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COMMON FIXED POINT THEOREMS IN G-FUZZY METRIC SPACES

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Abstract – In this paper, we obtain a unique common fixed point theorem for six weakly compatible mappings in G-fuzzy metric spaces.

Keywords – *G-metric Spaces, compatible mappings, G-fuzzy metric spaces*

1 Introduction

Mustafa and Sims [3] introduced a *G-metric* space and obtained some fixed point theorems in it. Some interesting references in *G-metric* spaces are [2-6,8]. We have generalized the result of Rao et al. [7]. Before giving our main results, we obtain a unique common fixed point theorem for six weakly compatible mappings in G-fuzzy metric spaces.

Definition 1.1 Let X be a nonempty set and let G $X \times X \times X \to [0,\infty)$ be a function satisfying the following properties

($G1$) $G(x, y, z) = 0$ if $x = y = z$,
($G2$) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
($G3$) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
($G4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables,
($G5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

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Then, the function G is called a generalized metric or a *G-metric* on X and the pair (X, G) is called a *G-metric* space.

Definition 1.2 The G-metric space (X, G) is called symmetric if $G(x, x, y) = G(x, y, y)$ for all $x, y \in X$.

Definition 1.3 A 3-tuple (X, G, *) is called a G- fuzzy metric space if X is an arbitrary nonempty set, * is a continuous t-norm, and G is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions for each $t, s > 0$

(i) $G(x, x, y, t) > 0$ for all $x, y \in X$ with $x \neq y$,
(ii) $G(x, x, y, t) \geq G(x, y, z, t)$ for all $x, y, z \in X$ with $y \neq z$,
(iii) $G(x, y, z, t) = 1$ if and only if $x = y = z$,
(iv) $G(x, y, z, t) = G(p(x, y, z), t)$, where p is a permutation function,
(v) $G(x, y, z, t + s) \geq G(a, y, z, t) * G(x, a, a, s)$ for all $x, y, z, a \in X$,
(vi) $G(x, y, z, \cdot)$ $(0,\infty) \rightarrow [0, 1]$ is continuous.

Definition 1.4 A G- fuzzy metric space (X,G,*) is said to be symmetric if

$$G(x, x, y, t) = G(x, y, y, t)$$

for all $x, y \in X$ and for each $t > 0$.

Example 1.5 Let X be a nonempty set and let G be a G- fuzzy metric on X. Denote $a*b=ab$ for all $a, b \in [0, 1]$. For each $t > 0$,

$$G(x, y, z, t) = \frac{t}{t+G(x,y,z,t)}$$

is a G- fuzzy metric on X. Let (X, G,*) be a G - fuzzy metric space. For $t > 0$, $0 < r < 1$, and $x \in X$, the set

$$B_G(x, r, t) = \{ y \in X \ \ G(x, y, y, t) > 1 - r\}$$

is called an open ball with center x and radius r. A subset A of X is called an open set if for each $x \in X$, there exist $t > 0$ and $0 < r < 1$ such that $B_G(x, r, t) \subseteq A$. A sequence $\{x_n\}$ in G-fuzzy metric space X is said to be G- convergent to $x \in X$ if $G(x_n, x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$ or each $t > 0$. It is called a G- Cauchy sequence if $G(x_n, x_n, x_m, t) \rightarrow 1$ as $n, m \rightarrow \infty$ for each $t > 0$. X is called G- complete if every G- Cauchy sequence in X is G- convergent in X.

Lemma 1.6 Let (X, G,*) be a G- fuzzy metric space. Then, $G(x, y, z, t)$ is nondecreasing with respect to t for all $x, y, z \in X$.

Lemma 1.7 Let (X, G, *) be a G- fuzzy metric space. If there exists $k \in (0, 1)$ such that

$$\min \{G(x, y, z, kt), G(u, v,w, kt)\} \geq \min \{G (x, y, z, t), G(u, v,w, t)\} \quad (1)$$

for all $x, y, z, u, v, w \in X$ and $t > 0$, then $x = y = z$ and $u = v = w$.

