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ON (k, h)-CONVEX STOCHASTIC PROCESSES

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Abstaract − We introduce the class of (k, h)-convex stochastic processes and we generalize results given for (k, h)-convex functions in [10] and h-convex stochastic process in [1], among them, Hermite-Hadamard and Fejér-type inequalities.

Keywords − *(k, h)-convex stochastic processes, $h-$convex stochastic processes, converse Jensen-type inequality, Fejér-type inequality, Hermite-Hadamard-type inequality.*

1 Introduction

In 1980, Nikodem [11] stated the line of investigation on stochastic convexity and later, several types of convex stochastic processes have been studied [1, 2, 4, 5, 6, 7, 8, 11, 12, 14] based in the classical convex notions for functions.

Micherda and Rajba, introduced in [10] the family of $(k, h)-$convex functions as the solutions of the functional inequality

$$f(k(t)x + k(1 - t)y) \leq h(t)f(x) + h(1 - t)f(y),$$

where $k, h : (0, 1) \to \mathbb{R}$ are given. The notion of (k, h)-convexity generalizes s-Orlicz convexity [3], subaditivity [9] and h-convexity [13].

In this paper, we introduce the notion of (k, h)-convex stochastic processes as a counterpart of the (k, h)-convex functions and a generalization of h-convex stochastic processes defined in [1]. Also, we prove properties of (k, h)-convex stochastic processes, among them, Hermite-Hadamard and Fejér-type inequalities.

Now, we would like to recall the context where the stochastic convexity is studied.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a *random variable* if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is a *stochastic process* if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

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If $h : (0,1) \to \mathbb{R}$ is a non-negative function, $h \not\equiv 0$, a stochastic process $X : I \times \Omega \to \mathbb{R}$ is *h-convex*, if for every $t_1, t_2 \in I$ and $\lambda \in (0,1)$, the following inequality holds

$$X(\lambda t_1 + (1-\lambda)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(1-\lambda)X(t_2, \cdot), \quad (a.e.).$$

When h is equal to the identity function, X is said to be *convex*, and additionally, if $\lambda = \frac{1}{2}$ then X is *Jensen-convex.*

Some examples and properties related with convex, Jensen-convex and h-convex stochastic processes can be readed in [1, 2, 8, 11, 14].

Now, for calculation, we need to introduce additional definitions:

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process such that $\mathbb{E}[X(t)]^2 < \infty$ for all $t \in I$, where $\mathbb{E}[X(t)]^2 < \infty$ denotes the expectation value of $X(t, \cdot)$. The stochastic process X is

1. *continuous in probability* in the interval I, if for all $t_0 \in I$, we have

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

   where $P - \lim$ denotes the limit in probability.

2. *mean-square continuous* in the interval I, if for all $t_0 \in I$

$$\lim_{t \to t_0} \mathbb{E}[(X(t) - X(t_0))^2] = 0.$$

Is important to note that mean-square continuity implies continuity in probability, but the converse implication is not true.

We say that the stochastic process X is *mean-square integrable* in $[a, b] \subseteq I$, if there exists a random variable Y such that for all normal sequence of partions of the interval $[a, b]$, $a = t_0 < t_1 < ... < t_n = b$, holds

$$\lim_{n \to \infty} \mathbb{E}\left[\sum_{k=1}^{n} X(\theta_k) \cdot (t_k - t_{k-1}) - Y\right]^2 = 0.$$

The random variable $Y : \Omega \to \mathbb{R}$ is the mean-square integral of the process X on $[a, b]$ and we can also write

$$Y(\cdot) = \int_a^b X(s, \cdot)ds, \quad (a.e).$$

Definition and properties of mean-square integral can be readed in [15].

2 (k, h)-convex Stochastic Processes

In order to extend the definition of h-convexity for stochastic processes, we introduce the notion of (k, h) stochastic convexity.

Given a function $k : (0,1) \to \mathbb{R}$, a set $D \subseteq \mathbb{R}$ is *k-convex* if $k(\lambda)t_1 + k(1-\lambda)t_2 \in D$ for all t_1, $t_2 \in D$ and $t \in (0,1)$.

In [10], k-convex sets were defined in real linear spaces and some examples for chosen functions k are given.

