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STABILITY OF THE FUNCTIONAL EQUATION IN FUZZY BANACH SPACES

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Abstaract − Different kind of stability have been studied concerning several areas of mathematics and fuzziness of such concepts, which is an extension of the former, are being introduced in recent times. The object of the present paper is to appraise generalization of the Hyers-Ulam-Rassias stability theorem for the functional equation

$$f(2x+y)+f(x+2y)=4f(x+y)+f(x)+f(y)$$

in fuzzy Banach spaces.

Keywords − *Fuzzy norm, functional equation, Hyers-Ulam stability, fuzzy Banach spaces.*

1 Introduction

In 1940, Ulam [18] first formulated stability for functional equation concerning group homomorphism and that was partially solved by Hyers [8] in the next year for Cauchy functional equations in Banach spaces and thereafter it was further generalized by Aoki [1]. The stability came in this way was known to be Hyers-Ulam stability. Later on the Hyers-Ulam stability was further generalized by Rassias [15]. The idea of such stability (which subsequently came to be known as Hyers-Ulam-Rassias stability) was generalized and extended to several areas of mathematics over the years. For instances, such stabilities were considered for differential equations [9], functional equations [7], isometries [5] etc.

After the introduction fuzzy set theory, it has been brought quick inroads to deal with uncertainty and vagueness for various problems in many branches of mathematics including functional analysis. In fact, when fuzzy norm on a linear space was first

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introduced by Katsaras [10], a great amendment has come forward in mathematical analysis and specially in functional analysis. Thereafter a few mathematicians have introduced and analyzed several notions of fuzzy norm from different points of views [3, 6, 13, 14, 16]. In particular, in 2003, Bag and Samanta [2] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek [12] type. Thus the notion of fuzzy Banach space came in this way and since then it is being used extensively to study the stability of functional equations, differential equation etc.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

is known as quadratic functional equation, since it is satisfied by the quadratic function $f(x) = cx^2$. The stability problem for the quadratic functional equation has been extensively investigated by a number of mathematicians [4, 11, 15, 17]. In this paper we now consider the functional equation

$$f(2x+y) + f(x+2y) = 4f(x+y) + f(x) + f(y) \tag{2}$$

which is also satisfied by the quadratic function $f(x) = cx^2$ but different from the functional equation (1). Here we like to deal with Hyers-Ulam-Rassias stability for the functional equation (2) in fuzzy Banach spaces.

2 Preliminary

We adopted some definitions and notations of fuzzy norm which will be needed in the sequel.

Definition 2.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$
(N1) $N(x, c) = 0$ for $c \leq 0$;
(N2) $x = 0$ if only if $N(x, c) = 1$ for all $c > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $\lim_{t\to\infty} N(x, t) = 1$.
Then the pair (X, N) is called a **fuzzy normed linear space**.

Example 2.2. Let $(X, \|.\|)$ be a normed linear space. then

$$N(x, t) = \frac{t}{t + k\|x\|}, \ t > 0$$
$$= 0, \ t \leq 0$$

is a fuzzy norm on X.

Definition 2.3. Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be **convergent** if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy normed space (X, N) is said to be **Cauchy** if for each $\varepsilon > 0$ and each $t > 0$, we can find some n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

Now we know that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and a complete fuzzy normed space is called a **fuzzy Banach space**.

3 Hyers-Ulam-Rassias Stability for the Functional Equation

Theorem 3.1. Let X be a linear space and f be a mapping from X to a fuzzy Banach space (Y, N) such that $f(0) = 0$. Suppose that ϕ is a function from X to a fuzzy normed space (Z, N') such that

$$N(f(2x + y) + f(x + 2y) - 4f(x + y) - f(x) - f(y), t + s) \geq \min\{N'(\phi(x), t), N'(\phi(y), s)\} \tag{3}$$

for all $x, y \in X$ and positive real numbers t, s. If $\phi(3x) = \alpha\phi(x)$ for some real number α with $0 < \alpha < 9$ then there exits a unique quadratic mapping $Q : X \to Y$ define by $Q(x) = \lim_{n\to\infty}\left(\frac{f(3^n x)}{9^n}\right)$ and satisfying

$$N(f(x) - Q(x), t) \geq M\left(x, t\frac{9 - \alpha}{18}\right) \tag{4}$$

Where

$$M(x, t) = \min\left\{N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(0), \frac{9t}{5}\right)\right\}.$$

Proof. Putting $y = x$ and $s = t$ in (3), we get

$$N(2f(3x) - 4f(2x) - 2f(x), 2t) \geq \min\{N'(\phi(x), t), N'(\phi(x), t)\}$$

$i.e.,$ $N(f(3x) - 2f(2x) - f(x), t) \geq \min\{N'(\phi(x), t), N'(\phi(x), t)\}$.