2 Main Result

Let Φ denote the set of all continuous non decreasing functions ϕ $[0,\infty) \to [0,\infty)$ such that $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$. It is clear that $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.

Theorem 2.1 Let (X, G, *) be a G- fuzzy metric space and S, T, R, f, g, h X → X be satisfying

(i) $S(X) \subseteq g(X)$, $T(X) \subseteq h(X)$ and $R(X) \subseteq f(X)$,
(ii) One of f(X), g(X) and h(X) is a complete subspace of X,
(iii) The pairs (S, f), (T, g) and (R, h) are weakly compatible, and

(iv) $$G(Sx, Ty, Rz, t) \geq \phi\left(\min\left\{\begin{matrix} G(fx, gy, hz, t) \\ \frac{1}{3}[G(fx, Sx, Ty, t) + G(gy, Ty, Rz, t) + G(hz, Rz, Sx, t)], \\ \frac{1}{4}[G(fx, Ty, hz, t) + G(Sx, gy, hz, t) + G(fx, gy, Rz, t)] \end{matrix}\right\}\right)$$

for all x, y, z ϵ X, where $\phi \in \Phi$.

Then either one of the pairs (S, f), (T, g), and (R, h) has a coincidence point or the maps S,T, R, f, g and h have a unique common fixed point in X.

Proof: Choose $x_0 \epsilon X$. By (i), there exist x_1, x_2, x_3, $\in$ X such that $Sx_0 = gx_1 = y_0$, $Tx_1 = hx_2 = y_1$ and $Rx_2 = fx_3 = y_2$. Inductively, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{3n} = Sx_{3n} = gx_{3n+1}$, $y_{3n+1} = Tx_{3n+1} = hx_{3n+2}$ and $y_{3n+2} = Rx_{3n+2} = fx_{3n+3}$, where n = 0,1,…

If $y_{3n} = y_{3n+1}$ then x_{3n+1} is a coincidence point of g and T.

If $y_{3n+1} = y_{3n+2}$ then x_{3n+2} is a coincidence point of h and R.

If $y_{3n+2} = y_{3n+3}$ then x_{3n+3} is a coincidence point of f and S.

Now assume that $y_n \neq y_{n+1}$ for all n. Denote $d_n = G(y_n, y_{n+1}, y_{n+2}, t)$. Putting $x = x_{3n}$, $y = x_{3n+1}$, $z = x_{3n+2}$ in (iv), we get

$$d_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2}, t)$$

$$= G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}, t)$$

$$\geq \phi\left(\min\left\{\begin{matrix} G(fx_{3n}, gx_{3n+1}, hx_{3n+2}, t), \frac{1}{3}[G(fx_n, Sx_{3n}, Tx_{3n+1}, t) + \\ G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, Sx_{3n,}, t)] \\ \frac{1}{4}[G(fx_{3n}, Tx_{3n+1}, hx_{3n+2}, t) + G(Sx_{3n}, gx_{3n+1}, hx_{3n+2}, t) \\ + G(fx_{3n}, gx_{3n+1}, Rx_{3n+2}, t)] \end{matrix}\right\}\right)$$

$$\geq \phi\left(\min\left\{\begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n+1}, t), \frac{1}{3}[G(y_{3n-1}, y_{3n}, y_{3n+1}, t) + \\ G(y_{3n}, y_{3n+1}, y_{3n+2}, t) + G(fy_{3n+1}, y_{3n+2}, y_{3n}, t)], \\ \frac{1}{4}[G(fy_{3n-1}, y_{3n+1}, y_{3n+1}, t) + G(y_{3n}, y_{3n}, y_{3n+1}, t) \\ +G(y_{3n-1}, y_{3n}, y_{3n+2}, t)] \end{array}\right\}\right)$$

$$\geq \phi\left(\min\left\{\begin{array}{c} d_{3n-1}, \frac{1}{3}[d_{3n-1} + d_{3n} + d_{3n}], \\ \frac{1}{4}[d_{3n-1} + d_{3n} + (d_{3n-1} + d_{3n})] \end{array}\right\}\right) \tag{2}$$

