http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 27.10.2015 *Year*: **2016**, *Number*: **10**, *Pages*: **45-53**
Published: 22.01.2016 *Original Article*[**]

PERFECTLY ω-IRRESOLUTE FUNCTIONS

İdris Zorlutuna <izorlu@cumhuriyet.edu.tr>

Department of Mathematics, Cumhuriyet University, 58140 Sivas, Turkey

Abstract − In this paper, we investigate a new form of continuity called perfect ω-irresoluteness and we use functions which have this type of continuity as a tool to set new characterizations of some properties of topological spaces.

Keywords − *ω-open, ω-irresolute, perfectly ω-irresolute.*

1 Introduction

An ω-closed set is a set which contains all its condensation points [5]. Since the advent of this notion, lots of topologist have studied on it and most of topological notions such as continuity, compactness, connectedness were generalized. Especially, some new strong and weak forms of continuity have been arised during the last years. One of these is ω-irresoluteness introduced by Al-Zoubi [4]. On the other hand, in 1984, Noiri [8] introduced and investigated the notion of perfect continuity of functions between topological spaces.

This paper devoted to investigate a new type of continuity is stronger than ω-irresoluteness and perfect continuity. In section 3, definition and fundamental properties are given. In section 4, we use perfectly ω-irresolute functions as a tool to set new characterizations of connectedness. Moreover some separation axioms related to ω-open sets are investigated. The last section deals with graphs of perfectly ω-irresolute functions.

2 Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation axiom is assumed unless explicitly stated and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f from a topological space (X, τ) into a topological space (Y, σ). Let A be a subset of a space X. A point $x \in X$ is called a condensation point of A if for each open set U with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed [5] if it contains all its condensation points. The

[**] *Edited by* Takashi Noiri *and* Naim Çağman *(Editor-in-Chief).*

complement of an ω-closed set is called ω-open. The family of all ω-open subsets of (X, τ) is denoted by τ_ω. It is known that τ_ω is a topology for X and $\tau \subset \tau_\omega$. For a subset A of (X, τ), the closure of A and the interior of A denoted by $Cl(A)$ and $Int(A)$, respectively. The closure of A with respect to τ_ω denoted by $\omega Cl(A)$. A is called regular closed [9] if $A = Cl(Int(A))$.

Let us recall the following definitions which we shall require later.

Definition 2.1. A function $f : X \to Y$ is called perfectly continuous [8] if $f^{-1}(V)$ is clopen in X for every open set V of Y.

Definition 2.2. A function $f : X \to Y$ is called ω-irresolute [4] if $f^{-1}(V)$ is ω-open in X for every ω-open set V of Y.

3 Perfectly ω-irresolute Functions

Definition 3.1. A function $f : X \to Y$ is said to be perfectly ω-irresolute if $f^{-1}(V)$ is clopen in X for every ω-open set V of Y.

Theorem 3.2. For a function $f : (X, \tau) \to (Y, \sigma)$, the followings are equivalent:
(1) f is perfectly ω-irresolute;
(2) for every ω-closed subset F of Y, $f^{-1}(F)$ is clopen in X;
(3) $f : (X, \tau) \to (Y, \sigma_\omega)$ is perfectly continuous.

Proof. (1)⇒(2). Let F be a ω-closed subset of Y. Then $Y \backslash F$ is an ω-open subset and by (1), $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$ is clopen in X. Hence $f^{-1}(F)$ is also clopen in X.

(2)⇒(3). Let $V \in \sigma_\omega$. Then $Y \backslash V$ is an ω-closed in Y and by (2), $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is clopen in X. Hence $f^{-1}(V)$ is also clopen in X.

(3)⇒(1). It can be shown easily. Recal that a space (X, τ) is said to be ω-space [1] if every ω-open set is open and is said to be locally ω-indiscrete [1] if every ω-open set is closed in X.

Then we have the following theorem which gives a characterization of locally ω-indiscrete ω-space. Its proof is clear.

Theorem 3.3. A space X is ω-space and locally ω-indiscrete if and only if the identity map of X is perfectly ω-irresolute.

Theorem 3.4. For a function $f : X \to Y$, the following are true.
(1) If f is perfectly ω-irresolute and $A \subseteq X$, then $f\mid_A : A \to Y$ is perfectly ω-irresolute.
(2) If $\{G_\alpha : \alpha \in I\}$ is a locally finite clopen cover of X and if for each α, $f_\alpha = f\mid_{G_\alpha}$ is perfectly ω-irresolute, then f is perfectly ω-irresolute.