Again putting $y = 0$ in (3), we get

$$N(f(2x) - 4f(x), 2t) \geq \min\{N'(\phi(x), t), N'(\phi(0), t)\}.$$

Now

$$\begin{aligned}
&N(f(3x) - 9f(x), 5t) \\
&= N(f(3x) - 2f(2x) - f(x) + 2f(2x) - 8f(x), t + 4t) \\
&\geq \min\{N(f(3x) - 2f(2x) - f(x), t), N(f(2x) - 4f(x), 2t)\}
\end{aligned}$$

$\geq \min\{N'(\phi(x), t), N'(\phi(x), t), N'(\phi(x), t), N'(\phi(0), t)\}$

or, $N\left(f(x) - \dfrac{f(3x)}{9}, \dfrac{5t}{9}\right)$

$\geq \min\{N'(\phi(x), t), N'(\phi(x), t), N'(\phi(x), t), N'(\phi(0), t)\}$,

$i.e., \ N\left(f(x) - \dfrac{f(3x)}{9}, t\right)$

$$\geq \min\left\{N'(\phi(x), \frac{9t}{5}), N'(\phi(x), \frac{9t}{5}), N'(\phi(x), \frac{9t}{5}), N'(\phi(0), \frac{9t}{5})\right\}$$

$$= M(x, t) \tag{5}$$

Where

$$M(x, t) = \min\left\{N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(x), \frac{9t}{5}\right), N'\left(\phi(0), \frac{9t}{5}\right)\right\}$$

Now from our assumption,

$$M(3x, t) = M\left(x, \frac{t}{\alpha}\right) \tag{6}$$

Replacing x by 3^x in (5) and using (6) we have

$N\left(\dfrac{f(3^n x)}{9^n} - \dfrac{f(3^{n+1} x)}{9^{n+1}}, \dfrac{\alpha^n t}{9^n}\right)$

$= N\left(f(3^n x) - \dfrac{f(3^{n+1} x)}{9}, \alpha^n t\right)$

$\geq M(3^n x, \alpha^n t) = M(x, t)$

Since

$$\frac{f(3^n x)}{9^n} - f(x) = \sum_{k=0}^{n-1}\left(\frac{f(3^{k+1}x)}{9^{k+1}} - \frac{f(3^k x)}{9^k}\right)$$

then we have

$$N\left(\frac{f(3^n x)}{9^n} - f(x), t\sum_{k=0}^{n-1}\frac{\alpha^k}{9^k}\right)$$

$$= N\left(\sum_{k=0}^{n-1}\left(\frac{f(3^{k+1} x)}{9^{k+1}} - \frac{f(3^k x)}{9^k}\right), t\sum_{k=0}^{n-1}\frac{\alpha^k}{9^k}\right)$$

$$= N\left(\frac{f(3x)}{9} - f(x) + \sum_{k=1}^{n-1}\left(\frac{f(3^{k+1} x)}{9^{k+1}} - \frac{f(3^k x)}{9^k}\right), t + t\sum_{k=1}^{n-1}\frac{\alpha^k}{9^k}\right)$$

$$\geq \min\left\{M(x, t), N\left(\sum_{k=1}^{n-1}\left(\frac{f(3^{k+1} x)}{9^{k+1}} - \frac{f(3^k x)}{9^k}\right), t\sum_{k=1}^{n-1}\frac{\alpha^k}{9^k}\right)\right\}$$

$$\geq \min\left\{ M(x,t),\, N\left(\frac{f(3^2x)}{9^2} - \frac{f(3x)}{9}, t\,\frac{\alpha}{9}\right), N\left(\frac{f(3^3x)}{9^3} - \frac{f(3^2x)}{9^2}, \frac{t\,\alpha^2}{9^2}\right),\right.$$
$$\left. N\left(\frac{f(3^4x)}{9^4} - \frac{f(3^3x)}{9^3}, \frac{t\,\alpha^3}{9^3}\right), \ldots, N\left(\frac{f(3^kx)}{9^k} - \frac{f(3^{k-1}x)}{9^{k-1}}, \frac{t\,\alpha^{k-1}}{9^{k-1}}\right)\right\}$$
$$\geq \min\{M(x,t), M(x,t), M(x,t), M(x,t), \ldots, M(x,t)\}$$
$$= M(x,t)$$

Therefore

$$N\left(\frac{f(3^nx)}{9^n} - f(x), t\right) \geq M\left(x, t\sum_{k=0}^{n-1}\frac{9^k}{\alpha^k}\right) \tag{7}$$

Replacing x by $3^m x$ in (7) we get

$$N\left(\frac{f(3^{n+m}x)}{9^{m+n}} - \frac{f(3^mx)}{9^m}, t\right) \geq M\left(x, t\sum_{k=m}^{m+n-1}\frac{9^k}{\alpha^k}\right) \tag{8}$$

Since $\lim_{t\to\infty} M(x,t) = 1$, taking limit $m \to \infty$,
the R. H. S. of (8) tends to 1 as $m \to \infty$.

