http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 08.08.2015 *Year*: **2015,** *Number*: **9**, *Pages*: **22-39**
Published: 14.12.2015 *Original Article*[**]

BIPOLAR (T,S)-FUZZY MEDIAL IDEAL OF BCI-ALGEBAS

Samy Mohammed Mostafa[1,*] <samymostafa@yahoo.com>
Abdelaziz Elazab Radwan[2] <zezoradwan@yahoo.com>
Amany El-menshawy[1] <amanyelmenshawy2010@gmail.com>
Reham Ghanem[1] <ghanemreham@yahoo.com>

[1]Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt.
[2]Department of Mathematics, Faculty of Science and Arts, Qassim University, Kingdom of Saudi Arabia

Abstract - In this paper, the concept bipolar (T,S)- fuzzy medial-ideals are introduced and several properties are investigated .Also, the relations between bipolar (T,S)- fuzzy medial-ideals and bipolar (T,S)- fuzzy BCI-ideals are given .The image and the pre-image of bipolar (T,S)- fuzzy medial-ideals under homomorphism of BCI-algebras are defined and how the image and the pre-image of bipolar (T,S)- fuzzy medial-ideals under homomorphism of BCI-algebras become bipolar (T,S)- fuzzy medial-ideals are studied. Moreover, the Cartesian product of bipolar (T,S)- fuzzy medial-ideals in Cartesian product BCI-algebras is established.

Keywords – Medial BCI-algebra, fuzzy medial-ideals, bipolar (T,S)-fuzzy medial-ideals, the pre-image of bipolar (T,S)- fuzzy medial-ideals in BCI-algebras, Cartesian product of bipolar (T,S)-fuzzy medial-ideals.

1. Introduction

In 1966 Iami and Iseki [5,6,7] introduced the notion of BCK-algebras Iseki [5,7] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK / BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature which has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideals theory of BCK/BCI-algebras . In 1956, Zadeh [17] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches. There are several kinds of fuzzy sets extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets etc. In [1,2,8,9,10,11,13] they introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0,1] to [-1,1]. On

[**]*Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
[*]*Corresponding Author.*

the other hand, triangular norm is a powerful tool in the theory research and application development of fuzzy sets [4,12]. Li [12] generalized the operators "$\wedge$" and "$\vee$" to T-norm and S-norm and defined the intuitionistic fuzzy groups of (T-S) - norms. as a generalization of the notion of fuzzy set. In 1991, Xi [16] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. In [14] J.Meng and Y.B.Jun studied medial BCI-algebras. Mostafa et al. [15] introduced the notion of medial ideals in BCI-algebras, they stated the fuzzification of medial ideals and investigated its properties. Now, in this note we use the notion of Bipolar valued fuzzy set to establish the notion of bipolar valued (T,S) - fuzzy medial ideals of BCI-algebras; then we obtained some related properties, which have been mentioned in the abstract .

2. Preliminaries

Now we review some definitions and properties that will be useful in our results.

Definition 2.1 [5,7] An algebraic system $(X,*,0)$ of type (2, 0) is called a BCI-algebra if it satisfying the following conditions:

(BCI-1) $((x*y)*(x*z))*(z*y)=0$,

(BCI-2) $(x*(x*y))*y=0$,

(BCI-3) $x*x=0$,

(BCI-4) $x*y=0$ and $y*x=0$ imply $x=y$.

For all x, y and $z\in X$. In a BCI-algebra X, we can define a partial ordering"$\leq$" by $x\leq y$ if and only if $x*y=0$.

In what follows, X will denote a BCI-algebra unless otherwise specified.

Definition 2.2 [14] A BCI-algebra $(X,*,0)$ of type (2, 0) is called a medial BCI-algebra if it satisfying the following condition: $(x*y)*(z*u)=(x*z)*(y*u)$, for all x,y,z and $u\in X$.

Lemma 2.3 [14] An algebra (X, $*$, 0) of type (2, 0) is a medial BCI-algebra if and only if it satisfies the following conditions:

(i) $x*(y*z)=z*(y*x)$

(ii) $x*0=x$

(iii) $x*x=0$

Lemma 2.4 [14] In a medial BCI-algebra X, the following holds:

$$x*(x*y)=y, \text{ for all } x,y\in X.$$

Lemma 2.5 Let X be a medial BCI-algebra, then $0*(y*x)=x*y$, for all $x,y\in X$.

Proof . Clear.

Definition 2.6 A non empty subset S of a medial BCI-algebra X is said to be medial sub-algebra of X, if $x*y\in S$, for all $x,y\in S$.

Definition 2.7 [5,7] A non-empty subset I of a BCI-algebra X is said to be a BCI-ideal of X if it satisfies:

(I_1) $0 \in I$,

(I_2) $x * y \in I$ and $y \in I$ implies $x \in I$ for all $x, y \in X$.

Definition 2.8 [15] A non empty subset M of a medial BCI-algebra X is said to be a medial ideal of X if it satisfies:

(M_1) $0 \in M$,

(M_2) $z * (y * x) \in M$ and $y * z \in M$ imply $x \in M$ for all x, y and $z \in X$.

Definition 2.9 [15] Let μ be a fuzzy set on a BCI-algebra X, then μ is called a fuzzy BCI-subalgebra of X if

(FS_1) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Definition 2.10 [15] Let X be a BCI-algebra. a fuzzy set μ in X is called a fuzzy BCI-ideal of X if it satisfies:

(FI_1) $\mu(0) \geq \mu(x)$,

(FI_2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$, for all x, y and $z \in X$.

Definition 2.11 [15] Let X be a medial BCI-algebra. A fuzzy set μ in X is called a fuzzy medial ideal of X if it satisfies:

(FM_1) $\mu(0) \geq \mu(x)$,

(FM_2) $\mu(x) \geq \min\{\mu(z * (y * x)), \mu(y * z)\}$, for all x, y and $z \in X$.