Definition 2.1. Let k, $h : (0,1) \to \mathbb{R}$ be two given functions and $D \subset \mathbb{R}$ a k-convex set. A stochastic process $X : D \times \Omega \to \mathbb{R}$ is (k,h)-convex if, for all t_1, $t_2 \in D$ and $\lambda \in (0,1)$,

$$X(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(1-\lambda)X(t_2, \cdot) \qquad (a.e.). \qquad (1)$$

If in (1) the equality holds, the stochastic process X is called (k,h)-affine.

This definition coincides in many important cases with other ones previously introduced, some of which are listed bellow.

Example 2.2. 1. For $k(\lambda) = \lambda$, the notion of (k,h)-convexity matches with the h-convexity one given in [1] (without the additional assumption of non negativity).

2. For $k(\lambda) = h(\lambda) = 1$, the class of (k,h)-convex stochastic processes consists in all stochastic process which are subadditive.

3. If $k(\lambda) = h(\lambda) = 1/2$ for all λ, then (1) gives the family of Jensen-convex stochastic processes.

4. Let k be defined by the formula

$$k(\lambda) = \begin{cases} 2\lambda, & \lambda \leq 1/2, \\ 0, & \lambda > 1/2. \end{cases}$$

Then X is a (k,k)-convex stochastic process if and only if it is starshaped, i.e., $X(\lambda t, \cdot) \leq \lambda X(t, \cdot)$ almost everywhere, for all $\lambda \in (0,1)$ and $t \in D$. In fact, fix t_1, $t_2 \in D$ and choose $\lambda \in (0,1)$. Then, assuming that X is a (k,k)-convex stochastic process, we get

$$X(\lambda t, \cdot) = X\left(k\left(\frac{\lambda}{2}\right)t + k\left(1 - \frac{\lambda}{2}\right)t, \cdot\right) \leq \lambda X(t, \cdot),$$

and

$$X(0, \cdot) = X\left(k\left(\frac{\lambda}{2}\right)t + k\left(\frac{\lambda}{2}\right)t, \cdot\right) = 0,$$

almost everywhere.

On the other hand, if X is starshaped, for anyone $t_1, t_2 \in D, \lambda \in (0,1)$ we obtain

$$X(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot) = \begin{cases} X(2\lambda t_1, \cdot) \leq 2\lambda X(t_1, \cdot), & \lambda \in (0, 1/2), \\ X(0, \cdot) \leq 0, & \lambda = 1/2, \\ X((2-2\lambda)t_2, \cdot) \leq (2-2\lambda)X(t_2, \cdot), & \lambda \in (1/2, 1). \end{cases}$$

Hence, (1) is satisfied for all $t \in D$ and $\lambda \in (0,1)$.

Hereinafter, we keep the notation used in the definition (2.1) for D, k and h.

3 Properties of (k,h)-convex Stochastic Processes

Many of the well-known properties of convex stochastic processes are satisfied by (k,h)-convex stochastic processes too. In the following propositions we present some basic properties for (k,h)-convex stochastic processes.

Proposition 3.1. If X, $Y : D \times \Omega \to \mathbb{R}$ be a (k, h)-convex stochastic processes and $c \geq 0$, then $X + Y$ and cX are also (k, h)-convex stochastic processes.

Proof. Let be t_1, $t_2 \in D$, $\lambda \in (0, 1)$ and $c \geq 0$. Then,

$$\begin{aligned}(X+Y)(k(\lambda)t_1 &+ k(1-\lambda)t_2, \cdot) \\ &= X(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot) + Y(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot) \\ &\leq h(\lambda)(X+Y)(t_1, \cdot) + h(1-\lambda)(X+Y)(t_2, \cdot), \quad (a.e).\end{aligned}$$

Also,

$$\begin{aligned}c(X(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot)) &\leq c[h(\lambda)X(t_1, \cdot) + h(1-\lambda)X(t_2, \cdot)] \\ &\leq h(\lambda)(cX)(t_1, \cdot) + h(1-\lambda)(cX)(t_2, \cdot), \quad (a.e).\end{aligned}$$

Proposition 3.2. Let k, h_1, $h_2 : (0, 1) \to \mathbb{R}$ be non negative functions and X, $Y : D \times \Omega \to \mathbb{R}$ non-negative stochastic processes such that:

$$(X(t_1, \cdot) - X(t_2, \cdot))(Y(t_1, \cdot) - Y(t_2, \cdot)) \geq 0, \tag{2}$$

for all t_1, $t_2 \in D$. If X is (k, h_1)-convex, Y is (k, h_2)-convex and $h(\lambda) + h(1 - \lambda) \leq c$ for all $\lambda \in (0, 1)$, where $h(\lambda) = \max\{h_1(\lambda), h_2(\lambda)\}$ and c is a fixed positive number, then the product XY is a (k, ch)-convex stochastic process.