Therefore $\left\{\frac{f(3^nx)}{9^n}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a complete fuzzy normed space, the sequence converges to some point $Q(x) \in Y$. So we can define a mapping $Q : X \to Y$ by $Q(x) := N \lim_{n\to\infty} \frac{f(3^nx)}{9^n}$ for all $n \in N$.
Also

$$N(Q(x) - f(x), t) = N\left(Q(x) - \frac{f(3^nx)}{9^n} + \frac{f(3^nx)}{9^n} - f(x), \frac{t}{2} + \frac{t}{2}\right)$$

$$\geq \min\left\{N\left(Q(x) - \frac{f(3^nx)}{9^n}, \frac{t}{2}\right), N\left(\frac{f(3^nx)}{9^n} - f(x), \frac{t}{2}\right)\right\}$$

$$\geq M\left(x, \frac{t}{2\sum\limits_{k=0}^{\infty}\left(\frac{\alpha}{9}\right)^k}\right) = M\left(x, \frac{t}{2\left(\frac{1}{1-\frac{\alpha}{9}}\right)}\right) = M\left(x, \frac{t(9-\alpha)}{18}\right)$$

To show that Q satisfies the functional equation (2), we replacing x by $3^n x$ and y by $3^n y$ in (3)

$$N\,(f(3^n(2x+y)) + f(3^n(x+2y)) - 4f(3^n(x+y))$$
$$- f(3^n(x)) - f(3^n(y)), t\,)$$
$$\geq \min\left\{N'(\phi(3^nx), \frac{t}{2}), N'(\phi(3^ny), \frac{t}{2})\right\}$$

$$\text{or},\ N\left(\frac{f(3^n(2x+y))}{9^n} + \frac{f(3^n(x+2y))}{9^n} - \frac{4f(3^n(x+y))}{9^n} - \right.$$
$$\left.\frac{f(3^n(x))}{9^n} - \frac{f(3^n(y))}{9^n}, \frac{t}{9^n}\right)$$

$$\geq \min\left\{ N'(\phi(3^n x), \frac{t}{2}), N'(\phi(3^n y), \frac{t}{2})\right\}$$

$$\text{or}, \ N\Bigg(\frac{f(3^n(2x+y))}{9^n} + \frac{f(3^n(x+2y))}{9^n} - \frac{4f(3^n(x+y))}{9^n} - \frac{f(3^n(x))}{9^n} - \frac{f(3^n(y))}{9^n}, t \Bigg)$$

$$\geq \min\left\{ N'(\phi(3^n x), \frac{9^n t}{2}), N'(\phi(3^n y), \frac{9^n t}{2})\right\}$$

$$= \min\left\{ N'\left(\phi(x), \left(\frac{9}{\alpha}\right)^n \frac{t}{2}\right), N'\left(\phi(y), \left(\frac{9}{\alpha}\right)^n \frac{t}{2}\right)\right\}$$

for all $x, y \in X, t > 0$.
As $0 < \alpha < 9$, taking limit $n \to \infty$ we get

$$N(Q(2x+y)+Q(x+2y)-4Q(x+y)-Q(x)-Q(y), t) = 1$$

Therefore

$$Q(2x+y)+Q(x+2y) = 4Q(x+y)+Q(x)+Q(y)$$

Hence Q satisfies (2).
Uniqueness : Let $T : X \to Y$ be an another quadratic mapping which satisfies (3). Since $Q(2x) = 4Q(x)$ and $Q(3x) = 2Q(2x) + Q(x) = 9Q(x)$. Therefore it can be proved by induction that $Q(3^n x) = 9^n Q(x)$. Now fix $x \in X$ and using $Q(3^n x) = 9^n Q(x)$ and $T(3^n x) = 9^n T(x)$ for all $x \in X$. Now

$$N(Q(x)-T(x), t) = N\left(\frac{Q(3^n x)}{9^n} - \frac{T(3^n x)}{9^n}, t\right)$$