If $d_{3n} \leq d_{3n-1}$ then from (1), we have $d_{3n} \geq \phi(d_{3n}) > d_{3n}$. It is a contradiction. Hence $d_{3n} \geq d_{3n-1}$. Now from (1), $d_{3n} \geq \phi(d_{3n-1})$. Similarly, by putting $x = x_{3n+3}$, $y = x_{3n+1}$, $z = x_{3n+2}$ and $x = x_{3n+3}$, $y = x_{3n+4}$, $z = x_{3n+2}$ in (iv), we get

$$d_{3n+1} \geq \phi(d_{3n}) \quad \text{and} \tag{3}$$

$$d_{3n+2} \geq \phi(d_{3n+1}) \tag{4}$$

Thus from (1), (2) and (3), we have

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+2}, t) &\geq \phi(G(y_{n-1}, y_n, y_{n+1}, t)) \\ &\geq \phi^2(G(y_{n-2}, y_{n-1}, y_n, t)) \\ &\quad \vdots \\ &\geq \phi^n(G(y_0, y_1, y_2, t)) \end{aligned} \tag{5}$$

we have $G(y_n, y_n, y_{n+1}, t) \geq G(y_n, y_{n+1}, y_{n+2}, t) \geq \phi^n(G(y_0, y_1, y_2, t))$. Now for $m > n$, we have

$$\begin{aligned} G(y_n, y_n, y_m, t) &\geq G(y_n, y_n, y_{n+1}, t) + G(y_{n+1}, y_{n+1}, y_{n+2}, t) + \ldots + G(y_{m-1}, y_{m-1}, y_m, t) \\ &\geq \phi^n(G(y_0, y_1, y_2, t)) + \phi^{n+1}(G(y_0, y_1, y_2, t)) + \ldots + \phi^{m-1}(G(y_0, y_1, y_2, t)) \\ &\to 1 \text{ as } n \to \infty, \end{aligned}$$

Since $\phi^n(t) \to 1$ as $n \to \infty$ for all $t > 0$. Hence $\{y_n\}$ is G- Cauchy. Suppose f(X) is G-complete. Then there exist p, $t \in X$ such that $y_{3n+2} \to p = f\,t$. Since $\{y_n\}$ is G- Cauchy, it follows that $y_{3n} \to p$ and $y_{3n+1} \to p$ as $n \to \infty$.

$G(St, Tx_{3n+1}, Rx_{3n+2}, t)$

$$\geq \phi\left(\min\left\{\begin{array}{c} G(ft, gx_{3n}, hx_{3n+2}, t), \frac{1}{3}[G(ft, St, Tx_{3n+1}, t) + \\ G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, St, t)], \\ \frac{1}{4}G(ft, Tx_{3n+1}, hx_{3n+2}, t) + G(St, gx_{3n+1}, hx_{3n+2}, t) \\ +G(ft, gx_{3n+1}, Rx_{3n+2}, t)] \end{array}\right\}\right)$$

Letting $n \to \infty$, we get

$$G(Sp, p, p, t) \geq \phi\left(\min\begin{Bmatrix} 1, \frac{1}{3}[G(p, St, p, t) + 1 + G(p, p, St, t)] \\ \frac{1}{4}[1 + G(St, p, p, t) + 1)] \end{Bmatrix}\right)$$