Proof. Fix t_1, $t_2 \in D$ and λ, $\beta \in (0, 1)$ such that $\lambda + \beta = 1$. First, note that if $(X(t_1, \cdot) - X(t_2, \cdot))(Y(t_1, \cdot) - Y(t_2, \cdot)) \geq 0$ holds almost everywhere, then:

$$X(t_1, \cdot)Y(t_2, \cdot) + Y(t_1, \cdot)X(t_2, \cdot) \leq X(t_1, \cdot)Y(t_1, \cdot) + Y(t_2, \cdot)X(t_2, \cdot), \quad (a.e).$$

Hence,

$$\begin{aligned}(XY)(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot) &\leq (h(\lambda)X(t_1, \cdot) + h(1-\lambda)X(t_2, \cdot)) \\ &\qquad \cdot (h(\lambda)Y(t_1, \cdot) + h(1-\lambda)Y(t_2, \cdot)) \\ &\leq (h(\lambda))^2(XY)(t_1, \cdot) \\ &\quad + h(\lambda)h(1-\lambda)[(XY)(t_1, \cdot) + (XY)(t_2, \cdot)] \\ &\quad + (h(1-\lambda))^2(XY)(t_2, \cdot) \\ &= (h(\lambda) + h(1-\lambda)) \\ &\quad \cdot [h(\lambda)(XY)(t_1, \cdot) + h(1-\lambda)XY(t_2, \cdot)] \\ &\leq ch(\lambda)(XY)(t_1, \cdot) + ch(1-\lambda)X(t_2, \cdot)], \quad (a.e).\end{aligned}$$

Proposition 3.3. Let $X : I \times \Omega \to \mathbb{R}$ be a (k, h)-convex stochastic process and $f : \mathbb{R} \to \mathbb{R}$ an increasing (h, h)-convex function. Then, $f \circ X : I \times \Omega \to \mathbb{R}$ is a (k, h)-convex stochastic process.

Proof. For arbitrary $t_1, t_2 \in I$ and $\lambda \in (0, 1)$, we have

$$\begin{aligned}f(X(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot)) &\leq f(h(\lambda)X(t_1, \cdot) + h(1-\lambda)X(t_2, \cdot)) \\ &\leq h(\lambda)f(X(t_1, \cdot)) + h(1-\lambda)f(X(t_2, \cdot)) \quad (a.e)\end{aligned}$$

In [8], Kotrys and Nikodem defined for every stochastic process X and random variable A, the sublevel set as follows

$$L_A = \{t \in D : X(t, \cdot) \leq A(\cdot), \quad (a.e.)\}.$$

In the following proposition we present a condition for h in way to the sublevel set L_A be k-convex for given (k, h)-convex stochastic process X and random variable A.

Proposition 3.4. Let $X : D \times \Omega \to \mathbb{R}$ be a (k, h)-convex stochastic process, with h a positive function. For every random variable $A : \Omega \to \mathbb{R}$, the sublevel set L_A is k-convex if the inequality $h(\lambda) + h(1 - \lambda) \leq 1$ holds for every $\lambda \in (0, 1)$.

Proof. Since X is (k, h)-convex, for t_1, $t_2 \in L_A$ and $\lambda \in (0, 1)$, we have:

$$\begin{aligned} X(k(\lambda)t_1 + k(1-\lambda)t_2, \cdot) &\leq h(\lambda)X(t_1, \cdot) + h(1-\lambda)X(t_2, \cdot) \\ &\leq h(\lambda)A(\cdot) + h(1-\lambda)A(\cdot) \\ &= (h(\lambda) + h(1-\lambda))A(\cdot) \leq A(\cdot), \quad (a.e.). \end{aligned}$$

Therefore, L_A is k-convex set.

Example 3.5. Considering $h(\lambda) = \lambda$ in the previous proposition, the result holds.

The proof of the following proposition follows immediately from the definitions.

