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(λ, μ)-FUZZY IDEALS OF ORDERED Γ-SEMIRINGS

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Abstaract — The notions of (λ, μ)-fuzzy ideals, (λ, μ)-fuzzy k-ideals, (λ, μ)-fuzzy k-bi-ideals, (λ, μ)-fuzzy k-quasi-ideals of an ordered Γ-semirings are introduced and some related properties are investigated. The concepts of k-regularity, k-intra-regularity are studied along with some of their characterizations.

Keywords — *Cartesian product, (λ, μ)-fuzzy ideal, homomorphism, k-intra-regular, k-regular, ordered Γ-semiring.*

1 Introduction

Uncertainties, which could be caused by information incompleteness, data randomness limitations of measuring instruments, etc., are pervasive in many complicated problems in biology, engineering, economics, environment, medical science and social science. We cannot successfully use the classical methods for these problems. To solve this, the concept of fuzzy sets was introduced by Zadeh [12] in 1965 where each element have a degree of membership and has been extensively applied to many scientific fields.

Semirings [3] which provide a common generalization of rings and distributive lattices arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, graph theory, automata theory, mathematical modelling and parallel computation systems etc.(for example, see [3], [4]). Semirings have also been proved to be an important algebraic tool in theoretical computer science, see for instance [4], for some detail and example. Many of the semirings have an order structure in addition to their algebraic structure and indeed the most interesting results concering them make use of the interplay between these two structures.

Ideals of semirings play an important role in the structure theory of ordered semirings and useful for many purposes. In this paper, like ordered semigroup [2, 5, 6, 8], it is an attempt to study how similar is the theory in terms of fuzzy for the case of ordered Γ-semiring [10], a generalization of ordered semirings [9] , since nowadays fuzzy research concerns standardization, axiomatization, extensions to lattice-valued fuzzy sets, critical comparison of the different so-called soft computing models that have

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been launched during the past three decennia for the representation and processing of incomplete information [7].

Here we first introduce (λ, μ)-fuzzy ideals of ordered Γ-semirings. After that we define intersection, cartesian product and composition of (λ, μ)-fuzzy ideals and use these to study regular (resp. intra-regular) ordered Γ-semirings.

2 Preliminaries

We recall the following definitions for subsequent use.

Definition 2.1. Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ ((a,α,b) $\mapsto$ aαb) satisfying the following conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$

(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$

(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$

(v) $0_S\alpha a = 0_S = a\alpha 0_S$

(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$

where $a, b, c \in S$, $\alpha, \beta \in \Gamma$, 0_S is the zero element of S and 0_Γ is the zero element of Γ.

For simplification we write 0 instead of 0_S and 0_Γ.

Definition 2.2. A left ideal I of Γ-semiring S is a nonempty subset of S satisfying the following conditions:

(i) If $a, b \in I$ then $a + b \in I$

(ii) If $a \in I$, $s \in S$ and $\gamma \in \Gamma$ then $s\gamma a \in I$

(iii) $I \neq S$.

A right ideal of S is defined in an analogous manner and an ideal of S is a nonempty subset which is both a left ideal and a right ideal of S.

Definition 2.3. An ordered semiring is a Γ-semiring S equipped with a partial order $\leq$ such that the operation is monotonic and constant 0 is the least element of S.

Definition 2.4. Let R, S be two Γ-semirings and $a, b \in R$, $\gamma \in \Gamma$. A function $f : R \to S$ is said to be a homomorphism if

(i) $f(a + b) = f(a) + f(b)$

(ii) $f(a\gamma b) = f(a)\gamma f(b)$

(iii) $f(0_R) = 0_S$ where 0_R and 0_S are the zeroes of R and S respectively.

Now we recall the definition and example of ordered ideal from [1]

Definition 2.5. A left (resp. right) ideal I of S is called a left (resp. right) ordered ideal, if for any $a \in S$, $b \in I$, $a \leq b$ implies $a \in I$ (i.e. $(I] \subseteq I$). I is called an ordered ideal of S if it is both a left and a right ordered ideal of S.

Example 2.6. Let $S = ([0,1], \vee, \cdot, 0)$ where [0,1] is the unit interval $a \vee b = \max\{a, b\}$ and $a \cdot b = (a + b - 1) \vee 0$ for $a, b \in [0, 1]$. Then it is easy to verify that S equipped with the usual ordering $\leq$ is an ordered semiring and $I = [0, \frac{1}{2}]$ is an ordered ideal of S.

Definition 2.7. [12] Let S be a non-empty set. A mapping $f : S \to [0, 1]$ is called a fuzzy subset of S.