$G(St, p, p, t) \geq \phi$ (G(St, p, p, t), since ϕ is non decreasing. Hence St = p. Thus p = f t = St. Since the pair (S, f) is weakly compatible, we have fp = Sp. Putting x = p, y = x_{3n+1}, z=x_{3n+2} in (iv), we get

$$G(Sp, Tx_{3n+1}, Rx_{3n+2}, t) \geq \phi\left(\min\begin{Bmatrix} G(fp, gx_{3n+1}, hx_{3n+2}, t), \frac{1}{3}[G(fp, Sp, Tx_{3n+1}, t) + \\ G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, Sp, t)], \\ \frac{1}{4}[G(fp, Tx_{3n+1}, hx_{3n+2}, t) + G(Sp, gx_{3n+1}, hx_{3n+2}, t) \\ +G(fp, gx_{3n+1}, Rx_{3n+2}, t)] \end{Bmatrix}\right)$$

Letting $n \to \infty$, we have

$$G(Sp, p, p, t) \geq \phi\left(\min\begin{Bmatrix} G(Sp, p, p, t), \frac{1}{3}[G(Sp, Sp, p, t) + 0 + G(p, p, Sp, t)], \\ \frac{1}{4}[G(Sp, p, p, t) + G(Sp, p, p, t) + G(Sp, p, p, t)] \end{Bmatrix}\right)$$

Since $G(Sp, Sp, p, t) \geq 2G(Sp, p, p, t)$, we have $G(Sp, p, p, t) \geq \phi(G(Sp, p, p, t))$. Thus Sp = p. Hence

$$f\,p = Sp = p. \tag{6}$$

Since p = Sp ϵ g(X), there exists v ϵ X such that p = gv. Putting x = p, y = v, z = x_{n+2} in (iv), we get

$$G(Sp, Tv, Rx_{n+2}, t) \geq \phi\left(\min\begin{Bmatrix} G(fp, gv, hx_{3n+2}, t), \frac{1}{3}[G(fp, Sp, Tv, t) + \\ G(gv, Tv, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, Sp, t)], \\ \frac{1}{4}[G(fp, Tv, hx_{3n+2}, t) + G(Sp, gv, hx_{3n+2}, t) \\ +G(fp, gv, Rx_{3n+2}, t)] \end{Bmatrix}\right)$$

Letting $n \to \infty$, we deduce that

$$G(p, Tv, p, t) \geq \phi\left(\min\begin{Bmatrix} 1, \frac{1}{3}[G(p, p, Tv, t) + G(p, Tv, p, t) + 1], \\ \frac{1}{4}[G(p, Tv, p, t) + 1 + 1] \end{Bmatrix}\right)$$
$$\geq \phi\,(G(p, Tv, p, t)),$$

since ϕ is non decreasing. Thus Tv = p, so that p = Tv = gv. Since the pair (T, g) is weakly compatible, we have Tp = gp.

$$G(Sp, Tp, Rx_{3n+2}, t) \geq \phi\left(\min\left\{\begin{array}{c} G(fp, gp, hx_{3n+2}, t), \frac{1}{3}[G(fp, Sp, Tp, t) + \\ G(gp, Tp, Rx_{3n+2}, t) + G(hx_{3n+2}, Rx_{3n+2}, Sp, t)], \\ \frac{1}{4}[G(fp, Tp, hx_{3n+2}, t) + G(Sp, gp, hx_{3n+2}, t) \\ +G(fp, gp, Rx_{3n+2}, t)] \end{array}\right\}\right)$$

Letting $n \to \infty$, we have

$$G(p, Tp, p, t) \geq \phi\left(\min\left\{\begin{array}{c} G(p, Tp, p, t), \frac{1}{3}[G(p, p, Tp, t) + G(Tp, Tp, p) + 1], \\ \frac{1}{4}[G(p, Tp, p, t) + G(p, Tp, p, t) + G(p, Tp, p, t)] \end{array}\right\}\right)$$