Definition 2.12 [12] A triangular norm (t-norm) is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies following conditions:

(T_1) boundary condition :T(x, 1) = x,

(T_2)) commutativity condition: T(x, y) = T(y, x),

(T_3) associativity condition :T(x, T(y, z)) = T(T(x, y), z),

(T_4) monotonicity : T(x, y) $\leq$ T(x, z) ,whenever y $\leq$ z for all x, y, z $\in$[0, 1].

A simple example of such defined *t*-norm is a function $T(\alpha, \beta) = \min\{\alpha, \beta\}$.

In the general case $T(\alpha, \beta) \leq \min\{\alpha, \beta\}$ and $T(\alpha, 0) = 0$ for all $\alpha, \beta \in [0, 1]$.

Definition 2.13 [4] Let X be a BCI-algebra. A fuzzy subset μ in X is called a fuzzy sub-algebra of X with respect to a t-norm T (briefly, a T-fuzzy sub-algbra of X) if $\mu(x) \geq T\{\mu(x * y), \mu(y)\}$, for all $x, y \in X$.

Definition 2.14 [12] A triangular conorm (t-conorm S) is a mapping $S : [0,1] \times [0,1] \to [0,1]$ that satisfies following conditions:

(S1) S(x, 0) = x,

(S2) S(x, y) = S(y, x),

(S3) S(x, S(y, z)) = S(S(x, y), z),

(S4) S(x, y) $\leq$ S(x, z) , whenever y $\leq$ z for all x, y, z $\in$[0, 1].

A simple example of such definition s-norm S is a function S(x, y) = max{x, y}.

Every S- conorm S has a useful property: $\max\{\alpha, \beta\} \le S(\alpha, \beta)$ for all $\alpha, \beta \in [0, 1]$.

Definition 2.15 [4] Let X be a BCI-algebra. a fuzzy set μ in X is called T- fuzzy BCI- ideal of X if it satisfies:

(TI$_1$) $\mu(0) \ge \mu(x)$,

(TI$_2$) $\mu(x) \ge T\{\mu(x * y), \mu(y)\}$, for all x, y and $z \in X$.

Definition 2.16 [4] Let X be a BCI-algebra. a fuzzy set λ in X is called S- fuzzy BCI- ideal of X if it satisfies:

(SI$_1$) $\lambda(0) \le \lambda(x)$,

(SI$_2$) $\lambda(x) \le S\{\lambda(x * y), \lambda(y)\}$, for all x, y and $z \in X$.

Definition 2.17 Let X be a BCI-algebra. A fuzzy set μ in X is called T- fuzzy medial -ideal of X if it satisfies:

(FM$_1$) $\mu(0) \ge \mu(x)$,

(FM$_2$) $\mu(x) \ge T\{\mu(z * (y * x)), \mu(y * z)\}$, for all x, y and $z \in X$.

Definition 2.18 Let X be a BCI-algebra. A fuzzy set λ in X is called S- fuzzy medial -ideal of X if it satisfies:

(FS$_1$) $\lambda(0) \le \lambda(x)$,

(FS$_2$) $\lambda(x) \le S\{\lambda(z * (y * x)), \lambda(y * z)\}$, for all x, y and $z \in X$.

Example 2.19 Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $*$ defined by the following table:

*	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	1	0	4	4
2	2	2	0	0	4	4
3	3	2	1	0	4	4
4	4	4	4	4	0	0
5	5	4	5	4	1	0

we can prove that $(X, *, 0)$ is a BCI-algebra. Let $T_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $T_m(\alpha, \beta) = \max\{\alpha + \beta - 1, 0\}$ for all $\alpha, \beta \in [0, 1]$ is a t-norm . By routine calculations, we known that a fuzzy set μ in X defined by

$$\mu(1) = 0.3 \text{ and } \mu(0) = \mu(2) = \mu(3) = \mu(4) = \mu(5) = 0.9$$

is a T_m-fuzzy medial-ideal , since

$$\mu_A(x) \ge T\{\mu_A(z * (y * x)), \mu_A(y * z)\}.$$

and $S_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S_m(\alpha, \beta) = \min\{1 - (\alpha + \beta), 1\}$. Then By routine calculations, we known that a fuzzy set λ in X defined by

λ (5) = 0.8 and λ (0) = λ (1) = λ (2) = λ (3) = λ (4) = 0.3

is a S_m-fuzzy medial-ideal ,because

$$\lambda_A(x) \le S\{\lambda_A(z*(y*x)), \lambda_A(y*z)\} \text{ , for all } x, y \text{ and } z \in X .$$

3. Bipolar (T,S)-fuzzy Medial Ideal

Now, we present some preliminaries on the theory of bipolar-valued fuzzy set.

Definition 3.1 [9] A bipolar valued fuzzy subset B in a nonempty set X is an object having the form $\mathrm{B} = \{(x, \mu^N(x), \mu^P(x)) \mid x \in X\}$ where $\mu^N : X \to [-1,0]$ and $\mu^P : X \to [0,1]$ are mappings. The positive membership degree $\mu^P(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\mathrm{B} = \{(x, \mu^N(x), \mu^P(x)) \mid x \in X\}$, and the negative membership degree $\mu^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of a bipolar-valued fuzzy set $\mathrm{B} = \{(x, \mu^N(x), \mu^P(x)) \mid x \in X\}$. For simplicity, we shall use the symbol $\mathrm{B} = (x, \mu^N, \mu^P)$ for bipolar fuzzy set $\mathrm{B} = \{(x, \mu^N(x), \mu^P(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

Definition 3.2 [9] Let $\mathrm{B} = (x, \mu^N, \mu^P)$ be a bipolar fuzzy set and $(s,t) \in [-1,0] \times [0,1]$.
The set $\mathrm{B}_s^N = \{x \in X : \mu^N(x) \le s\}$ and $\mathrm{B}_t^P = \{x \in X : \mu^P(x) \ge t\}$ which are called the negative s-cut and the positive t-cut of $\mathrm{B} = (x, \mu^N, \mu^P)$, respectively.