Definition 2.8. The union and intersection of two fuzzy subsets f and σ of a set S, denoted by $f \cup \sigma$ and $f \cap \sigma$ respectively, are defined by

$$(f \cup \sigma)(x) = f(x) \vee \sigma(x) \ \text{ for all } x \in S$$

$$(f \cap \sigma)(x) = f(x) \wedge \sigma(x) \ \text{ for all } x \in S.$$

3 (λ, μ)-fuzzy ideals with some operations

Throughout this paper unless otherwise mentioned S denote the ordered Γ-semiring with identity 1, χ_S denote its characteristic function and we will always assume that $0 \leq \lambda < \mu \leq 1$.

Definition 3.1. Let f and g be two fuzzy subsets of an ordered Γ-semiring S. We define two compositions of f and g as follows:

$$\begin{aligned} f \circ_1 g(x) &= \vee\{\underset{x+y_1\alpha z_1 \leq y_2\beta z_2}{\wedge} \{f(y_1), f(y_2), g(z_1), g(z_2)\}\} \\ &= 0, \text{ if } x \text{ cannot be expressed as } x + y_1\alpha z_1 \leq y_2\beta z_2 \\ &\quad \text{where } x, y_1, y_2, z_1, z_2 \in S \text{ and } \alpha, \beta \in \Gamma \end{aligned}$$

and

$$\begin{aligned} fo_2g(x) &= \vee[\underset{i}{\wedge}\{\wedge\{f(a_i), f(c_i), g(b_i), g(d_i)\}\}] \\ &\quad {}_{x+\sum_{i=1}^{n} a_i\alpha_i b_i \leq \sum_{i=1}^{n} c_i\beta_i d_i} \\ &= 0, \text{ if } x \text{ cannot be expressed as above} \\ &\quad \text{where } x, z, a_i, b_i, c_i, d_i \in S \text{ and } \alpha_i, \beta_i \in \Gamma. \end{aligned}$$

Definition 3.2. Let f be a non-empty fuzzy subset of an ordered Γ-semiring S (i.e. $f(x) \neq 0$ for some $x \in S$). Then f is called a (λ, μ)-fuzzy left ideal [resp. (λ, μ)-fuzzy right ideal] of S if

(i) $f(x + y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$

(ii) $f(x\gamma y) \vee \lambda \geq f(y) \wedge \mu$ [resp. $f(x\gamma y) \vee \lambda \geq f(x) \wedge \mu$] and

(iii) $x \leq y$ implies $f(x) \vee \lambda \geq f(y) \wedge \mu$.

for all $x, y \in S$ and $\gamma \in \Gamma$.
A (λ, μ)-fuzzy ideal of an ordered Γ-semiring S is a non-empty (λ, μ)-fuzzy subset of S which is a (λ, μ)-fuzzy left ideal as well as a (λ, μ)-fuzzy right ideal of S.
Note that for (λ, μ)-fuzzy k-ideal the following additional relation must be holds: For $x, a, b \in S$ with $x + a \leq b \Rightarrow f(x) \vee \lambda \geq f(a) \wedge f(b) \wedge \mu$.

Example 3.3. Let $S = \Gamma = \{0, a, b\}$ with the ordered relation $0 \prec b \prec a$. Define operations on S by following:

$\oplus$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

and

$\odot$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Then $(S, \oplus, \odot)$ forms an ordered semiring.
We now define a fuzzy subset μ of S by $\mu(0) = 1,\ \mu(b) = 0.2,\ $ and $\mu(a) = 0.1$, then μ will be a $(0.5, 0.8)$-fuzzy right ideal of S.

Theorem 3.4. Let S be an ordered Γ-semiring and f be a (λ, μ)-fuzzy right (resp. left) ideal of S. Then $I_a = \{b \in S | f(b) \vee \lambda \geq f(a) \wedge \mu\}$ is a right (resp. left) ideal of S for every $a \in S$.

Proof. Let f be a (λ, μ)-fuzzy right ideal of S and $a \in S$. Then $I_a \neq \phi$ because $a \in I_a$ for every $a \in S$. Let $b, c \in I_a$, $\gamma \in \Gamma$ and $x \in S$. Since $b, c \in I_a$, $f(b) \vee \lambda \geq f(a) \wedge \mu$ and $f(c) \vee \lambda \geq f(a) \wedge \mu$. Now

$$\begin{aligned} f(b+c) \vee \lambda &\geq f(b) \wedge f(c) \wedge \mu \ [\because f \text{ is a } (\lambda, \mu)\text{-fuzzy right ideal}] \\ &\geq f(a) \wedge \mu. \end{aligned}$$

which implies $b + c \in I_a$.
Also $f(b\gamma x) \vee \lambda \geq f(b) \wedge \mu \geq f(a) \wedge \mu$ i.e. $b\gamma x \in I_a$.
Let $b \in I_a$ and $S \ni x \leq b$. Then $f(x) \vee \lambda \geq f(b) \wedge \mu \geq f(a) \wedge \mu \Rightarrow x \in I_a$.
Thus I_a is a right ideal of S.
Similarly we can prove the result for left ideal also. □

Proposition 3.5. Intersection of a non-empty collection of (λ, μ)-fuzzy right (resp. left) ideals is also a (λ, μ)-fuzzy right (resp. left) ideal of S.