Proposition 3.6. If h_1, h_2 are functions such that $h_2 \geq h_1$, then every non-negative (k, h_1)-convex stochastic process is also (k, h_2)-convex stochastic process.

Remark 3.7. Note that if D is a k-convex subset of X and $X : D \times \Omega \to \mathbb{R}$ is a (k, h)-affine stochastic process, then the image of X not necessarily is an h-convex set in $\mathbb{R}$. For instance, if $D = \Omega = [0, 1]$, k, h are the identity function and X is defined by

$$X(t, \omega) = \begin{cases} 0, & \text{if} \quad t \neq \omega, \\ 1, & \text{if} \quad t = \omega. \end{cases}$$

then $X(D \times \Omega) = \{0, 1\}$ is not an h-convex subset of $\mathbb{R}$.

In the following theorem we present conditions under the inequality

$$X(k(\lambda)t_1 + k(\beta)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot),$$

holds almost everywhere, for all λ, $\beta > 0$ such that $\lambda + \beta \leq 1$.

In the following theorem definitions of supermultiplicative and submultiplicative functions are needed. We recall these notions:

Definition 3.8. A function $f : (0, 1) \to \mathbb{R}$ is said to be supermultiplicative if for all $x, y \in (0, 1)$,

$$f(x)f(y) \leq f(xy), \tag{3}$$

If inequality (3) is reversed, then f is a submultiplicative function. Moreover, if the equality holds in (3), f is multiplicative.

Theorem 3.9. Let be $k, h : (0,1) \to \mathbb{R}$ non-negative functions and $D \subseteq \mathbb{R}$ a $k-$convex set such that $0 \in D$. If k is submultiplicative, h is supermultiplicative and $X : D \times \Omega \to \mathbb{R}$ is a (k,h)-convex and non-decreasing stochastic process such that $X(0,\cdot) = 0$, then the inequality

$$X(k(\lambda)t_2 + k(\beta)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot),$$

hold almost everywhere, for all λ, $\beta > 0$ such that $\lambda + \beta \leq 1$.

Proof. If $\lambda + \beta = 1$, the inequality holds from (k,h)-convex stochastic process definition. Let λ, $\beta > 0$ be numbers such that $\lambda + \beta = \gamma$ with $\gamma < 1$. Let us define numbers $a := \frac{\lambda}{\gamma}$ and $b := \frac{\beta}{\gamma}$. Then, $a + b = 1$ and fixed $t_1, t_2 \in D$, we have the following inequality:

$$\begin{aligned}
X(k(a\gamma)t_1 + k(b\gamma)t_2, \cdot) &\leq X(k(a)k(\gamma)t_1 + k(b)k(\gamma)t_2, \cdot)\\
&\leq h(a)X(k(\gamma)t_1, \cdot) + h(b)X(k(\gamma)t_2, \cdot)\\
&= h(a)X(k(\gamma)t_1 + k(1-\gamma)0, \cdot)\\
&\quad +h(b)X(k(\gamma)t_1 + k(1-\gamma)0, \cdot)\\
&\leq h(a)[h(\gamma)X(t_1, \cdot) + h(1-\gamma)X(0, \cdot)]\\
&\quad +h(b)[h(\gamma)X(t_1, \cdot) + h(1-\gamma)X(0, \cdot)]\\
&= h(a)h(\gamma)X(t_1, \cdot) + h(b)h(\gamma)X(t_2, \cdot)\\
&\leq h(a\gamma)X(t_1, \cdot) + h(b\gamma)X(t_2, \cdot)\\
&= h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot), \quad (a.e).
\end{aligned}$$

Theorem 3.10. Let k, h be non-negative functions and $D \subseteq \mathbb{R}$ a k-convex set such that $0 \in D$. If $X : D \times \Omega \to \mathbb{R}$ is a non-negative stochastic process such that

$$X(k(\lambda)t_1 + k(\beta)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot) \ \ (a.e), \tag{4}$$

holds for any t_1, $t_2 \in D$ and $\lambda, \beta > 0$ with $\lambda+\beta \leq 1$ and $h(\lambda) < \frac{1}{2}$ for some $\lambda \in (0, \frac{1}{2})$, then $X(0, \cdot) = 0$.