Definition3.3 [9] A bipolar fuzzy set $\mathrm{B} = (x, \mu^N, \mu^P)$ in a BCI-algebra (X,∗, 0) is called a bipolar (T,S)- fuzzy BCI-subalgebra of X if it satisfies the following condition:
For all $x, y \in X$, $\mu^N(x*y) \le S\{\mu^N(x), \mu^N(y)\}$, $\mu^P(x*y) \ge T\{\mu^P(x), \mu^P(y)\}$.

Lemma 3.4 If $\mathrm{B} = (x, \mu^N, \mu^P)$ is a bipolar (T,S)- fuzzy BCI-subalgebra of X , then

$$\mu^N(0) \le \mu^N(x) \text{ and } \mu^P(0) \ge \mu^P(x)$$

Proof: Put x= y in Definition3.3and use (BCI-3) . The proof is complete.

Defintion 3.5 A bipolar fuzzy set $\mathrm{B} = (x, \mu^N, \mu^P)$ in X is called a bipolar (T,S)- fuzzy BCI-ideal of X if it satisfies the following condition: for all $x, y \in X$
(b_1) $\mu^N(0) \le \mu^P(x)$ and $\mu^N(0) \ge \mu^P(x)$,
(b_2) $\mu^N(x) \le S\{\mu^N(x*y), \mu^N(y)\}$, $\mu^P(x) \ge T\{\mu^P(x*y), \mu^P(x)\}$.

Defintion 3.6 A bipolar fuzzy set $\mathrm{B} = (x, \mu^N, \mu^P)$ in X is called a bipolar (T,S)- fuzzy medial -ideal of X if it satisfies the following condition: for all $x, y, z \in X$
(b_1) $\mu^N(0) \le \mu^P(x)$ and $\mu^N(0) \ge \mu^P(x)$,

(B_2) $\mu^N(x) \leq S\{\mu^N(z*(y*x)), \mu^N(y*z)\}$, $\mu^P(x) \geq T\{\mu^P(z*(y*x)), \mu^P(y*z)\}$.

Example 3.7 Let X = {0, 1, 2, 3} be a set with a binary operation $*$ define by the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	2	0
2	2	0	0	0
3	3	2	1	0

	0	1	2	3
μ^N	**-0.7**	**-0.7**	**- 0.6**	**- 0.4**
μ^P	**0.6**	**0.5**	**0.3**	**0.3**

Let $T : [0,1]\times[0,1] \to [0,1]$ be a function defined by T (α, β) = max {α + β − 1, 0} for all α, β ∈ [0, 1] and $S : [-1,0]\times[-1,0] \to [-1,0]$ be a function defined by $S(\tilde{\alpha}, \tilde{\beta}) = \min\{1 + (\tilde{\alpha} + \tilde{\beta}), 0\}$, $\forall\ \tilde{\alpha}, \tilde{\beta} \in [-1,0]$ .By routine calculations, we know that $\mathrm{B} = (x, \mu^N, \mu^P)$ is bipolar (T,S)- fuzzy medial-ideal of X .

Proposition 3.8 Every a bipolar fuzzy(T,S)- medial-ideal of X is a bipolar fuzzy(T,S)-BCI-ideal of X .

Proof: clear.

Proposition 3.9 If $\mathrm{B} = (x, \mu^N, \mu^P)$ is a bipolar (T,S)- fuzzy medial-ideal of X and $x \leq y$, then $\mu^N(x) \leq \mu^N(y)$ and $\mu^P(x) \geq \mu^P(y)$.

Proof: If $x \leq y$, then $x * y = 0$, since $\mathrm{B} = (x, \mu^N, \mu^P)$ is a bipolar fuzzy(T,S)- medial-ideal of X , we get

$$\mu^N(x) \leq S\{\mu^N(0*(y*x)), \mu^N(y)\} \overset{by\quad Lemma\, 2.5}{=} S\{\mu^N(x*y), \mu^N(y)\}$$
$$= S\{\mu^N(0), \mu^N(y)\} = \mu^N(y).$$

And

$$\mu^P(x) \geq T\{\mu^P(0*(y*x)), \mu^P(y)\} \overset{by\quad Lemma\, 2.5}{=} T\{\mu^P(x*y), \mu^P(y)\}$$
$$= T\{\mu^P(0), \mu^P(y)\} = \mu^N(y).$$

Theorem 3.10 Every bipolar fuzzy (T,S)- medial-ideal of X is a bipolar (T,S)- fuzzy BCI-sub-algebra of X .

Proof . Let $B=(x,\mu^N,\mu^P)$ be bipolar (T,S)- fuzzy medial-ideal of X . Since $x*y\le x$, for all $x,y\in X$, then $\mu^N(x*y)\le\mu^N(x)$, $\mu^P(x*y)\ge\mu^P(x)$ .Put z = 0 in (b_1), (B_2), we have

$\mu^N(0)\le\mu^P(x)$ and $\mu^N(0)\ge\mu^P(x)$,

$$\mu^N(x*y)\le\mu^N(x)\le S\{\mu^N(0*(y*x),\mu^N(y*0)\}=S\{\mu^N(0*(y*x),\mu^N(y*0)\}$$
$$S\{\mu^N(x*y),\mu^N(y)\}\le S\{\mu^N(x),\mu^N(y)\}\ ,\ and$$

$$\mu^P(x*y)\ge\mu^P(x)\ge T\{\mu^P(0*(y*x),\mu^P(y*0)\}=T\{\mu^P(0*(y*x),\mu^P(y*0)\}$$
$$T\{\mu^P(x*y),\mu^P(y)\}\ge T\{\mu^P(x),\mu^P(y)\}.$$

Then $B=(x,\mu^N,\mu^P)$ is bipolar (T,S)- fuzzy sub-algebra of *X*.

Proposition 3.11 Let $B=(x,\mu^N,\mu^P)$ be a bipolar (T,S)- fuzzy medial-ideal of X . If the inequality $x*y\le z$ holds in X, then

$\mu^N(x)\le S\{\mu^N(y),\mu^N(z)\}$ and $\mu^P(x)\ge T\{\mu^P(y),\mu^P(z)\}$ for all $x,y,z\in X$.