Proof. Let $\{f_i : i \in I\}$ be a non-empty family of (λ, μ)-fuzzy right ideals of S and $x, y \in S$, $\gamma \in \Gamma$.
Then

$$\begin{aligned} (\bigcap_{i \in I} f_i)(x+y) \vee \lambda &= \bigwedge_{i \in I} \{f_i(x+y) \vee \lambda\} \geq \bigwedge_{i \in I} \{f_i(x) \wedge f_i(y) \wedge \mu\} \\ &= \wedge\{\bigwedge_{i \in I} f_i(x), \bigwedge_{i \in I} f_i(y)\} \wedge \mu = (\bigcap_{i \in I} f_i)(x) \wedge (\bigcap_{i \in I} f_i)(y) \wedge \mu. \end{aligned}$$

Again

$$(\bigcap_{i \in I} f_i)(x\gamma y) \vee \lambda = \bigwedge_{i \in I} \{f_i(x\gamma y) \vee \lambda\} \geq \bigwedge_{i \in I} f_i(x) \wedge \mu = (\bigcap_{i \in I} f_i)(x) \wedge \mu.$$

Suppose $x \leq y$. Then $f_i(x) \vee \lambda \geq f_i(y) \wedge \mu$ for all $i \in I$ which implies $(\bigcap_{i \in I} f_i)(x) \vee \lambda \geq (\bigcap_{i \in I} f_i)(y) \wedge \mu$. Hence $\bigcap_{i \in I} f_i$ is a (λ, μ)-fuzzy right ideal of S.
Similarly we can prove the result for (λ, μ)-fuzzy left ideal also. □

Proposition 3.6. Let $f : R \to S$ be a morphism of ordered Γ-semirings i.e. Γ-semiring homomorphism satisfying additional condition $a \leq b \Rightarrow f(a) \leq f(b)$. Then if ϕ is a (λ, μ)-fuzzy left ideal of S, then $f^{-1}(\phi)$ [11] is also a (λ, μ)-fuzzy left ideal of R.

Proof. Let $f : R \to S$ be a morphism of ordered semirings.
Let ϕ be a (λ, μ)-fuzzy left ideal of S.
Now $f^{-1}(\phi)(0_R) \vee \lambda = \phi(0_S) \vee \lambda \geq \phi(x^{'}) \neq 0$ for some $x^{'} \in S$.
Therefore $f^{-1}(\phi)$ is non-empty.
Now, for any $r, s \in R$ and $\gamma \in \Gamma$

$$\begin{aligned} f^{-1}(\phi)(r+s) \vee \lambda &= \phi(f(r+s)) \vee \lambda = \phi(f(r)+f(s)) \vee \lambda \\ &\geq \phi(f(r)) \wedge \phi(f(s)) \wedge \mu = (f^{-1}(\phi))(r) \wedge (f^{-1}(\phi))(s) \wedge \mu. \end{aligned}$$

Again

$$\begin{aligned} (f^{-1}(\phi))(r\gamma s) \vee \lambda &= \phi(f(r\gamma s)) \vee \lambda = \phi(f(r)\gamma f(s)) \vee \lambda \\ &\geq \phi(f(s)) \wedge \mu = (f^{-1}(\phi))(s) \wedge \mu. \end{aligned}$$

Also if $r \leq s$, $f(r) \leq f(s)$. Then

$$(f^{-1}(\phi))(r) \vee \lambda = \phi(f(r)) \vee \lambda \geq \phi(f(s)) \wedge \mu = (f^{-1}(\phi))(s) \wedge \mu.$$

Thus $f^{-1}(\phi)$ is a (λ, μ)-fuzzy left ideal of R.

□

Definition 3.7. Let f and g be fuzzy subsets of X. The cartesian product of f and g is defined by $(f \times g)(x, y) = f(x) \wedge g(y)$ for all $x, y \in X$.

Theorem 3.8. Let f and g be (λ, μ)-fuzzy left ideals of an ordered Γ-semiring S. Then $f \times g$ is a (λ, μ)-fuzzy left ideal of $S \times S$.

