set. Then G $\cup$ (X\(int(G))*) is a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$).

Proof. Let G be a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$). Suppose that H is a δ-open set such that G $\cup$ (X\(int(G))*) $\subseteq$ H. We have X\H $\subseteq$ X\(G $\cup$ (X\(int(G))*)) = (X\G) $\cap$ (int(G))* = (int(G))*\G. Since X\H is a δ-closed set and G is a weakly $\mathcal{I}_{g\delta}$-closed set, it follows from Theorem 2.4 that X\H = $\emptyset$. Hence, X = H. Thus, X is the only δ-open set containing G $\cup$ (X\(int(G))*). Consequently, G $\cup$ (X\(int(G))*) is a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$).

Corollary 2.11. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X be a weakly $\mathcal{I}_{g\delta}$-closed set. Then (int(G))*\G is a weakly $\mathcal{I}_{g\delta}$-open set in (X, τ, $\mathcal{I}$).

Proof. Since X\((int(G))*\G) = G $\cup$ (X\(int(G))*), it follows from Theorem 2.10 that (int(G))*\G is a weakly $\mathcal{I}_{g\delta}$-open set in (X, τ, $\mathcal{I}$).

Theorem 2.12. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. The following properties are equivalent:

1. G is a $*$-closed and δ-open set,
2. G is $\mathcal{I}$-R closed and δ-open set,
3. G is a weakly $\mathcal{I}_{g\delta}$-closed and δ-open set.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Since G is δ-open and weakly $\mathcal{I}_{g\delta}$-closed, cl*(int(G)) $\subseteq$ G and so G = cl*(int(G)). Then G is $\mathcal{I}$-R closed and hence it is $*$-closed.

Proposition 2.13. Every pre*$_{\mathcal{I}}$-closed set is weakly $\mathcal{I}_{g\delta}$-closed but not conversely.

Proof. Let H $\subseteq$ G and G be a δ-open set in X. Since H is pre*$_{\mathcal{I}}$-closed, cl*(int(H)) $\subseteq$ H $\subseteq$ G. Hence H is weakly $\mathcal{I}_{g\delta}$-closed set.

Example 2.14. Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = {a, b, c}, τ = {$\emptyset$, X, {c}, {a, c}} and $\mathcal{I}$ = {$\emptyset$, {c}}. Then {a, c} is weakly $\mathcal{I}_{g\delta}$-closed set but not pre*$_{\mathcal{I}}$-closed.

3 Further Properties

Theorem 3.1. Let (X, τ, $\mathcal{I}$) be an ideal topological space. The following properties are equivalent:

1. Each subset of (X, τ, $\mathcal{I}$) is a weakly $\mathcal{I}_{g\delta}$-closed set,
2. G is pre*$_{\mathcal{I}}$-closed for each δ-open set G in X.

Proof. (1) ⇒ (2) : Suppose that each subset of (X, τ, $\mathcal{I}$) is a weakly $\mathcal{I}_{g\delta}$-closed set. Let G be a δ-open set in X. Since G is weakly $\mathcal{I}_{g\delta}$-closed, then we have cl*(int(G)) ⊆ G. Thus, G is pre*$_{\mathcal{I}}$-closed.

(2) ⇒ (1) : Let G be a subset of (X, τ, $\mathcal{I}$) and H be a δ-open set such that G ⊆ H. By (2), we have cl*(int(G)) ⊆ cl*(int(H)) ⊆ H. Thus, G is a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$).

Theorem 3.2. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If G is a weakly $\mathcal{I}_{g\delta}$-closed set and G ⊆ H ⊆ cl*(int(G)), then H is a weakly $\mathcal{I}_{g\delta}$-closed set.

Proof. Let H ⊆ K and K be a δ-open set in X. Since G ⊆ K and G is a weakly $\mathcal{I}_{g\delta}$-closed set, then cl*(int(G)) ⊆ K. Since H ⊆ cl*(int(G)), then cl*(int(H)) ⊆ cl*(int(G)) ⊆ K. Thus, cl*(int(H)) ⊆ K and hence, H is a weakly $\mathcal{I}_{g\delta}$-closed set.

Corollary 3.3. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If G is a weakly $\mathcal{I}_{g\delta}$-closed and open set, then cl*(G) is a weakly $\mathcal{I}_{g\delta}$-closed set.

Proof. Let G be a weakly $\mathcal{I}_{g\delta}$-closed and open set in (X, τ, $\mathcal{I}$). We have G ⊆ cl*(G) ⊆ cl*(G) = cl*(int(G)). Hence, by Theorem 3.2, cl*(G) is a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$).