Proof . Let $x,y,z\in X$ be such that $x*y\le z$. Thus, put $z=0$ in (Defintion3.6), (using lemma2.5 and lemma 3.9), we get,

$$\mu^N(x)\le S\{\mu^N(0*(y*x),\mu^N(y*0)\}=\ S\{\mu^N(x*y),\mu^N(y)\}\le\ \overbrace{S\{\mu^N(z),\mu^N(y)\}}^{\sin ce\ \ \mu^N(x*y)\le\mu^N(z)}.$$

Similarly we have,

$$\mu^P(x)\ge T\{\mu^P(0*(y*x),\mu^P(y*0)\}=T\{\mu^P(x*y),\mu^P(y)\}\ge\ \overbrace{T\{\mu^P(z),\mu^P(y)\}}^{\sin ce\ \ \mu^P(x*y)\ge\mu^P(z)}.$$

Theorem 3.12 Let $B=(x,\mu^N,\mu^P)$ be a bipolar (T,S)- fuzzy sub-algebra of X .such that $\mu^P(x)\ge T\{\mu^P(y),\mu^P(z)\}$, $\mu^N(x)\le S\{\mu^N(y),\mu^N(z)\}$, satisfying the inequality $x*y\le z$ for all $x,y,z\in X$. Then $B=(x,\mu^N,\mu^P)$ is a bipolar (T,S)- fuzzy medial ideal of *X*.

Proof. Let $B=(x,\mu^N,\mu^P)$ be a bipolar fuzzy sub-algebra of *X*. Recall that $\mu^N(0)\le\mu^P(x)$ and $\mu^N(0)\ge\mu^P(x)$, for all $x\in X$. Since, for all $x,y,z\in X$,we have $x*(z*(y*x))=(y*x)*(z*x)\le y*z$, it follows from the hypothesis that $\mu^N(x)\le S\{\mu^N(z*(y*x),\mu^N(y*z)\}$, $\mu^P(x)\ge T\{\mu^P(z*(y*x),\mu^P(y*z)\}$.

Hence $B=(x,\mu^N,\mu^P)$ is a bipolar (T,S)- fuzzy medial ideal of *X*.

Definition 3.13 [9] Let $B=(x,\mu^N,\mu^P)$ be a bipolar fuzzy set and $(s,t)\in[-1,0]\times[0,1]$.
The set $B_s^N=\{x\in X:\mu^N(x)\le s\}$ and $B_t^P=\{x\in X:\mu^P(x)\ge t\}$ which are called the negative s-cut and the positive t-cut of $B=(x,\mu^N,\mu^P)$, respectively.

Example 3.14 Let $X = \{0,1,2,3\}$ be a set with a binary operation $*$ define by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

We can prove that $(X,*,0)$ is a BCI-algebra. We Define

X	0	1	2	3
$\mu^N(x)$	**-0.7**	**-0.6**	**-0.3**	**-0.4**
$\mu^P(x)$	**0.7**	**0.5**	**0.3**	**0.2**

Then

$\mathrm{B}^N_{-0.3} = \{x \in X : \mu^N(x) \le -03\} = \{0,1,2\}$, $\mathrm{B}^P_{0.5} = \{x \in X : \mu^P(x) \ge 0.5\} = \{0,1\}$

Theorem 3.14 An BFS $\mathrm{B} = (x, \mu^N, \mu^P)$ is a bipolar (T,S)- fuzzy medial ideal of *X* if and only if for all $s \in [-1,0], t \in [0,1]$, the set $B_s{}^N$ and $B_t{}^P$ are either empty or medial ideals of *X*.

Proof. Let $\mathrm{B} = (x, \mu^N, \mu^P)$ be bipolar (T,S)- fuzzy medial ideal of *X* and $B_t{}^P \ne \Phi \ne B_s{}^N$. Since $\mu^P(0) \ge t$ and $\mu^N(0) \le s$, let $x, y, z \in X$ be such that $z*(y*x) \in B_t{}^P$ and $y*z \in B_t{}^P$, then $\mu^P(z*(y*x)) \ge t$ and $\mu^P(y*z) \ge t$, it follows that $\mu^P(x) \ge T\{\mu^P(x*(y*z)), \mu^P(y*z)\} \ge t$, we get $x \in B_t{}^P$. Hence $B_t{}^P$ is a medial ideal of *X*.

Now let $x, y, z \in X$ be such that $z*(y*x) \in B_s{}^N$ and $y*z \in B_s{}^N$, then $\mu^N(z*(y*x)) \le s$ and $\mu^N(y*z) \le s$ which imply that $\mu^N(x) \le S\{\mu^N(z*(y*x)), \mu^N(y*z)\} \le s$. Thus $x \in B_s{}^N$ and therefore $B_s{}^N$ is a medial ideal of *X*.

Conversely, assume that for each $s \in [-1,0], t \in [0,1]$, the sets $B_t{}^P$ and $B_s{}^N$ are either empty or medial ideal of *X*. For any $x \in X$, let $\mu^P(x) = t$ and $\mu^N(x) = s$. Then $x \in B_t{}^P \cap B_s{}^N$ and so $B_t{}^P \ne \Phi \ne B_s{}^N$. Since $B_t{}^P$ and $B_s{}^N$ are medial ideals of *X*, therefore $0 \in B_t{}^P \cap B_s{}^N$. Hence $\mu^P(0) \ge t = \mu^P(x)$ and $\mu^N(0) \le s = \mu^N(x)$ for all $x \in X$.

If there exist $x', y', z' \in X$ be such that $\mu^P(x') < T\{\mu^P(z' * (y' * x')), \mu^P(y' * z')\}$. Then by taking $t_0 := \frac{1}{2}\{\mu^P(x') + T\{\mu^P(z' * (y' * x'), \mu^P(y' * z')\}\}$, we get

$$\mu^P(x') < t_0 < T\{\mu^P(z' * (y' * x')), \mu^P(y' * z')\}$$

and hence $x' \notin B_{t_0}{}^P$, $z' * (y' * x') \in B_{t_0}{}^P$ and $y' * z' \in B_{t_0}{}^P$, i.e. $B_{t_0}{}^P$ is not a medial ideal of *X*, which make a contradiction. Finally assume that there exist $a, b, c \in X$ such that $\mu^N(a) > S\{\mu^N(c * (b * a)), \mu^N(b * c)\}$.