$$= N(Q(3^n x) - T(3^n x), 9^n t)$$

$$\geq \min\left\{ N\left(Q(3^n x) - f(3^n x), \frac{9^n t}{2}\right), N\left(T(3^n x) - f(3^n x), \frac{9^n t}{2}\right)\right\}$$

$$\geq \min\left\{ M\left(3^n x, \frac{9^n t(9-\alpha)}{2\times 18}\right), M\left(3^n x, \frac{9^n t(9-\alpha)}{2\times 18}\right)\right\}$$

$$= \min\left\{ M\left(x, \frac{t(9-\alpha)}{2\times 18}\left(\frac{9}{\alpha}\right)^n\right), M\left(x, \frac{t(9-\alpha)}{2\times 18}\left(\frac{9}{\alpha}\right)^n\right)\right\}$$

$$= M\left(x, \frac{t(9-\alpha)}{2\times 18}\left(\frac{9}{\alpha}\right)^n\right)$$

for all $x \in X$ and $t > 0$. Since $0 < \alpha < 9$, and $\lim_{n\to\infty} \left(\frac{9}{\alpha}\right)^n = \infty$ therefore right hand side of the inequality tend to 1 as $n \to \infty$. Hence $Q(x) = T(x)$ for all $x \in X$. This completes the proof of the theorem. □

Corollary 3.2. Let $\delta > 0$ and X be a linear space, (Y, N') be a fuzzy Banach space. If let $f : X \to Y$ be a mapping and z_0 is a fixed vector of a fuzzy normed space (Z, N'') such that

$$N'(f(2x + y) + f(x + 2y) - 4f(x + y) - f(x) - f(y), t + s)$$
$$\geq \min\{N''(\delta z_0, t), N''(\delta z_0, s)\} \tag{9}$$

for all $x, y \in X$ and positive real numbers t, s. Then there exits a unique quadratic mapping $Q : X \to Y$ define by $Q(x) = \lim_{n\to\infty}\left(\frac{f(3^n x)}{9^n}\right)$ and satisfying

$$N'(f(x) - Q(x), t) \geq N''\left(z_0, \frac{9t}{5\delta}\right) \tag{10}$$

Proof. Define $\phi(x) = \delta z_0$, then the proof is followed by the previous Theorem. □

Corollary 3.3. Let $\varepsilon \geq 0$ and X be a linear space, (Y, N'') be a fuzzy Banach space. If let $f : X \to Y$ be a mapping such that

$$N''(f(2x + y) + f(x + 2y) - 4f(x + y) - f(x) - f(y), t) \geq \epsilon$$

for all $x, y \in X$ and positive real numbers t. Then there exits a unique quadratic mapping $Q : X \to Y$ define by $Q(x) = \lim_{n\to\infty}\left(\frac{f(3^n x)}{9^n}\right)$ and satisfying

$$N''(f(x) - Q(x), \frac{5}{8}t) \geq \epsilon$$

Proof. The proof is same as that of the previous theorem. □

Example 3.4. Let X be a normed algebra. Define $f : (X, N) \to (X, N')$ by $f(x) = x^2 + \|x\|x_0$, and
$\phi(x, y) = (\|2x + y\| + \|x + 2y\| - 4\|x + y\| - \|x\| - \|y\|)x_0$ where x_0 is a unit vector in X. Then

$$N(f(2x + y) + f(x + 2y) - 4f(x + y) - f(x) - f(y), t + s)$$
$$\geq \min\{N'(\phi(x), t), N'(\phi(y), s)\}$$

Also $\phi(3x, 3y) = 3\phi(x, y)$ for each $x, y \in X$. Hence all the conditions of Theorem (3.1) holds for $\alpha = 1$. Therefore fuzzy difference between $Q(x) = \lim_{n\to\infty}\left(\frac{f(3^n x)}{9^n}\right) = x^2$ and $f(x)$ is equal to [using Example 2.2]

$$N(f(x) - Q(x), t) = N(\|x\|x_0, t)$$
$$= \frac{t}{t + \|x\|} = N(x, t) \geq N'(x, t) \geq M\left(x, \frac{4}{9}t\right).$$

4 Conclusion

In this article, to establish Hyers-Ulam-Rassias stability we have used the functional equation (2) having quadratic function as its one particular solution. But this equation is not known as quadratic functional equation. So, natural question arises, whether the functional equation (2) can be derived from the quadratic functional equation (1) or the equation (1) can be derived from the equation (2). In fact, what could be the general solution of the equation (2). Can we establish the Theorem (3.1) for complex valued function f of complex variable satisfying the equation (2)? So, in our view, this article has good prospect for future work.

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