Proof. The proof of (1) is clear. We will only prove (2).

Let F be a ω-open subset of Y. Since each f_α is perfectly ω-irresolute, each $f_\alpha^{-1}(F)$ is clopen in G_α and hence in X. Thus $f^{-1}(F) = \cup\{f_\alpha^{-1}(F) : \alpha \in I\}$ is open in X. On the other hand, since the family $\{G_\alpha : \alpha \in I\}$ is locally finite, $\{f_\alpha^{-1}(F) : \alpha \in I\}$ is a locally finite family of closed sets in X. Hence $f^{-1}(F)$ being the union of a locally finite collection of closed sets is closed in X. Consequently, $f^{-1}(F)$ is clopen in X.

Definition 3.5. A function $f : X \to Y$ is called
(1) ω-continuous [6] $f^{-1}(V)$ is ω-open in X for every open set V of Y.
(2) slightly ω-continuous [7] $f^{-1}(V)$ is ω-open in X for every clopen set V of Y.
(3) contra ω-irresolute if $f^{-1}(V)$ is ω-closed in X for every ω-open set V of Y.

Theorem 3.6. The followings hold for functions $f : X \to Y$ and $g : Y \to Z$:
(1) If $f : X \to Y$ is perfectly ω-irresolute and $g : Y \to Z$ is ω-irresolute, then $g \circ f : X \to Z$ is perfectly ω-irresolute.
(2) If $f : X \to Y$ is perfectly ω-irresolute and $g : Y \to Z$ is ω-continuous, then $g \circ f : X \to Z$ is perfectly continuous.
(3) If $f : X \to Y$ is slightly ω-continuous and $g : Y \to Z$ is perfectly ω-irresolute, then $g \circ f : X \to Z$ is ω-irresolute.
(4) If $f : X \to Y$ is perfectly ω-irresolute and $g : Y \to Z$ is contra ω-irresolute, then $g \circ f : X \to Z$ is perfectly ω-irresolute.

Proof. (1) Let V be any ω-open set in Z. By the ω-irresoluteness of g, $g^{-1}(V)$ is ω-open. Since f is perfectly ω-irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is clopen in X. Therefore, $g \circ f$ is perfectly ω-irresolute.

The others can be proved similarly.

Theorem 3.7. If $f : X \to Y$ is a surjective open and closed function and $g : Y \to Z$ is a function such that $gof : X \to Z$ is perfectly ω-irresolute function, then g is perfectly ω-irresolute function.

Proof. Let V be any ω-open set in Z. Since gof is perfectly ω-irresolute, $(gof)^{-1}(V)$ is clopen in X. Since f is surjective open and closed, $f((g \circ f)^{-1}(V)) = f((f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is clopen in Y. Therefore, g is perfectly ω-irresolute.

It is easy to show that perfect ω-irresoluteness implies perfect continuity and ω-irresoluteness. The following theorems are about reverse of these implications and they can be proved directly.

Theorem 3.8. Let X be a locally ω-indiscrete and ω-space. Then for any topological space Y, a function $f : X \to Y$ is perfectly ω-irresolute if and only if f is ω-irresolute.

Theorem 3.9. Let Y be an ω-space. Then for any topological spaces X, a function $f : X \to Y$ is perfectly ω-irresolute if and only if f is perfectly continuous.

Theorem 3.10. For a function $f : X \to Y$, the following properties are equivalent:
(1) f is contra ω-irresolute;
(2) for every ω-closed F of Y, $f^{-1}(F)$ is ω-open in X;
(3) for every $x \in X$ and for every ω-closed set F containing $f(x)$, there exists an ω-open set U containing x such that $f(U) \subseteq F$.

Proof. (1)$\Leftrightarrow$(2). These follow from equality $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ for each subset F of Y.

(2)$\Rightarrow$(3). Let F be an ω-closed set containing $f(x)$. Then by (2), $f^{-1}(F)$ is ω-open in X containing x. If we choose $U = f^{-1}(F)$, proof is completed.

(3)$\Rightarrow$(2). Obvious.

A space X is called anti locally countable (see [3]) if every nonempty open set is uncountable. It is shown in [7] that in an anti locally countable space X, U is clopen in X iff U is ω-open and ω-closed in X. Then we have the following corollary.