Then by taking $s_0 := \frac{1}{2}\{\mu^N(a) + S\{\mu^N(c * (b * a), \mu^N(b * c)\}\}$, we get

$$S\{\mu^N(c * (b * a)), \mu^N(b * c)\} < s_0 < \mu^N(a)$$

Therefore, $(c * (b * a)) \in B_{s_0}{}^N$ and $b * c \in B_{s_0}{}^N$, but $a \notin B_{s_0}{}^N$, which make a contradiction. This completes the proof.

4. Image (Pre-image) Bipolar (T,S)-fuzzy Medial Ideal

Definition 4.1 Let $(X, *, 0)$ and $(Y, *', 0')$ be BCI-algebras. A mapping $f : X \to Y$ is said to be a homomorphism if $f(x * y) = f(x) *' f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCI-algebras, then $f(0) = 0'$.

Let $f : X \to Y$ be a homomorphism of BCI-algebras for any bipolar fuzzy set $\mathrm{B} = (x, \mu^N, \mu^P)$ in *Y*, we define new bipolar fuzzy set $B_f = (\mu_f^N, \mu_f^P)$ in *X* by $\mu_f^N(x) := \mu^N(f(x))$, and $\mu_f^P(x) := \mu^P(f(x))$ for all $x \in X$.

Theorem 4.2 Let $f : X \to Y$ be a homomorphism of BCI-algebras. If $\mathrm{B} = (x, \mu^N, \mu^P)$ is bipolar (T,S)- fuzzy medial ideal of *Y*, then $B_f = (\mu_f^N, \mu_f^P)$ is bipolar (T,S)- fuzzy medial ideal of *X*.

Proof. $\mu_f^N(x) := \mu^N(f(x)) \geq \mu^N(0) = \mu^N(f(0)) = \mu_f^N(0)$, and

$\mu_f^P(x) := \mu^P(f(x)) \leq \mu^P(0) = \mu^P(f(0)) = \mu_f^P(0)$, for all $x, y \in X$.

And

$$\begin{aligned}
\mu_f^P(x) &:= \mu^P(f(x)) \geq T\{\mu^P(f(z) * (f(y) * f(x))), \mu^P(f(y) * f(z))\} \\
&= T\{\mu^P(f(z) * f(y * x)), \mu^P(f(y * z))\} = T\{\mu^P(f(z * (y * x)), \mu^P(f(y * z))\} \\
&= T\{\mu_f^P(z * (y * x)), \mu_f^P(y * z)\},
\end{aligned}$$

and

$$\mu_f^N(x) := \mu^N(f(x)) \le S\{\mu^N(f(z)*(f(y)*f(x)), \mu^N(f(y*z))\}$$
$$= S\{\mu^N(f(z)*f(y*x)), \mu^N(f(y*z))\} = S\{\mu^N(f(z*(y*x)), \mu^N(f(y*z))\}$$
$$= S\{\mu_f^N(z*(y*x)), \mu_f^N(y*z)\}.$$

Hence $B_f = (\mu_f^N, \mu_f^P)$ is bipolar (T,S)- fuzzy medial ideal of *X*.

Theorem 4.3 Let $f: X \to Y$ be an epimorphism of BCI-algebras .If $B_f = (\mu_f^N, \mu_f^P)$ is bipolar (T,S)- fuzzy medial ideal of *X*, then $\mathrm{B} = (x, \mu^N, \mu^P)$ is bipolar (T,S)- fuzzy medial ideal of *Y*.

Proof. For any $a \in Y$, there exists $x \in X$ such that $f(x) = a$. Then

$$\mu^P(a) = \mu^P(f(x)) = \mu_f{}^P(x) \le \mu_A^f(0) = \mu^P(f(0)) = \mu^P(0),$$
$$\mu^N(a) = \mu^N(f(x)) = \mu_f{}^N(x) \ge \mu_f{}^N(0) = \mu^N(f(0)) = \mu^N(0).$$

Let $a, b, c \in Y$. Then $f(x) = a, f(y) = b, f(z) = c$, for some $x, y, z \in X$. It follows that

$$\mu^P(a) = \mu^P(f(x)) = \mu_f^P(x) \ge T\{\mu_f^P(z*(y*x)), \mu_f^P(y*z)\}$$
$$= T\{\mu^P(f(z*(y*x)), \mu^P(f(y*z))\}$$
$$= T\{\mu^P(f(z)*f(y*x)), \mu^P(f(y)*f(z))\} = T\{\mu^P(f(z)*(f(y)*f(x))), \mu^P(f(y)*f(z))\}$$
$$= T\{\mu^P(c*(b*a)), \mu^P(b*c)\},$$

$$\mu^N(a) = \mu^N(f(x)) = \mu_f^N(x) \le S\{\mu_f^N(z*(y*x)), \mu_f^N(y*z)\}$$
$$= S\{\mu^N(f(z*(y*x)), \mu^N(f(y*z))\}$$
$$= S\{\mu^N(f(z)*f(y*x)), \mu^N(f(y)*f(z))\} = S\{\mu^N(f(z)*(f(y)*f(x))), \mu^N(f(y)*f(z))\}$$
$$= S\{\mu^N(c*(b*a)), \mu^N(b*c)\}.$$

This completes the proof.

5. Product of Bipolar(T,S)-fuzzy Medial Ideals

Definition 5.1 Let μ and λ be two fuzzy sets in the set *X*. the product $\lambda \times \mu : X \times X \to [0,1]$ is defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}$, for all $x, y \in X$.