Proof. Let us suppose that exists $w \in \Omega$ with $X(0, \omega) \neq 0$, then $X(0, \omega) > 0$ and putting $t_1 = t_2 = 0$ in the inequality (4), we get

$$X(0, \omega) \leq h(\lambda)X(0, \omega) + h(\beta)X(0, \omega),$$

for λ, $\beta > 0$ such that $\lambda + \beta \leq 1$. Putting $\lambda = \beta$, $\lambda \in (0, \frac{1}{2})$ and dividing by $X(0, \omega)$, we obtain $1 \leq h(\lambda) + h(\lambda) = 2h(\lambda)$ for all $\lambda \in (0, \frac{1}{2})$. That is, $\frac{1}{2} \leq h(\lambda)$ for all $\lambda \in (0, \frac{1}{2})$, what is a contradiction with the assumption of theorem.

In the following proposition we present a Schur-type inequality.

Proposition 3.11. If k, $h : (0,1) \to \mathbb{R}$ are non-negative functions, with $k(\lambda) \geq \lambda$, h submultiplicative and $X : D \times \Omega \to \mathbb{R}$ is a non-decreasing (k,h)-convex stochastic process, then the following inequality holds:

$$h(t_3 - t_2)X(t_1, \cdot) - h(t_3 - t_1)X(t_2, \cdot) + h(t_2 - t_1)X(t_3, \cdot) \geq 0, \ \ (a.e.), \tag{5}$$

for $t_1, t_2, t_3 \in D$, such that $t_1 < t_2 < t_3$ and $t_3 - t_1, t_3 - t_2, t_2 - t_1 \in D$.

Proof. Consider $t_1, t_2, t_3 \in D$ be numbers wich satisfy assumptions of the proposition. Then,

$$\frac{t_3 - t_2}{t_3 - t_1}, \frac{t_2 - t_1}{t_3 - t_1} \in (0, 1),$$

and

$$\frac{t_3 - t_2}{t_3 - t_1} + \frac{t_2 - t_1}{t_3 - t_1} = 1.$$

Also, since h is supermultiplicative and non-negative, we have

$$h(t_3 - t_2) = h\left(\frac{t_3 - t_2}{t_3 - t_1}.(t_3 - t_1)\right) \geq h\left(\frac{t_3 - t_2}{t_3 - t_1}\right) h(t_3 - t_1),$$

$$h(t_2 - t_1) = h\left(\frac{t_2 - t_1}{t_3 - t_1}.(t_3 - t_1)\right) \geq h\left(\frac{t_2 - t_1}{t_3 - t_1}\right) h(t_3 - t_1),$$

Let $h(t_3 - t_1) > 0$. Because $k(\lambda) \geq \lambda$, X is non-decreasing and (k, h)-convex, X satisfies:

$$X(\lambda z_1 + (1-\lambda)z_2, \cdot) \leq X(k(\lambda)z_1 + k(1-\lambda)z_2, \cdot) \leq h(\lambda)X(z_1, \cdot) + h(1-\lambda)X(z_2, \cdot), \quad (a.e),$$

for all $z_1, z_2 \in D, \lambda \in (0, 1)$. In particular, for $\lambda = \frac{t_3 - t_2}{t_3 - t_1}$, $z_1 = t_1, z_2 = t_3$, we have $t_2 = \lambda z_1 + (1 - \lambda)z_2$ and

$$\begin{aligned} X(t_2, \cdot) &\leq h\left(\frac{t_3 - t_2}{t_3 - t_1}\right) X(t_1, \cdot) + h\left(\frac{t_2 - t_1}{t_3 - t_1}\right) X(t_3, \cdot) \\ &\leq \frac{h(t_3 - t_2)}{h(t_3 - t_1)} X(t_1, \cdot) + \frac{h(t_2 - t_1)}{h(t_3 - t_1)} X(t_3, \cdot), \quad (a.e). \end{aligned} \tag{6}$$

Finally, multiplying by $h(t_3 - t_1)$, we obtain the following

$$h(t_3 - t_1)X(t_2, \cdot) \leq h(t_3 - t_2)X(t_1, \cdot) + h(t_2 - t_1)X(t_3, \cdot), \quad (a.e).$$

That is,

$$0 \leq h(t_3 - t_2)X(t_1, \cdot) - h(t_3 - t_1)X(t_2, \cdot) + h(t_2 - t_1)X(t_3, \cdot), \quad (a.e).$$

The following theorem is an converse Jensen-type inequality.