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} (f \times g)((x_1, x_2) + (y_1, y_2)) \vee \lambda &= (f \times g)(x_1 + y_1, x_2 + y_2) \vee \lambda \\ &= (f(x_1 + y_1) \wedge g(x_2 + y_2)) \vee \lambda \\ &= (f(x_1 + y_1) \vee \lambda) \wedge (g(x_2 + y_2) \vee \lambda) \\ &\geq (f(x_1) \wedge f(y_1) \wedge \mu) \wedge (g(x_2) \wedge g(y_2) \wedge \mu) \\ &= (f \times g)(x_1, x_2) \wedge (f \times g)(y_1, y_2) \wedge \mu \end{aligned}$$

and

$$\begin{aligned} (f \times g)((x_1, x_2)\gamma(y_1, y_2)) \vee \lambda &= (f \times g)(x_1\gamma y_1, x_2\gamma y_2) \vee \lambda \\ &= (f(x_1\gamma y_1) \wedge g(x_2\gamma y_2)) \vee \lambda \\ &= (f(x_1\gamma y_1) \vee \lambda) \wedge (g(x_2\gamma y_2) \vee \lambda) \\ &\geq f(y_1) \wedge g(y_2) \wedge \mu = (f \times g)(y_1, y_2) \wedge \mu. \end{aligned}$$

Also if $(x_1, x_2) \leq (y_1, y_2)$, then

$$\begin{aligned} (f \times g)(x_1, x_2) \vee \lambda &= (f(x_1) \wedge g(x_2)) \vee \lambda = (f(x_1) \vee \lambda) \wedge (g(x_2) \vee \lambda) \\ &\geq (f(y_1) \wedge \mu) \wedge (g(y_2) \wedge \mu) = f(y_1) \wedge g(y_2) \wedge \mu \\ &= (f \times g)(y_1, y_2) \wedge \mu. \end{aligned}$$

Therefore $f \times g$ is a (λ, μ)-fuzzy left ideal of $S \times S$. □

Theorem 3.9. Let f be a (λ,μ)-fuzzy subset in an ordered Γ-semiring S. Then f is a (λ,μ)-fuzzy left ideal of S if and only if $f\times f$ is a (λ,μ)-fuzzy left ideal of $S\times S$.

Proof. Assume that f is a (λ,μ)-fuzzy left ideal of S. Then by Theorem 3.8, $f\times f$ is a (λ,μ)-fuzzy left ideal of $S\times S$.
Conversely, suppose that $f\times f$ is a (λ,μ)-fuzzy left ideal of $S\times S$. Let $x_1,x_2,y_1,y_2\in S$ and $\gamma\in\Gamma$. Then

$$\begin{aligned}(f(x_1+y_1)\wedge f(x_2+y_2))\vee\lambda &= (f\times f)(x_1+y_1,x_2+y_2)\vee\lambda\\ &= (f\times f)((x_1,x_2)+(y_1,y_2))\vee\lambda\\ &\geq (f\times f)(x_1,x_2)\wedge(f\times f)(y_1,y_2)\wedge\mu\\ &= (f(x_1)\wedge f(x_2)\wedge\mu)\wedge(f(y_1)\wedge f(y_2)\wedge\mu).\end{aligned}$$

Now, putting $x_1=x,\ x_2=0,\ y_1=y$ and $y_2=0$, in this inequality and noting that $f(0)\geq f(x)$ for all $x\in S$, we obtain $f(x+y)\vee\lambda\geq f(x)\wedge f(y)\wedge\mu$.
Next, we have

$$\begin{aligned}(f(x_1\gamma y_1)\wedge f(x_2\gamma y_2))\vee\lambda &= (f\times f)(x_1\gamma y_1,x_2\gamma y_2)\vee\lambda\\ &= (f\times f)((x_1,x_2)\gamma(y_1,y_2))\vee\lambda\\ &\geq (f\times f)(y_1,y_2)\wedge\mu\\ &= f(y_1)\wedge f(y_2)\wedge\mu.\end{aligned}$$

Taking $x_1=x,\ y_1=y$ and $y_2=0$, we obtain $f(x\gamma y)\vee\lambda\geq f(y)\wedge\mu$.
Also if $(x_1,x_2)\leq(y_1,y_2)$, then $(f(x_1)\wedge f(x_2))\vee\lambda\geq f(y_1)\wedge f(y_2)\wedge\mu$. Now, putting $x_1=x,\ x_2=0,\ y_1=y$ and $y_2=0$, in this inequality we have $f(x)\vee\lambda\geq f(y)\wedge\mu$.
Hence f is a (λ,μ)-fuzzy left ideal of S. □

Theorem 3.10. If $f_1,\ f_2$ be any two (λ,μ)-fuzzy k-ideals of an ordered semiring S then $f_1o_2f_2$ is a (λ,μ)-fuzzy ideal of S.