Theorem 3.4. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G ⊆ X. If G is a nowhere dense set, then G is a weakly $\mathcal{I}_{g\delta}$-closed set.

Proof. Let G be a nowhere dense set in X. Since int(G) ⊆ int(cl(G)), then int(G) = ∅. Hence, cl*(int(G)) = ∅. Thus, G is a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$).

Remark 3.5. The reverse of Theorem 3.4 is not true in general as shown in the following example.

Example 3.6. In Example 2.14, {a, c} is a weakly $\mathcal{I}_{g\delta}$-closed set but not a nowhere dense set.

Remark 3.7. The intersection of two weakly $\mathcal{I}_{g\delta}$-closed sets in an ideal topological space need not be a weakly $\mathcal{I}_{g\delta}$-closed set.

Example 3.8. Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = {a, b, c, d}, τ = {∅, X, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}} and $\mathcal{I}$ = {∅, {d}}. Then A = {a, b, d} and B = {a, b, c} are weakly $\mathcal{I}_{g\delta}$-closed sets but their intersection {a, b} is not a weakly $\mathcal{I}_{g\delta}$-closed set.

Theorem 3.9. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G ⊆ X. Then G is a weakly $\mathcal{I}_{g\delta}$-open set if and only if H ⊆ int*(cl(G)) whenever H ⊆ G and H is a δ-closed set.

Proof. Let H be a δ-closed set in X and H ⊆ G. It follows that X\H is a δ-open set and X\G ⊆ X\H. Since X\G is a weakly $\mathcal{I}_{g\delta}$-closed set, then cl*(int(X\G)) ⊆ X\H. We have X\int*(cl(G)) ⊆ X\H. Thus, H ⊆ int*(cl(G)).

Conversely, let K be a δ-open set in X and X\G ⊆ K. Since X\K is a δ-closed set such that X\K ⊆ G, then X\K ⊆ int*(cl(G)). We have X\int*(cl(G)) = cl*(int(X\G)) ⊆ K. Thus, X\G is a weakly $\mathcal{I}_{g\delta}$-closed set. Hence, G is a weakly $\mathcal{I}_{g\delta}$-open set in (X, τ, $\mathcal{I}$).

Theorem 3.10. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G ⊆ X. If G is a weakly $\mathcal{I}_{g\delta}$-closed set, then cl*(int(G))\G is a weakly $\mathcal{I}_{g\delta}$-open set in (X, τ, $\mathcal{I}$).

Proof. Let G be a weakly $\mathcal{I}_{g\delta}$-closed set in (X, τ, $\mathcal{I}$). Suppose that H is a δ-closed set such that H $\subseteq$ cl*(int(G))\G. Since G is a weakly $\mathcal{I}_{g\delta}$-closed set, it follows from Theorem 2.5 that H = $\emptyset$. Thus, we have H $\subseteq$ int*(cl(cl*(int(G))\G)). It follows from Theorem 3.9 that cl*(int(G))\G is a weakly $\mathcal{I}_{g\delta}$-open set in (X, τ, $\mathcal{I}$).

Theorem 3.11. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_{g\delta}$-open set, then H = X whenever H is a δ-open set and int*(cl(G)) $\cup$ (X\G) $\subseteq$ H.

Proof. Let H be a δ-open set in X and int*(cl(G)) $\cup$ (X\G) $\subseteq$ H. We have X\H $\subseteq$ (X\int*(cl(G))) $\cap$ G = cl*(int(X\G))\(X\G). Since X\H is a δ-closed set and X\G is a weakly $\mathcal{I}_{g\delta}$-closed set, it follows from Theorem 2.5 that X\H = $\emptyset$. Thus, we have H = X.

Theorem 3.12. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If G is a weakly $\mathcal{I}_{g\delta}$-open set and int*(cl(G)) $\subseteq$ H $\subseteq$ G, then H is a weakly $\mathcal{I}_{g\delta}$-open set.

Proof. Let G be a weakly $\mathcal{I}_{g\delta}$-open set and int*(cl(G)) $\subseteq$ H $\subseteq$ G. Since int*(cl(G)) $\subseteq$ H $\subseteq$ G, then int*(cl(G)) = int*(cl(H)). Let K be a δ-closed set and K $\subseteq$ H. We have K $\subseteq$ G. Since G is a weakly $\mathcal{I}_{g\delta}$-open set, it follows from Theorem 3.9 that K $\subseteq$ int*(cl(G)) = int*(cl(H)). Hence, by Theorem 3.9, H is a weakly $\mathcal{I}_{g\delta}$-open set in (X, τ, $\mathcal{I}$).

Corollary 3.13. Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_{g\delta}$-open and closed set, then int*(G) is a weakly $\mathcal{I}_{g\delta}$-open set.