Theorem 3.12. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be positive real numbers such that $\sum_{i=1}^{n} \lambda_i = 1$ and $(m, M) \subseteq I$. If k, $h : (0, 1) \to \mathbb{R}$ is a non negative with $k(\lambda) \geq \lambda$ and h supermultiplicative function, and $X : I \times \Omega \to \mathbb{R}$ is an (k, h)-convex stochastic process, then for any $t_1, t_2, ..., t_n \in [m, M]$, the following inequality holds almost everywhere

$$\begin{aligned} \sum_{i=1}^{n} h(\lambda_i)X(t_i, \cdot) &\leq X(m, \cdot) \sum_{i=1}^{n} h(\lambda_i) h\left(\frac{M - t_i}{M - m}\right) \\ &\quad + X(M, \cdot) \sum_{i=1}^{n} h(\lambda_i) h\left(\frac{t_i - m}{M - m}\right). \end{aligned}$$

Proof. Fix $i \in \{1, ..., n\}$. Putting $t_1 = m, t_2 = t_i, t_3 = M$ and $\lambda = \left(\frac{M-t_i}{M-m}\right) \in [0,1]$ in the inequality (6), we get

$$X(t_i, \cdot) \leq h\left(\frac{M-t_i}{M-m}\right) X(m, \cdot) + h\left(\frac{t_i - m}{M-m}\right) X(M, \cdot), \quad (a.e).$$

Since h is non negative, we have that multiplying by $h(\lambda_i)$:

$$\begin{aligned} h(\lambda_i)X(t_i, \cdot) \ &\leq \ h(\lambda_i)h\left(\frac{M-t_i}{M-m}\right) X(m, \cdot) \\ &\quad + h(\lambda_i)h\left(\frac{t_i - m}{M-m}\right) X(M, \cdot). \end{aligned}$$

Adding all inequalities for $i = 1, ..., n$, we complete the proof.

4 Main Results

We will prove the main results of this paper which consists in some new Fejér and Hermite-Hadamard-type inequalities for (k, h)-convex stochastic processes. From now, we suppose that all mean-square integrals considered bellow exist.

Theorem 4.1. (First Fejér-type inequality) If there are $X : D \times \Omega \to \mathbb{R}$ a (k, h)-convex stochastic process with $h(1/2) > 0$, $a < b$ such that $[a, b] \subset D$ and $G : [a, b] \times \Omega \to \mathbb{R}$ a non-negative and symmetric respect $\frac{a+b}{2}$ mean-square integrable stochastic process, then the following inequality holds almost everywhere:

$$\frac{X(k(1/2)(a+b), \cdot)}{2h(1/2)} \int_a^b G(t, \cdot)dt \leq \int_a^b X(t, \cdot)G(t, \cdot)dt, \quad (a.e). \tag{7}$$

Proof. From the definition with $\lambda = 1/2$, $t_1 = wa + (1-w)b$ and $t_2 = (1-w)a + wb$ with $w \in [0, 1]$, then

$$\begin{aligned} X\left(k\left(\frac{1}{2}\right)(a+b), \cdot\right) \ &= \ X\left(k\left(\frac{1}{2}\right)t_1 + k\left(\frac{1}{2}\right)t_2, \cdot\right) \\ &= \ X\left(k\left(\frac{1}{2}\right)(wa + (1-w)b) + k\left(\frac{1}{2}\right)((1-w)a + wb), \cdot\right) \\ &\leq \ h\left(\frac{1}{2}\right) X(wa + (1-w)b, \cdot) \\ &\quad + h\left(\frac{1}{2}\right) X((1-w)a + wb, \cdot), \quad (a.e). \end{aligned} \tag{8}$$

Multiplying both sides of the inequality (8) for $G(t_1, \cdot) = G(t_2, \cdot)$, almost everywhere and integrate it with respect to w, getting:

$$\begin{aligned} &X\left(k\left(\frac{1}{2}\right)(a+b), \cdot\right) \cdot \int_0^1 G(wa + (1-w)b, \cdot)dw \\ &\quad \leq h\left(\frac{1}{2}\right)\left[\int_0^1 X(wa + (1-w)b, \cdot)G(wa + (1-w)b, \cdot)dw \right. \\ &\quad\quad \left. + \int_0^1 X((1-w)a + wb, \cdot)G((1-w)a + wb, \cdot)dw\right], \end{aligned}$$

almost everywhere. This implies

$$X\left(k\left(\frac{1}{2}\right)(a+b),\cdot\right)\cdot\frac{1}{b-a}\int_a^b G(t,\cdot)dt \;\leq\; h\left(\frac{1}{2}\right)\cdot 2\cdot\frac{1}{b-a}\int_a^b X(t,\cdot)G(t,\cdot)dt,$$

which completes the proof.