Proof. Let $f_1,\ f_2$ be any two (λ,μ)-fuzzy k-ideals of an ordered semiring S and $x,\ y\in S$ and $\gamma\in\Gamma$. Then

$$\begin{aligned}&(f_1o_2f_2)(x+y)\vee\lambda\\ &= \bigvee_{x+y+\sum a_i\alpha_ib_i\leq\sum c_i\beta_id_i}\{\wedge\{f_1(a_i),f_2(b_i),f_1(c_i),f_2(d_i)\}\}\vee\lambda\\ &\geq \bigvee_{x+y+\sum x_{1i}\alpha_{1i}x_{2i}+\sum y_{1i}\alpha_{2i}y_{2i}\leq\sum x_{3i}\beta_{1i}x_{4i}+\sum y_{3i}\beta_{2i}y_{4i}}\{\wedge\{f_1(x_{1i}),f_1(x_{3i}),f_1(y_{1i}),f_1(y_{3i}),\quad f_2(x_{2i}),f_2(x_{4i}),f_2(y_{2i}),f_2(y_{4i})\}\}\vee\lambda\\ &\geq \wedge\{\bigvee_{x+\sum x_{1i}\alpha_{1i}x_{2i}\leq\sum x_{3i}\beta_{1i}x_{4i}}\{\wedge\{f_1(x_{1i}),f_1(x_{3i}),f_2(x_{2i}),f_2(x_{4i})\}\},\bigvee_{y+\sum y_{1i}\alpha_{2i}y_{2i}\leq\sum y_{3i}\beta_{2i}y_{4i}}\{\wedge\{f_1(y_{1i}),f_1(y_{3i}),f_2(y_{2i}),f_2(y_{4i})\}\}\}\wedge\mu\\ &= (f_1o_2f_2)(x)\wedge(f_1o_2f_2)(y)\wedge\mu.\end{aligned}$$

Now assuming $f_1,\ f_2$ are as (λ,μ)-fuzzy right ideals we have

$$\begin{aligned}(f_1o_2f_2)(x\gamma y)\vee\lambda &= \bigvee_{x\gamma y+\sum a_i\gamma_ib_i\leq\sum c_i\delta_id_i}\{\wedge\{f_1(a_i),f_2(b_i),f_1(c_i),f_2(d_i)\}\}\vee\lambda\\ &\geq \bigvee_{x\gamma y+\sum x_{1i}\alpha_ix_{2i}\gamma y\leq\sum x_{3i}\beta_ix_{4i}\gamma y}\{\wedge\{f_1(x_{1i}),f_1(x_{3i}),\quad f_2(x_{2i}\gamma y),f_2(x_{4i}\gamma y)\}\}\vee\lambda\\ &\geq \bigvee_{x+\sum x_{1i}\alpha_ix_{2i}\leq\sum x_{3i}\beta_ix_{4i}}\{\wedge\{f_1(x_{1i}),f_1(x_{3i}),f_2(x_{2i}),f_2(x_{4i})\}\}\wedge\mu\\ &= (f_1o_2f_2)(x)\wedge\mu.\end{aligned}$$

Similarly, assuming $f_1,\ f_2$ are as (λ,μ)-fuzzy left k-ideals we can show that $(f_1o_2f_2)(x\gamma y)\geq(f_1o_2f_2)(y)$.

Now suppose $x \leq y$. Then $f_1(x) \vee \lambda \geq f_1(y) \wedge \mu$ and $f_2(x) \vee \lambda \geq f_2(y) \wedge \mu$.

$$\begin{aligned}(f_1 o_2 f_2)(x) \vee \lambda &= \underset{x+\sum x_{1i}\alpha_i x_{2i} \leq \sum x_{3i}\beta_i x_{4i}}{\vee}\{\wedge\{f_1(x_{1i}), f_1(x_{3i}), f_2(x_{2i}), f_2(x_{4i})\}\} \vee \lambda \\ &\geq \underset{x+\sum y_{1i}\gamma_{1i}y_{2i} \leq y+\sum y_{1i}\gamma_{2i}y_{2i} \leq \sum y_{3i}\gamma_{3i}y_{4i}}{\vee}\{\wedge\{f_1(y_{1i}), f_1(y_{3i}), f_2(y_{2i}), f_2(y_{4i})\}\} \wedge \mu \\ &= \underset{y+\sum y_{1i}\gamma_{1i}y_{2i} \leq \sum y_{3i}\gamma_{3i}y_{4i}}{\vee}\{\wedge\{f_1(y_{1i}), f_1(y_{3i}), f_2(y_{2i}), f_2(y_{4i})\}\} \wedge \mu \\ &= (f_1 o_2 f_2)(y) \wedge \mu.\end{aligned}$$

Hence $f_1 o_2 f_2$ is a (λ, μ)-fuzzy ideal of S. □

4 (λ, μ)-fuzzy ideals of regular ordered Γ-semiring

Definition 4.1. A fuzzy subset f of an ordered semiring S is called (λ, μ)-fuzzy bi-ideal if for all $x, y \in S$ and $\alpha,\ \beta \in \Gamma$ we have

(i) $f(x+y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$

(ii) $f(x\alpha y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$

(iii) $f(x\alpha y\beta z) \vee \lambda \geq f(x) \wedge f(z) \wedge \mu$

(iv) $x \leq y \Rightarrow f(x) \vee \lambda \geq f(y) \wedge \mu$.