Corollary 3.11. Let (X, τ) be an anti locally countable space. Then for any topological spaces (Y, σ), a function $f : (X, \tau) \to (Y, \sigma)$ is perfectly ω-irresolute if and only if f is ω-irresolute and contra ω-irresolute.

A space X is called ω-regular [7] if for each closed set F and each point $x \in X - F$, there exist disjoint ω-open sets U and V such that $x \in U$ and $F \subseteq V$. It is shown in [2] that a space X is ω-regular if and only if for every point x of X and every open set V containing x, there exists an ω-open set U such that $x \in U \subseteq \omega Cl(U) \subseteq V$.

Theorem 3.12. Let X be an anti local countable space and let Y be an ω-regular space. For a function $f : X \to Y$, the following properties are equivalent:
(1) f is perfectly ω-irresolute;
(2) for every ω-open set V in Y, $f^{-1}(V)$ is regular closed in X;
(3) for every ω-open set V in Y, $f^{-1}(V)$ is closed in X;
(4) f is contra-ω-irresolute.

Proof. The implications (1)⇒(2)⇒(3)⇒(4) are trivial. If we show that f is ω-irresolute by Corollary 3.11, we have the proof of the implication (4)⇒(1). Let $x \in X$ be an arbitrary point and V be an ω-open set of Y containing $f(x)$. Since Y is ω-regular, there exists an ω-open set W in Y such that $f(x) \in \omega Cl(W) \subseteq V$. Since f is contra-ω-irresolute, there exists an ω-open set U_x containing x such that $f(U_x) \subseteq \omega Cl(W)$. Then $U_x \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is ω-open in X.

4 Applications

Note that (X, τ_ω) is always a T_1-space for any given space (X, τ) [3]. Hence we have the following results.

Theorem 4.1. If $f : (X, \tau) \to (Y, \sigma)$ is a perfectly ω-irresolute function, then f is constant on each component of X.

Proof. Let a and b be two points of X that lie in same component C of X. Assume that $f(a) \neq f(b)$. Since (Y, σ_ω) is T_1- space, there exists an $U \in \sigma_\omega$ containing say $f(a)$ but not $f(b)$. By perfect ω-irresoluteness of f, $f^{-1}(U)$ and $X - f^{-1}(U)$ are disjoint clopen sets containing a and b, respectively. This is a contradiction with the fact that C is a component containing a and b. Hence we have the result.

Corollary 4.2. If $f : X \to Y$ is a perfectly ω-irresolute function and if A is non-empty connected subset of X, then $f(A)$ is a single point.

Theorem 4.3. A space X is connected if and only if every perfectly ω-irresolute function from a space X into any space Y is constant.

Proof. The first part of the proof is clear by Theorem 4.1. For the seconf part, assume that X is not connected. Then there exists a proper non-empty clopen subset A of X. Let $Y = \{u, v\}$ and σ be discrete topology on Y. Then the function $f : X \to Y$ defined by $f(x) = u$ if $x \in A$, $f(x) = v$ if $x \notin A$ is non-constant and perfectly ω-irresolute. This is a contradiction by Theorem 4.1. Hence X must be connected.

Note that the topological space consisting of two points with the discrete topology is usually denoted by 2.

Corollary 4.4. For a topological space X, the following are equivalent :
(1) X is connected;
(2) Every perfectly ω-irresolute function $f : X \to 2$ is constant;
(3) There is no perfectly ω-irresolute function $f : X \to 2$ is surjective.

Definition 4.5. A space X is said to be ultra Hausdorff [9] (resp. ω-T_2 [3]) if every two distinct points of X can be separated by disjoint clopen (resp. ω-open) sets.

Theorem 4.6. If $f : X \to Y$ is a perfectly ω-irresolute injection, then X is ultra Hausdorff.

Proof. Let x_1, $x_2 \in X$ and $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$. Since Y is always $\omega - T_1$, there exists an ω-open set U containing say $f(x_1)$ but not $f(x_2)$. By perfect ω-irresoluteness of f, $f^{-1}(U)$ and $X - f^{-1}(U)$ are disjoint clopen sets containing x_1 and x_2, respectively. Thus X is ultra Hausdorff.