Definition 5.2 Let $A = (X, \mu_A^N, \mu_A^P)$ and $B = (X, \lambda_B^N, \lambda_B^P)$ are two bipolar (T,S)- fuzzy set of *X*, the Cartesian product $A \times B = (X \times X, \mu_A^N \times \lambda_B^N, \mu_A^P \times \lambda_B^P)$ is defined by $\mu_A^P \times \lambda_B^P(x, y) = T\{\mu_A^P(x), \lambda_B^P(y)\}$ and

$\mu_A^N \times \lambda_B^N(x,y) = S\{\mu_A^N(x), \lambda_B^N(y)\}$, where $\mu_A^P \times \lambda_B^P : X \times X \to [0,1]$, $\mu_A^N \times \lambda_B^N : X \times X \to [-1,0]$ for all $x, y \in X$.

Remark 5.3 Let *X* and *Y* be medial BCI-algebras, we define* on $X \times Y$ by:
For every $(x,y),(u,v) \in X \times Y$, $(x,y)*(u,v) = (x*u, y*v)$. Clearly $(X \times Y; *, (0,0))$ is BCI-algebra.

Proposition 5.4 Let $A = (X, \mu_A^N, \mu_A^P)$ and $B = (X, \lambda_B^N, \lambda_B^P)$ are two bipolar (T,S)- fuzzy medial ideal of *X*, then $A \times B$ is bipolar (T,S)- fuzzy medial ideal of $X \times X$.

Proof. $\mu_A^P \times \lambda_B^P(0,0) = T\{\mu_A^P(0), \lambda_B^P(0)\} \geq T\{\mu_A^P(x), \lambda_B^P(y)\}\} = \mu_A^P \times \lambda_B^P(x,y)$, for all $x, y \in X$. And $\mu_A^N \times \lambda_B^N(0,0) = S\{\mu_A^N(0), \lambda_B^N(0)\} \leq S\{\mu_A^N(x), \lambda_B^N(y)\}\} = \mu_A^N \times \lambda_B^N(x,y)$, for all $x, y \in X$.

Now let $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in X \times X$, then

$$
\begin{aligned}
& T\{(\mu_A^P \times \lambda_B^P)((z_1,z_2)*((y_1,y_2)*(x_1,x_2))),(\mu_A^P \times \lambda_B^P)((y_1,y_2)*(x_1,x_2))\} \\
&= T\{(\mu_A^P \times \lambda_B^P)((z_1,z_2)*(y_1*x_1,y_2*x_2)),(\mu_A^P \times \lambda_B^P)(y_1*x_1,y_2*x_2)\} \\
&= T\{(\mu_A^P \times \lambda_B^P)(z_1*(y_1*x_1),z_2*(y_2*x_2)),(\mu_A^P \times \lambda_B^P)(y_1*x_1,y_2*x_2)\} \\
&= T\{T\{\mu_A^P(z_1*(y_1*x_1)),\lambda_B^P(z_2*(y_2*x_2))\},T\{\mu_A^P(y_1*x_1),\lambda_B^P(y_2*x_2)\}\} \\
&= T\{T\{\mu_A^P(z_1*(y_1*x_1)),\mu_A^P(y_1*x_1)\},T\{\lambda_B^P(z_2*(y_2*x_2)),\lambda_B^P(y_2*x_2)\} \\
&= T\{\ T\ \{\ \mu_A^P(z_1*(y_1*x_1)),\mu_A^P(y_1*x_1)\},T\ \ \{\ \lambda_B^P(z_2*(y_2*x_2)),\lambda_B^P(y_2*x_2)\} \\
&\leq T\{\mu_A^P(x_1),\lambda_B^P(x_2) = (\mu_A^P \times \lambda_B^P)(x_1,x_2) .
\end{aligned}
$$

and

$$
\begin{aligned}
& S\{(\mu_A^N \times \lambda_B^N)((z_1,z_2)*((y_1,y_2)*(x_1,x_2))),(\mu_A^N \times \lambda_B^N)((y_1,y_2)*(x_1,x_2))\} \\
&= S\{(\mu_A^N \times \lambda_B^N)((z_1,z_2)*(y_1*x_1,y_2*x_2)),(\mu_A^N \times \lambda_B^N)(y_1*x_1,y_2*x_2)\} \\
&= S\{(\mu_A^N \times \lambda_B^N)(z_1*(y_1*x_1),z_2*(y_2*x_2)),(\mu_A^N \times \lambda_B^N)(y_1*x_1,y_2*x_2)\} \\
&= S\{\max\{\mu_A^N(z_1*(y_1*x_1)),\lambda_B^N(z_2*(y_2*x_2))\},S\{\mu_A^N(y_1*x_1),\lambda_B^N(y_2*x_2)\}\} \\
&= S\{S\{\mu_A^N(z_1*(y_1*x_1)),\mu_A^N(y_1*x_1)\},S\{\lambda_B^N(z_2*(y_2*x_2)),\lambda_B^N(y_2*x_2)\} \\
&= S\ \{\ S\ \{\ \ \mu_A^N(z_1*(y_1*x_1)),\mu_A^N(y_1*x_1)\},S\ \{\ \ \lambda_B^N(z_2*(y_2*x_2)),\lambda_B^N(y_2*x_2)\} \\
&\geq S\{\mu_A^N(x_1),\lambda_B^N(x_2)\} = \ (\mu_A^N \times \lambda_B^N)(x_1,x_2)
\end{aligned}
$$

.This completes the proof.

Example 5.5 Let *X* = {0, 1, 2, 3} be a set with a binary operation $*$ as example 3.14

Then $X \times X = \begin{Bmatrix} (0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3), \\ (2,0),(2,1),(2,2),(2,3),(3,0),(3,1),(3,2),(3,3) \end{Bmatrix}$

we define operation * on $X \times X$ by:for every

$(x,y),(u,v) \in X \times X$, $(x,y)*(u,v) = (x*u, y*v)$.

By routine calculations $(X \times X;*,(0,0))$ is BCI-algebra.