Definition 4.2. A (λ, μ)-fuzzy subset f of an ordered Γ-semiring S is called (λ, μ)-fuzzy quasi-ideal if for all $x, y \in S$ we have

(i) $f(x+y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$

(ii) $((f o_2 \chi_S) \cap (\chi_S o_2 f))(x) \wedge \mu \leq f(x) \vee \lambda$

(iii) $x \leq y \Rightarrow f(x) \vee \lambda \geq f(y) \wedge \mu$.

Proposition 4.3. Intersection of a non-empty collection of (λ, μ)-fuzzy bi-ideals of S is also a (λ, μ)-fuzzy bi-ideal of S.

Proof. The proof follows by routine verifications. □

Proposition 4.4. Let $\{f_i : i \in I\}$ be a family of bi-ideals of S such that $f_i \subseteq f_j$ or $f_j \subseteq f_i$ for $i, j \in I$. Then $\underset{i \in I}{\cup} f_i$ is a (λ, μ)-fuzzy bi-ideal of S.

Proof. Straightforward. □

Lemma 4.5. In an ordered Γ-semiring every (λ, μ)-fuzzy quasi ideals are (λ, μ)-fuzzy bi-ideals.

Proof. Let f be a (λ, μ)-fuzzy quasi ideal of S. It is sufficient to prove that $f(x\alpha y\beta z) \vee \lambda \geq f(x) \wedge f(z) \wedge \mu$ for all $x,\ y,\ z \in S$ and $\alpha,\ \beta \in \Gamma$. Since f is a (λ, μ)-fuzzy quasi ideal of S, we have

$$\begin{aligned}f(x\alpha y\beta z) \vee \lambda &\geq ((f o_2 \chi_S) \cap (\chi_S o_2 \mu))(x\alpha y\beta z) \wedge \mu \\ &= \{(f o_2 \chi_S)(x\alpha y\beta z) \wedge (\chi_S o_2 f)(x\alpha y\beta z)\} \wedge \mu \\ &= \wedge\{\underset{x\alpha y\beta z+\sum_{i=1}^{n} a_i\gamma_i b_i \leq \sum_{i=1}^{n} c_i\delta_i d_i}{\vee}(f(a_i) \wedge f(c_i)), \underset{x\alpha y\beta z+\sum_{i=1}^{n} a_i\gamma_i b_i \leq \sum_{i=1}^{n} c_i\delta_i d_i}{\vee}(f(b_i) \wedge f(d_i))\} \wedge \mu \\ &\geq f(0) \wedge f(x) \wedge f(0) \wedge f(z)) \wedge \mu (\text{since } x\alpha y\beta z + 0\gamma 0 + 0 = x\alpha y\beta z + 0) \\ &= f(x) \wedge f(z) \wedge \mu.\end{aligned}$$

Similarly, we can show that $f(x\alpha y) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$ for all $x, y \in S$ and $\alpha \in \Gamma$. $\square$

Definition 4.6. An ordered Γ-semiring S is said to be k-regular if for each $x \in S$, there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x \leq x\gamma b\delta x$.

Definition 4.7. An ordered Γ-semiring S is said to be k-intra-regular if for each $x \in S$, there exist $z, a_i, a_i^{'}, b_i, b_i^{'} \in S$, $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}, \beta_{1i}, \beta_{2i}, \beta_{3i} \in \Gamma$, $i \in \mathbf{N}$, the set of natural numbers, such that $x + \sum_{i=1}^{n} a_i\alpha_{1i}x\alpha_{2i}x\alpha_{3i}a_i^{'} \leq \sum_{i=1}^{n} b_i\beta_{1i}x\beta_{2i}x\beta_{3i}b_i^{'}$.

Theorem 4.8. Let S be a k-regular ordered semiring and $x \in S$. Then

(i) $f(x) \wedge \mu \leq (fo_2\chi_S o_2 f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-bi-ideal f of S.

(ii) $f(x) \wedge \mu \leq (fo_2\chi_S o_2 f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-quasi-ideal f of S.

Proof. Let S be a k-regular ordered semiring and x be any element of S. Suppose f be any (λ, μ)-fuzzy k-bi-ideal of S. Since S is k-regular there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x \leq x\gamma b\delta x$. Now

$$
\begin{aligned}
&(fo_2\chi_S o_2 f)(x) \vee \lambda \\
&= \bigvee_{x+\sum_{i=1}^{n} a_i\alpha_i b_i \leq \sum_{i=1}^{n} c_i\beta_i d_i} (\wedge\{(fo_2\chi_S)(a_i), (fo_2\chi_S)(c_i), f(b_i), f(d_i)\}) \vee \lambda \\
&\geq \wedge\{(fo_2\chi_S)(x\alpha a), (fo_2\chi_S)(x\gamma b), f(x)\} \wedge \mu \\
&= \wedge\{ \bigvee_{x\alpha a+\sum_{i=1}^{n} a_i\alpha_{1i}b_i \leq \sum_{i=1}^{n} c_i\beta_{1i}d_i}(\wedge\{(f(a_i), f(c_i))\}), \bigvee_{x\gamma b+\sum_{i=1}^{n} a_i\alpha_{2i}b_i \leq \sum_{i=1}^{n} c_i\beta_{2i}d_i}(\wedge\{(f(a_i), f(c_i))\}), f(x)\} \wedge \mu \\
&\geq \wedge\{f(x), f(x), f(x)\} \wedge \mu \\
&\qquad (\text{since } x\alpha a + x\alpha a\beta x\alpha a \leq x\gamma b\delta x\alpha a \text{ and } x\gamma b + x\alpha a\beta x\gamma b \leq x\gamma b\delta x\gamma b) \\
&= f(x) \wedge \mu.
\end{aligned}
$$