Some important results are obtained as consequence of the previous result, among them, a Hermite-Hadamard-type inequality for (k,h)-convex stochastic processes, as the following corollary shows.

Corollary 4.2. Let $X : D\times\Omega\to\mathbb{R}$ be a (k,h)- convex stochastic process with $h(1/2)>0$ and fixed $a<b$ such that $[a,b]\subset D$. Then

$$\frac{X(k(1/2)(a+b),\cdot)}{2h(1/2)}\leq\frac{1}{b-a}\int_a^b X(t,\cdot)dt,\quad (a.e). \tag{9}$$

Remark 4.3. 1. If X is an h-convex stochastic process, then (7) gives the following inequality

$$\frac{1}{2h(1/2)}X\left(\frac{a+b}{2},\cdot\right)\int_a^b G(t,\cdot)dt\leq\int_a^b X(t,\cdot)G(t,\cdot)dt.$$

2. For every convex stochastic process X the following Fejér-type inequality is valid by Theorem 4.1,

$$X\left(\frac{a+b}{2},\cdot\right)\int_a^b G(t,\cdot)dt\leq\int_a^b X(t,\cdot)G(t,\cdot)dt.$$

In particular, for $G(t,\cdot)=1$ we get the Hermite-Hadamard inequality

$$X\left(\frac{a+b}{2},\cdot\right)\leq\frac{1}{b-a}\int_a^b X(t,\cdot)dt.$$

3. From (7) and (9) we recover the left-hand sides of the classical Fejér and Hermite-Hadamard-type inequalities for Jensen-convex stochastic processes.

Theorem 4.4. (Second Fejér-type inequality) Let be $k,h:(0,1)\to\mathbb{R}$ given functions such that $h(1/2)>0$ and $k(w)+k(1-w)=0$ for all $w\in[0,1]$. If $X : D\times\Omega\to\mathbb{R}$ is a (k,h)-convex stochastic, a, $b\in D$, $a<b$ and $G:[a,b]\times\Omega\to\mathbb{R}$ is a non-negative and symmetric respect to $\frac{a+b}{2}$ mean-square integrable stochastic process, then the following inequality holds almost everywhere:

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1 X\left(k\left(\frac{1}{2}\right)[k(t)+k(1-t)](a+b),\cdot\right)G(ta+(1-t)b,\cdot)dt\\ \leq\int_0^1 X(k(t)a+k(1-t)b,\cdot)G(at+(1-t)b,\cdot)dt\\ \leq[X(a,\cdot)+X(b,\cdot)]\int_0^1 h(t)G(at+(1-t)b,\cdot)dt.\end{aligned} \tag{10}$$

Proof. By definition (1) with $t_1 = k(w)a + k(1-w)b$, $t_2 = k(1-w)a + k(w)b$ and $t = 1/2$, we have the following inequality almost everywhere:

$$X\left(k\left(\frac{1}{2}\right)[k(w)+k(1-w)]\cdot(a+b),\cdot\right) = X\left(k\left(\frac{1}{2}\right)t_1 + k\left(\frac{1}{2}\right)t_2,\cdot\right)$$
$$\leq h\left(\frac{1}{2}\right)[X(k(w)a+k(1-w)b,\cdot)+X(k(1-w)a+k(w)b,\cdot)]. \qquad (11)$$

As in the proof of the previous theorem, we multiply both sides of the inequality (11) by $G(wa+(1-w)b,\cdot) = G((1-w)a+wb,\cdot)$, and we integrate the new inequality over $(0,1)$, getting

$$\int_0^1 X\left(k\left(\frac{1}{2}\right)[k(w)+k(1-w)]\cdot(a+b),\cdot\right)G(wa+(1-w)b,\cdot)dt$$
$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1 X(k(w)a+k(1-w)b,\cdot)G(wa+(1-w)b,\cdot)dw\right.$$
$$\left.+\int_0^1 X(k(1-w)a+k(w)b,\cdot)G(wa+(1-w)b,\cdot)dw\right]$$
$$\leq 2h\left(\frac{1}{2}\right)\cdot\int_0^1 X(k(1-w)a+k(w)b,\cdot)G(wa+(1-w)b,\cdot)dw, \quad (a.e).$$

From this we obtain the first desired inequality.