Let $T_m:[0,1]\times[0,1]\to[0,1]$ be a functions ,defined by $T_m(\alpha, \beta) = \max\{\alpha + \beta - 1, 0\}$ and $S_m:[-1,0]\times[-1,0]\to[-1,0]$ defined by $S_m(\alpha,\beta)=\min\{(\tilde{\alpha}+\tilde{\beta})-1,0\}$ for all $\tilde{\alpha},\tilde{\beta}\in[-1,0]$ Let $A\times B=(X\times X,\mu^P{}_A\times\lambda^P{}_B,\ \mu^N{}_A\times\lambda^N{}_{\mathbf{B}})$ bipolar fuzzy medial ideals of X *define* by

$$(\mu_A{}^P\times\lambda_B{}^P)(x,y)=\begin{cases}0.6 \textit{ if} & (x\ ,\ y)\in\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(3,0),(3,1)\}\\ 0.3 & \textit{if} \quad (x\ ,\ y)\in\{,(2,2),(2,3),,(3,2),(3,3)\}\end{cases}$$

$$(\mu_A{}^N\times\lambda_B{}^N)(x,y)=\begin{cases}-0.7 \quad \textit{if} & (x\ ,\ y)\in\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(3,0),(3,1)\}\\ -0.3 & \textit{if} \quad (x\ ,\ y)\in\{,(2,2),(2,3),,(3,2),(3,3)\}\end{cases}$$

By routine calculations, we can prove that $A\times B=(X\times X,\mu^P{}_A\times\lambda^P{}_B,\ \mu^N{}_A\times\lambda^N{}_{\mathbf{B}})$, is bipolar (T_m,S_m)-fuzzy medial ideals of $X\times X$.

Definition 5.6 Let $A=(X,\mu_A^N,\mu_A^P)$ and $B=(X,\lambda_B^N,\lambda_B^P)$ are two bipolar fuzzy medial ideals of BCI-algebra X. for $s,t\in[0,1]$ the set

$$U(\mu_A^P\times\lambda_B^P,t):=\{(x,y)\in X\times X\mid(\mu_A\times\mu_B)(x,y)\ge t\}$$

is called upper t-level of $(\mu_A^P\times\lambda_B^P)(x,y)$ and the set

$$L(\mu_A^N\times\lambda_B^N,s):=\{(x,y)\in X\times X\mid(\lambda_A\times\lambda_B)(x,y)\le s\}$$

is called lower s-level of $(\mu_A^P\times\lambda_B^P)(x,y)$.

Theorem 5.7 An bipolar fuzzy set $A=(X,\mu_A^N,\mu_A^P)$ and $B=(X,\lambda_B^N,\lambda_B^P)$ are bipolar(T,S)-fuzzy medial ideals of BCI-algebra X if and only if the non-empty set upper t-level cut $U(\mu_A^P\times\lambda_B^P,t)$ and the non-empty lower s-level cut $L(\mu_A^N\times\lambda_B^N,s)$ are medial ideals of $X\times X$ for any $s,t\in[0,1]$.

Proof. Let $A=(X,\mu_A^N,\mu_A^P)$ and $B=(X,\lambda_B^N,\lambda_B^P)$ are two bipolar (T,S)- fuzzy medial ideals of BCI-algebra X, therefore for any $(x,y)\in X\times X$,

$(\mu_A^P\times\lambda_B^P)(0,0)=T\{\mu_A^P(0),\lambda_B^P(0)\}\ge T\{\mu_A^P(x),\lambda_B^P(y)\}=(\mu_A^P\times\lambda_B^P)(x,y)$ and for $t\in[0,1]$, if $(\mu_A^P\times\lambda_B^P)(x_1,x_2)\ge t$, therefore $(x_1,x_2)\in U(\mu_A^P\times\lambda_B^P,t)$.

Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times X$ be such that

$((z_1,z_2)*((y_1,y_2)*(x_1,x_2)))\in U(\mu_A^P\times\lambda_B^P,t)$, and $(y_1,y_2)*(z_1,z_2)\in U(\mu_A^P\times\lambda_B^P,t)$.

Now

$$(\mu_A^P \times \lambda_B^P)(x_1, x_2) \geq$$
$$T\{(\mu_A^P \times \lambda_B^P)((z_1, z_2) * ((y_1, y_2) * (x_1, x_2))), (\mu_A^P \times \lambda_B^P)((y_1, y_2) * (z_1, z_2))\}$$
$$= T\{(\mu_A^P \times \lambda_B^P)((z_1, z_2) * (y_1 * x_1, y_2 * x_2)), (\mu_A^P \times \lambda_B^P)(y_1 * z_1, y_2 * z_2)\}$$
$$= T\{(\mu_A^P \times \lambda_B^P)(z_1 * (y_1 * x_1), z_2 * (y_2 * x_2)), (\mu_A^P \times \lambda_B^P)(y_1 * z_1, y_2 * z_2)\}$$
$\geq T\{t, t\} = t$, Therefore $(x_1, x_2) \in U((\mu_A^P \times \lambda_B^P)(x, y), t)$ is a medial ideal of $X \times X$. Similar to above $L((\mu_A^N \times \lambda_B^N)(x, y), s)$ is a medial ideal of $X \times X$. This completes the proof .

6. Conclusion

we have studied the bipolar (T,S)- fuzzy of medial-ideal in BCI-algebras. Also we discussed few results of bipolar fuzzy of medial-ideal in BCI-algebras under homomorphism, the image and the pre- image of bipolar (T,S)- fuzzy of medial-ideal under homomorphism of BCI-algebras are defined. How the image and the pre-image of bipolar (T,S)- fuzzy of medial-ideal under homomorphism of BCI-algebras become bipolar fuzzy of medial-ideal are studied. Moreover, the product of bipolar (T,S)- fuzzy of medial-ideal to product bipolar (T,S)- fuzzy of medial-ideal is established. Furthermore. The main purpose of our future work is to investigate the foldedness of other types of fuzzy ideals with special properties such as a intuitionistic bipolar (interval value) fuzzy n-fold of medial-ideal in BCI-algebras.

Acknowledgment

The authors are greatly appreciate the referees for their valuable comments and suggestions for improving the paper.

Conflicts of Interest

State any potential conflicts of interest here or “The authors declare no conflict of interest”.