Proof. Let G be a weakly $\mathcal{I}_{g\delta}$-open and closed set in (X, τ, $\mathcal{I}$). Then int*(cl(G)) = int*(G) $\subseteq$ int*(G) $\subseteq$ G. Thus, by Theorem 3.12, int*(G) is a weakly $\mathcal{I}_{g\delta}$-open set in (X, τ, $\mathcal{I}$).

Definition 3.14. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called $Q_{\mathcal{I}}$-set if A = M$\cup$N where M is δ-closed and N is pre*$_{\mathcal{I}}$-open.

Remark 3.15. Every pre*$_{\mathcal{I}}$-open (resp. δ-closed) set is $Q_{\mathcal{I}}$-set but not conversely.

Example 3.16. Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = {a, b, c, d}, τ = {$\emptyset$, X, {a}, {b}, {a, b}, {b, c}, {a, b, c}} and $\mathcal{I}$ = {$\emptyset$, {d}}. Then {b, d} is a $Q_{\mathcal{I}}$-set but it is neither pre*$_{\mathcal{I}}$-open nor δ-closed.

Theorem 3.17. For a subset H of (X, τ, $\mathcal{I}$), the following are equivalent.

1. H is pre*$_{\mathcal{I}}$-open.
2. H is a $Q_{\mathcal{I}}$-set and weakly $\mathcal{I}_{g\delta}$-open.

Proof. (1) $\Rightarrow$ (2): By Remark 3.15, H is a $Q_{\mathcal{I}}$-set. By Proposition 2.13, H is weakly $\mathcal{I}_{g\delta}$-open.

(2) $\Rightarrow$ (1): Let H be a $Q_{\mathcal{I}}$-set and weakly $\mathcal{I}_{g\delta}$-open. Then there exist a δ-closed set M and a pre*$_{\mathcal{I}}$-open set N such that H =M$\cup$N. Since M $\subseteq$ H and H is weakly $\mathcal{I}_{g\delta}$-open, by Theorem 3.9, M $\subseteq$ int*(cl(H)). Also, we have N $\subseteq$ int*(cl(N)). Since N $\subseteq$ H, N $\subseteq$ int*(cl(N)) $\subseteq$ int*(cl(H)). Then H = M$\cup$N $\subseteq$ int*(cl(H)). So H is pre*$_{\mathcal{I}}$-open.

The following example shows that the concepts of weakly $\mathcal{I}_{g\delta}$-open set and $Q_{\mathcal{I}}$-set are independent.

Example 3.18. In Example 3.16, {c} is weakly $\mathcal{I}_{g\delta}$-open set but not $Q_{\mathcal{I}}$-set. Also {d} is $Q_{\mathcal{I}}$-set but not weakly $\mathcal{I}_{g\delta}$-open set.

Remark 3.19. The following diagram holds for any ideal topological space:

$$\begin{array}{ccc} \mathcal{I}_{g\delta}\text{-closed set} & \longrightarrow & \text{weakly } \mathcal{I}_{g\delta}\text{-closed set} \\ \downarrow & & \downarrow \\ \mathcal{I}_{rg}\text{-closed set} & \longrightarrow & \text{weakly } \mathcal{I}_{rg}\text{-closed set} \end{array}$$

None of the implications is reversible as shown in the following examples and in [6].

Example 3.20. Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = {a, b, c, d}, τ = {∅, X, {a}, {b}, {a, b}, {b, c}, {a, b, c}} and $\mathcal{I}$ = {∅}. Then {a, b} is $\mathcal{I}_{rg}$-closed set but not $\mathcal{I}_{g\delta}$-closed.

Example 3.21. In Example 3.20, {a, b} is weakly $\mathcal{I}_{rg}$-closed set but not weakly $\mathcal{I}_{g\delta}$-closed.

Example 3.22. In Example 3.20, {c} is weakly $\mathcal{I}_{g\delta}$-closed set but not $\mathcal{I}_{g\delta}$-closed.

4 gδ-pre*$_{\mathcal{I}}$-normal Spaces

Definition 4.1. An ideal topological space (X, τ, $\mathcal{I}$) is said to be gδ-pre*$_{\mathcal{I}}$-normal if for every pair of disjoint δ-closed subsets A, B of X, there exist disjoint pre*$_{\mathcal{I}}$-open sets U, V of X such that A ⊆ U and B ⊆ V.

Theorem 4.2. The following properties are equivalent for a space (X, τ, $\mathcal{I}$).