To prove the second one, we need to use the definition of (k,h)-convexity with $x = a$ and $y = b$. Namely, we have:

$$X(k(t)a+k(1-t)b,\cdot) \leq h(t)X(a,\cdot)+h(1-t)X(b,\cdot), \quad (a.e),$$

witch, by symmetry of $G(t,\cdot)$, implies

$$\int_0^1 X\left(k(t)a+k(1-t)b,\cdot\right)G(ta+(1-t)b,\cdot)dt$$
$$\leq X(a,\cdot)\int_0^1 h(t)G(wa+(1-w)b,\cdot)dw$$
$$+X(b,\cdot)\int_0^1 h(1-t)G((1-w)a+wb,\cdot)dw$$
$$= [X(a,\cdot)+X(b,\cdot)]\int_0^1 h(t)G(wa+(1-w)b,\cdot)dw, \quad (a.e),$$

and the proof is complete.

As a corollary, we obtain the second Hermite-Hadamard inequality for (k,h)-convex stochastic processes.

Corollary 4.5. Let $X : D\times\Omega \to \mathbb{R}$ be a (k,h)-convex stochastic process where $h(1/2) > 0$ and choose $a, b \in D$ such that $a < b$. Then

$$\frac{1}{h(1/2)}\int_0^1 X\left(k\left(\frac{1}{2}\right)[k(t)+k(1-t)](a+b),\cdot\right)dt$$
$$\leq \int_0^1 X\left(k(t)a+k(1-t)b,\cdot\right)dt \leq [X(a,\cdot)+X(b,\cdot)]\int_0^1 h(t)dt.$$

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References

[1] D. Barráez, L. González, N. Merentes, A. Moros, *On h-convex stochastic processes*, Mathematica Aeterna, 5 (4) (2015) 571-581.

[2] L. González, D. Kotrys, K. Nikodem, *Separation by convex and strongly convex stochastic processes*, (2015) (subbmited for publication).

[3] H. Hudzik, L. Maligranda, *Some remarks on* $s-$*convex functions*, Aequationes Math. 48 (1) (1994) 100-111.

[4] D. Kotrys, *Remarks on strongly convex stochastic processes*, Aequat. Math. 86 (2012) 91-98.

[5] D. Kotrys, *Hermite-Hadamard inequality for convex stochastic processes*, Aequat. Math. 83 (2012) 143-151.

[6] D. Kotrys, *Some characterizations of strongly convex stochastic processes*, Mathematica Aeterna 4 (8) (2014), 855-861.

[7] D. Kotrys, *Remarks on Jensen, Hermite-Hadamard and Fejér inequalities for strongly convex stochastic processes*, Mathematica Aeterna 5 (1) (2015), 95-104.

[8] D. Kotrys, K. Nikodem, *Quasiconvex stochastic processes and a separation theorem*, Aequat. Math. 89 (2015), 41-48.

[9] J. Matkowski, T. Świątkowski, *On subadditive functions*, Proc. Amer. Math. Soc, 119 (1993), 187-197.

[10] B. Micherda, T. Rajba, *On some Hermite-Hadamard inequalities for* (k, h)*-convex functions*, Mathematical Inequalities & Applications, 12 (4) (2012) 931-940.

[11] K. Nikodem, *On convex stochastic processes*, Aequat. Math. 20 (1980) 184-197.

[12] S. Maden, E. Set, M. Tomar, *s-convex stochastic processes in the second sense*, Turkish Journal of Analysis and Number Theory 2 (6) (2014) 202-207.

[13] S. Varošanec, *On* $h-$*convexity*, J. Math. Anal. Appl. 326 (2007) 303-311.

[14] M. Shaked, J. G. Shanthikumar, *Stochastic convexity and its applications*, Adv. in Appl.Prob. 20, (1988) 427-446.

[15] K. Sobczyk, *Stochastic differential equations with applications to physics and engineering*, Kluwer Academic Publishers B. V. (1991).