This implies that $f(x) \wedge \mu \leq (fo_2\chi_S o_2 f)(x) \vee \lambda$.
$(i) \Rightarrow (ii)$ is straight forward from Lemma 4.5. $\square$

Theorem 4.9. Let S be a k-regular ordered semiring and $x \in S$. Then

(i) $(f \cap g)(x) \wedge \mu \leq (fo_2 go_2 f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-bi-ideal f and every (λ, μ)-fuzzy k-ideal g of S.

(ii) $(f \cap g)(x) \wedge \mu \leq (fo_2 go_2 f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-quasi-ideal f and every (λ, μ)-fuzzy k-ideal g of S.

Proof. Assume that S is a k-regular ordered semiring. Let f and g be any (λ, μ)-fuzzy k-bi-ideal and (λ, μ)-fuzzy k-ideal of S, respectively and x be any element of S. Since S is k-regular, there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x \leq x\gamma b\delta x$.

Then

$$
\begin{aligned}
&(fo_2go_2f)(x) \vee \lambda \\
&= \vee_{x+\sum_{i=1}^{n} a_i\alpha_i b_i \leq \sum_{i=1}^{n} c_i\beta_i d_i} (\wedge\{(fo_2g)(a_i), (fo_2g)(c_i), f(b_i), f(d_i)\}) \vee \lambda \\
&\geq \wedge\{(fo_2g)(x\alpha a), (fo_2g)(x\gamma b), f(x)\} \wedge \mu \\
&= \wedge\{\vee_{x\alpha a+\sum_{i=1}^{n} a_i\alpha_i b_i \leq \sum_{i=1}^{n} c_i\beta_i d_i}(\wedge\{(f(a_i), f(c_i), g(b_i), g(d_i))\}), \vee_{x\gamma b+\sum_{i=1}^{n} a_i\gamma_i b_i \leq \sum_{i=1}^{n} c_i\delta_i d_i}(\wedge\{(f(a_i), f(c_i), g(b_i), g(d_i))\}), f(x)\} \wedge \mu \\
&\geq \wedge\{\wedge\{f(x), f(a\beta x\alpha a), g(b\delta x\alpha a)\}, \wedge\{f(x), g(a\beta x\gamma b), g(b\delta x\gamma b)\}, f(x)\} \wedge \mu \\
&\qquad (\text{since } x\alpha a + x\alpha a\beta x\alpha a \leq x\gamma b\delta x\alpha a \text{ and } x\gamma b + x\alpha a\beta x\gamma b \leq x\gamma b\delta x\gamma b) \\
&\geq f(x) \wedge g(x) \wedge \mu = (f \cap g)(x) \wedge \mu.
\end{aligned}
$$

(i)⇒(ii) is straight forward from Lemma 4.5. □

Theorem 4.10. Let S is both k-regular and k-intra-regular an ordered semiring and $x \in S$. Then

(i) $f(x) \wedge \mu = (fo_2f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-bi-ideal f of S

(ii) $f(x) \wedge \mu = (fo_2f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-quasi-ideal f of S.

Proof. Suppose S is both k-regular and k-intra-regular ordered semiring. Let $x \in S$ and f be any fuzzy k-bi-ideal of S. Since S is both k-regular and k-intra-regular there exist $a_i, b_i, c_i, d_i \in S$, $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}, \alpha_{4i}, \alpha_{5i}, \beta_{1i}, \beta_{2i}, \beta_{3i}, \beta_{4i}, \beta_{5i} \in \Gamma$, $i \in \mathbf{N}$ such that $x + \sum_{i=1}^{n} x\alpha_{1i}a_i\alpha_{2i}x\alpha_{3i}x\alpha_{4i}b_i\alpha_{5i}x \leq \sum_{i=1}^{n} x\beta_{1i}c_i\beta_{2i}x\beta_{3i}x\beta_{4i}d_i\beta_{5i}x$. Therefore