Since $G(Tp, Tp, p, t) \geq 2G(Tp, p, p, t)$, we have ,$G(p, Tp, p, t) \geq \phi\,(G(p, Tp, p, t))$. Thus $Tp = p$. Hence

$$gp = Tp = p. \tag{7}$$

Since $p = Tp \in h(X)$, there exists $w \in X$ such that $p = hw$. Putting $x = p$, $y = p$, $z = w$ in (iv), we get

$$G(Sp, Tp, Rw, t) \geq \phi\left(\min\left\{\begin{array}{c} G(fp, gp, hw, t), \frac{1}{3}[G(fp, Sp, Tp, t) + \\ G(gp, Tp, Rw, t) + G(hw, Rw, Sp, t)], \\ \frac{1}{4}[G(fp, Tp, hw, t) + G(Sp, gp, hw, t) \\ +G(fp, gp, Rw, t)] \end{array}\right\}\right)$$

$$G(p, p, Rw, t) \geq \phi\left(\min\left\{\begin{array}{c} 1, \frac{1}{3}[1 + G(p, p, Rw, t) + G(p, Rw, p, t)], \\ \frac{1}{4}[1 + 1 + G(p, p, Rw, t)] \end{array}\right\}\right) \geq \phi\,(G(p, p, Rw, t)),$$

since ϕ is non decreasing. Thus $Rw = p$, so that $p = hw = Rw$. Since the pair (R, h) is weakly compatible, we have $Rp = hp$. Putting $x = p$, $y = p$, $z = p$ in (iv), we get,

$$G(p, p, Rp, t) = G(Sp, Tp, Rp, t) \geq \phi\left(\min\left\{\begin{array}{c} G(fp, gp, Rp, t), \frac{1}{3}[1 + \\ G(p, p. Rp, t) + G(Rp, Rp, p, t)], \\ \frac{1}{4}[G(p, p, Rp, t) + G(p, p, Rp, t) \\ +G(p, p, Rp, t)] \end{array}\right\}\right)$$

Since $G(Rp, Rp, p, t) \geq 2G(p, p, Rp, t)$, we have

$$G(p, p, Rp, t) \geq \phi(G(p, p, Rp, t)). \tag{8}$$

Thus $Rp = p$, so that $Rp = hp = p$. From (6), (7) and (8), it follows that p is a common fixed point of S, T, R, f ,g and h. Uniqueness of common fixed point follows easily from (iv). Similarly, we can prove the theorem when g(X) or h(X) is a complete subspace of X.

Corollary 2.2 Let (X, G, *) be a G -fuzzy metric space and S, T, R, f, g, h, $X \to X$ be satisfying

(i) $S(X) \subseteq g(X)$, T(X) and $R(X) \subseteq f(X)$,
(ii) One of f(X) , g(X) and h(X) is a complete subspace of X,
(iii) The pairs (S, f) ,(T, g) and (R, h) ate weakly compatible and
(iv) $G(Sx, Ty, Rz, t) \geq \phi\ (G(fx, gy, hz, t))$ for all x, y, z ϵ X, where $\phi\ \epsilon\ \Phi$.

Then the maps S, T, R, f, g and h have a unique fixed point in X.

Corollary 2.3 Let(X, G, *) be a complete G - fuzzy metrics space and S, T, R $X \to X$ be satisfying $G(Sx, Ty, Rz, t) \geq \phi\ (G(x, y, z, t))$ for all x, y, z ϵ X, where $\phi \in \Phi$. Then the maps S, T and R have a unique common fixed point, p ϵ X and S,T and R are G-continuous at p.

Proof: There exists p ϵ X such that p is the unique common fixed point of S, T and R as in Theorem 2.1. Let $\{y_n\}$ be any sequence in X which G –converges to p. Then

$G(Sy_n, Sp, Sp, t) = G(Sy_n, Tp, Rp, t) \leq \phi\ (G(y_n, p, p, t)) \to 1$ as $n \to \infty$.

Hence S is G - continuous at p. Similarly, we can show that T and R are also G-continuous at p.

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