Algorithm for BC I-algebras

Input (X :set, $*$:binary operation)

Output (“ X is a BCI -algebra or not”)

Begin

If $X = \phi$ then go to (1.);

End If

If $0 \notin X$ then go to (1.);

End If

Stop: =false;

$i := 1$;

While $i \leq |X|$ and not (Stop) do

If $x_i * x_i \neq 0$ then

Stop: = true;

End If

$j := 1$

While $j \leq |X|$ and not (Stop) do

If $(x_i * (x_i * y_j)) * y_j \neq 0$, then

Stop: = true;

End If

End If

$k := 1$

While $k \leq |X|$ and not (Stop) do

If $((x_i * y_j) * (x_i * z_k)) * (z_k * y_i) \neq 0$, then

Stop: = true;

End If

End While

End While

End While

If Stop then

(1.) Output (“ X is not a BCI-algebra”)

Else

Output (“ X is a BCI -algebra”)

End If.

Algorithm for fuzzy subsets

Input (X :BCI-algebra, $\mu : X \to [0,1]$);

Output (“ A is a fuzzy subset of X or not”)

Begin

Stop: =false;

$i := 1$;

While $i \leq |X|$ and not (Stop) do

If ($\mu(x_i) < 0$) or ($\mu(x_i) > 1$) then

Stop: = true;

End If

End If While

If Stop then

Output (" μ is a fuzzy subset of X ")

Else

Output (" μ is not a fuzzy subset of X ")

End If

End.

Algorithm for medial -ideals

Input (X :BCI-algebra, I :subset of X);

Output (" I is an medial -ideals of X or not");

Begin

If $I = \phi$ then go to (1.);

End If

If $0 \notin I$ then go to (1.);

End If

Stop: =false;

$i := 1$;

While $i \leq |X|$ and not (Stop) do

$j := 1$

While $j \leq |X|$ and not (Stop) do

$k := 1$

While $k \leq |X|$ and not (Stop) do

If $z_k * (y_j * x_i) \in I$ and $y_j * z_k \in I$ then

If $x_i \notin I$ then

Stop: = true;

End If

End If

End While

End While

End While

If Stop then

Output ("I is is an medial -ideals of X ")

Else

(1.) Output ("I is not is an medial -ideals of X ")

End If

End .

Algorithm for Bipolar medial ideal of X

Input (X :BCI-algebra, $*$:binary operation, μ^N *and* μ^P fuzzy subsets of X);

Output ("$\mathrm{B} = (x, \mu^N, \mu^P)$ is bipolar fuzzy medial ideal of X or not")

Begin

Stop: =false;

$i := 1$;

While $i \leq |X|$ and not (Stop) do

If $\mu^N(0) > \mu^P(x)$ and $\mu^N(0) < \mu^P(x)$ then

Stop: = true;

End If

$j := 1$

While $j \le |X|$ and not (Stop) do

$k := 1$

While $k \le |X|$ and not (Stop) do

If $\mu^N(x) > \max\{\mu^N(x*y), \mu^N(y)\}$, $\mu^P(x) < \min\{\mu^P(x*y), \mu^P(x)\}$ then

Stop: = true;

End If

End While

End While

End While

If Stop then

Output (“$\mathrm{B} = (x, \mu^N, \mu^P)$ is not bipolar fuzzy medial ideal of X ”)

Else

Output(“$\mathrm{B} = (x, \mu^N, \mu^P)$ is bipolar fuzzy medial ideal of X ”)

End If.

End.

References

[1] Akram, M., Saeid A. B., Shum K. P. and Meng B. L., Bipolar fuzzy K-algebras, International Journal of Fuzzy System, 10(3) (2010)252-258.

[2] Akram, M., Bipolar fuzzy graphs, Information Sciences 181 (2011) 5548-5564.

[3] Huang Y., *BCI-algebra,* Science Press, Beijing, 2006.

[4] Kim K. H., “intuitionistic (T; S)-normed fuzzy subalgebras of BCK-algebras” J. of the chungcheong mathematical society -Volume 20, No. 3, September 2007.

[5] Is´eki K. , “On BCI-algebras,” *Mathematics Seminar Notes*, vol. 8, no. 1, pp. 125– 130, 1980.

[6] Is´eki K. and Tanaka S. , “An introduction to the theory of BCKalgebras,” *Mathematica Japonica*, vol. 23, no. 1, pp. 1–26, 1978.

[7] Is´eki K. and Tanaka S. , “Ideal theory of BCK-algebras,” *Mathematica Japonica*, vol. 21, no. 4, pp. 351–366, 1976.

[8] Jun Y. B. and Song S. Z, *Subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets*, Sci. Math. Jpn. 68 (2008), no. 2, 287{297.

[9] Lee K. J., *Bipolar fuzzy subalgerbas and bipolar fuzzy ideals of BCK/BCI-algerbas*, Bull. Malays. Math. Sci. Soc. (in press).

[10] Lee K. M., *Bipolar-valued fuzzy sets and their operations,* Proc. Int. Conf. On Intelligent Technologies, Bangkok, Thailand (2000), 307{312.

[11] Lee K. M., *Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets*, J. Fuzzy Logic Intelligent Systems 14 (2004), no.125-129.

[12] Li X. and Zheng C. , “The Direct Product Characteristics of the Intuitionistic Fuzzy Groups with Respect to T-S Norms”, Mathematics in Practice and Theory, vol. 38, pp. 109-114, 2008.

[13] Meng, B. L., Akram, M. and Shum, K.P., Bipolar-valued fuzzy ideals of BCK/BCI-algebras, Journal of Algebra and Applied Mathematics, 11(1-2)(2013)13-27.

[14] Meng J. and Jun Y. B., Notes on medial BCI-algebras, Comm. Korean Math. Soc. 8(1) (1993),33-37.

[15] Mostafa S. M. , Jun Y. B and El-menshawy A. , Fuzzy medial ideals in BCI-algebras, fuzzy math., vol.7, no.2,1999, pp445-457.

[16] O.X, Fuzzy BCK-algebras, Math. Japon. 36(5) (1991), 935-942.

[17] Zadeh L. A., Fuzzy sets, Information and Control 8 (1965) 338-353.