$$
\begin{aligned}
(fo_2f)(x) \vee \lambda &= \vee_{x+\sum_{i=1}^{n} a_i\alpha_i b_i \leq \sum_{i=1}^{n} c_i\beta_i d_i}[\wedge_i\{\wedge\{f(a_i), f(c_i), f(b_i), f(d_i)\}\}] \vee \lambda \\
&\geq \wedge_i[\wedge\{f(x\alpha_{1i}a_i\alpha_{2i}x), f(x\alpha_{4i}b_i\alpha_{5i}x), f(x\beta_{1i}c_i\beta_{2i}x), f(x\beta_{4i}d_i\beta_{5i}x)\}] \wedge \mu \\
&\qquad x+\sum_{i=1}^{n} x\alpha_{1i}a_i\alpha_{2i}x\alpha_{3i}x\alpha_{4i}b_i\alpha_{5i}x \leq \sum_{i=1}^{n} x\beta_{1i}c_i\beta_{2i}x\beta_{3i}x\beta_{4i}d_i\beta_{5i}x \\
&\geq f(x) \wedge \mu.
\end{aligned}
$$

Now $(fo_2f)(x) \vee \lambda \leq (fo_2\chi_S)(x) \wedge \mu \leq f(x) \wedge \mu$. Hence $f(x) \wedge \mu = (fo_2f)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-bi-ideal f of S.
$(i) \Rightarrow (ii)$ is straightforward from the Lemma 4.5. □

Theorem 4.11. Let S is both k-regular and k-intra-regular ordered semiring and $x \in S$. Then

(i) $(f \cap g)(x) \wedge \mu \leq (fo_2g)(x) \vee \lambda$ for all (λ, μ)-fuzzy k-bi-ideals f and g of S.

(ii) $(f \cap g)(x) \wedge \mu \leq (fo_2g)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-bi-ideals f and every (λ, μ)-fuzzy k-quasi-ideal g of S.

(iii) $(f \cap g)(x) \wedge \mu \leq (fo_2g)(x) \vee \lambda$ for every (λ, μ)-fuzzy k-quasi-ideals f and every (λ, μ)-fuzzy k-bi-ideal g of S.

(iv) $(f \cap g)(x) \wedge \mu \leq (fo_2g)(x) \vee \lambda$ for all (λ, μ)-fuzzy k-quasi-ideals f and g of S.

Proof. Assume that S is both k-regular and k-intra-regular ordered semiring. Let $x \in S$ and f, g be any (λ, μ)-fuzzy k-bi-ideals of S. Since S is both k-regular and k-intra-regular there exist $a_i, b_i, c_i, d_i \in S$, $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}, \alpha_{4i}, \alpha_{5i}$, $\beta_{1i}, \beta_{2i}, \beta_{3i}, \beta_{4i}, \beta_{5i} \in \Gamma$, $i \in \mathbf{N}$ such that $x + \sum_{i=1}^{n} x\alpha_{1i}a_i\alpha_{2i}x\alpha_{3i}x\alpha_{4i}b_i\alpha_{5i}x \leq \sum_{i=1}^{n} x\beta_{1i}c_i\beta_{2i}x\beta_{3i}x\beta_{4i}d_i\beta_{5i}x$. Therefore

$$
\begin{aligned}
&(fo_2g)(x) \vee \lambda \\
&= \vee[\bigwedge_i\{\wedge\{f(a_i), f(c_i), g(b_i), g(d_i)\}\}] \vee \lambda \\
&\quad {\scriptstyle x+\sum_{i=1}^{n} a_i\alpha_i b_i \leq \sum_{i=1}^{n} c_i\beta_i d_i} \\
&\geq \bigwedge_i[\wedge\{f(x\alpha_{1i}a_i\alpha_{2i}x), g(x\alpha_{4i}b_i\alpha_{5i}x), f(x\beta_{1i}c_i\beta_{2i}x), g(x\beta_{4i}d_i\beta_{5i}x\}] \wedge \mu \\
&\geq f(x) \wedge g(x) \wedge \mu = (f \cap g)(x) \wedge \mu.
\end{aligned}
$$

$(i) \Rightarrow (ii) \Rightarrow (iv)$ and $(i) \Rightarrow (iii) \Rightarrow (iv)$ are obvious from Lemma 4.5. □

5 Conclusion

In this paper, the concept of (λ, μ)-fuzzy ideals and k-ideals of an ordered Γ-semiring is introduced and studied along with some operation on them and some of their characterizations are obtained. Actually the main aim of studying the concept of (λ, μ)-fuzzy set is to restrict ourself to the interval (λ, μ) with $0 \leq \lambda < \mu \leq 1$. The case for which $\lambda = 0$ and $\mu = 1$, the results will coincide with the fuzzy ideals of ordered Γ-semiring. We can similarly obtain the parallel results for fuzzy h-ideal also. The future work may be focused on prime(semiprime) fuzzy ideal, prime(semiprime) fuzzy h-ideal, fuzzy h-bi(quasi, interior)-ideal, prime(semiprime)fuzzy h-bi(quasi, interior)-ideal etc..

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