1. X is gδ-pre*$_{\mathcal{I}}$-normal;
2. for any disjoint δ-closed sets A and B, there exist disjoint weakly $\mathcal{I}_{g\delta}$-open sets U, V of X such that A ⊆ U and B ⊆ V;
3. for any δ-closed set A and any δ-open set B containing A, there exists a weakly $\mathcal{I}_{g\delta}$-open set U such that A ⊆ U ⊆ cl*(int(U)) ⊆ B.

Proof. (1) ⇒ (2): The proof is obvious.

(2) ⇒ (3): Let A be any δ-closed set of X and B any δ-open set of X such that A ⊆ B. Then A and X\B are disjoint δ-closed sets of X. By (2), there exist disjoint weakly $\mathcal{I}_{g\delta}$-open sets U, V of X such that A ⊆ U and X\B ⊆ V. Since V is weakly $\mathcal{I}_{g\delta}$-open set, by Theorem 3.9, X\B ⊆ int*(cl(V)) and U∩int*(cl(V)) = ∅. Therefore we obtain cl*(int(U)) ⊆ cl*(int(X\V)) and hence A ⊆ U ⊆ cl*(int(U)) ⊆ B.

(3) ⇒ (1): Let A and B be any disjoint δ-closed sets of X. Then A ⊆ X\B and X\B is δ-open and hence there exists a weakly $\mathcal{I}_{g\delta}$-open set G of X such that A ⊆ G ⊆ cl*(int(G)) ⊆ X\B. Put U = int*(cl(G)) and V = X\cl*(int(G)). Then U and V are disjoint pre*$_{\mathcal{I}}$-open sets of X such that A ⊆ U and B ⊆ V. Therefore X is gδ-pre*$_{\mathcal{I}}$-normal.

Definition 4.3. A function f : (X, τ, $\mathcal{I}$) → (Y, σ) is said to be weakly $\mathcal{I}_{g\delta}$-continuous if f^{-1}(V) is weakly $\mathcal{I}_{g\delta}$-closed in X for every closed set V of Y.

Definition 4.4. A function f : (X, τ, $\mathcal{I}$) → (Y, σ, $\mathcal{J}$) is called weakly $\mathcal{I}_{g\delta}$-irresolute if f^{-1}(V) is weakly $\mathcal{I}_{g\delta}$-closed in X for every weakly $\mathcal{J}_{g\delta}$-closed of Y.

Definition 4.5. A function f : (X, τ) $\rightarrow$ (Y, σ) is said to be δ-closed [4, 12] if f(V) is δ-closed in Y for every δ-closed set V of X.

Definition 4.6. A topological space (X, τ) is said to be δ-normal if for every pair of disjoint δ-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A $\subseteq$ U and B $\subseteq$ V.

Theorem 4.7. Let f : X $\rightarrow$ Y be a weakly $\mathcal{I}_{g\delta}$-continuous δ-closed injection. If Y is δ-normal, then X is gδ-pre$^*_{\mathcal{I}}$-normal.

Proof. Let A and B be disjoint δ-closed sets of X. Since f is δ-closed injection, f(A) and f(B) are disjoint δ-closed sets of Y. By the δ-normality of Y, there exist disjoint open sets U and V in Y such that f(A) $\subseteq$ U and f(B) $\subseteq$ V. Since f is weakly $\mathcal{I}_{g\delta}$-continuous, then f^{-1}(U) and f^{-1}(V) are weakly $\mathcal{I}_{g\delta}$-open sets of X such that A $\subseteq$ f^{-1}(U) and B $\subseteq$ f^{-1}(V). Therefore X is gδ-pre$^*_{\mathcal{I}}$-normal by Theorem 4.2.

Theorem 4.8. Let f : X $\rightarrow$ Y be a weakly $\mathcal{I}_{g\delta}$-irresolute δ-closed injection. If Y is gδ-pre$^*_{\mathcal{I}}$-normal, then X is gδ-pre$^*_{\mathcal{I}}$-normal.

Proof. Let A and B be disjoint δ-closed sets of X. Since f is δ-closed injection, f(A) and f(B) are disjoint δ-closed sets of Y. Since Y is gδ-pre$^*_{\mathcal{I}}$-normal, by Theorem 4.2, there exist disjoint weakly $\mathcal{J}_{g\delta}$-open sets U and V in Y such that f(A) $\subseteq$ U and f(B) $\subseteq$ V. Since f is weakly $\mathcal{I}_{g\delta}$-irresolute, then f^{-1}(U) and f^{-1}(V) are disjoint weakly $\mathcal{I}_{g\delta}$-open sets of X such that A $\subseteq$ f^{-1}(U) and B $\subseteq$ f^{-1}(V). Therefore X is gδ-pre$^*_{\mathcal{I}}$-normal.

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