The quasi-topology denoted by τ_q on X is the topology having as base the clopen subsets of (X, τ). A subset A of X is called quasi open if $A \in \tau_q$. The complement of a quasi-open set is called quasi-closed [9].

Theorem 4.7. Let Y be ω-T_2 space.
(1) If f, $g : X \to Y$ are perfectly ω-irresolute functions, then the set $A = \{x \in X : f(x) = g(x)\}$ is quasi-closed in X.
(2) If $f : X \to Y$ is perfectly ω-irresolute function, then the subset $E = \{(x, y) : f(x) = f(y)\}$ is quasi-closed in $X \times X$.

Proof. (1). Let $x \notin A$. Then $f(x) \neq g(x)$. Since Y is ω-T_2, there exist disjoint ω-open sets V_1 and V_2 in Y such that $f(x) \in V_1$ and $g(x) \in V_2$. Since f and g are perfectly ω-irresolute, $f^{-1}(V_1)$ and $g^{-1}(V_2)$ are clopen sets. Put $U = f^{-1}(V_1) \cap g^{-1}(V_2)$. Then U is clopen set containing x and $U \cap A = \varnothing$. Hence we have $U \subseteq X - A$. This shows that $X - A$ is quasi-open or equivalently A is quasi-closed.

(2). Let $(x, y) \notin E$. Then $f(x) \neq f(y)$. Since Y is ω-T_2, there exist disjoint ω-open sets V_1 and V_2 containing $f(x)$ and $f(y)$ respectively. Since f is perfectly ω-irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are clopen sets. Then for the clopen set $U = f^{-1}(V_1) \times f^{-1}(V_2)$ containing (x, y), we have $U \cap E = \varnothing$ i.e. $U \subseteq (X \times X) - E$. This shows that $(X \times X) - E$ is quasi-open or equivalently E is quasi-closed.

Definition 4.8. [5]A function $f : X \to Y$ is called
(a) ω-closed if for each closed set K in X, $f(K)$ is ω-closed in Y.
(b) ω-open if for each open set U in X, $f(U)$ is ω-open in Y.

Theorem 4.9. A function $f : X \to Y$ is ω-closed (resp. ω-open) if and only if for each subset S of Y and for each open (resp. closed) subset U of X with $f^{-1}(S) \subseteq U$, there exists an ω-open (resp. ω-closed) set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. We only prove for the ω-closedness. The other is entirely analogous.

($\Rightarrow$): Suppose that f is ω-closed. Let S be any subset of Y and U be an open subset of X with $f^{-1}(S) \subseteq U$. Since f is ω-closed, $Y - f(X - U)$ is an ω-open set in Y. Then for the set $V = Y - f(X - U)$, we have $S \subseteq V$ and $f^{-1}(V) = f^{-1}(Y - f(X - U)) = X - f^{-1}(f(X - U)) \subseteq U$.

($\Leftarrow$): Let K be any closed subset of X and $S = Y - f(K)$. Then $f^{-1}(S) \subseteq X - K$. By hypothesis, there exists an ω-open set V in Y containing S such that $f^{-1}(V) \subseteq X - K$. Then, we have $K \subseteq X - f^{-1}(V)$ and $Y - V = f(K)$. Since, $Y - V$ is ω-closed, $f(K)$ is ω-closed and thus f is an ω-closed function. A topological space

X is called ω-normal [7] if for every pair of disjoint closed subsets F_1 and F_2 of X, there exists disjoint ω-open sets U and V such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

Theorem 4.10. If $f : X \to Y$ is a continuous ω-closed surjection and if X is normal space, then Y is ω-normal.

Proof. Let F_1 and F_2 be disjoint closed sets of Y. Since f is continuous and X is normal, there exist disjoint open sets U and V such that $f^{-1}(F_1) \subseteq U$ and $f^{-1}(F_2) \subseteq V$. By Theorem 4.9, there exist ω-open sets G and H such that $F_1 \subseteq G$, $F_2 \subseteq H$ and $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \varnothing$ and hence $G \cap H = \varnothing$. This shows that Y is ω-normal. The following theorem shows that we can get same result under different hypothesis.

Theorem 4.11. If $f : X \to Y$ is perfectly ω-irresolute, ω-open bijection and X is a normal space, then Y is ω-normal.

Proof. Let F_1 and F_2 be disjoint closed sets in Y. Since F_1 and F_2 are also ω-closed and f is perfectly ω-irresolute, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint clopen and so closed sets in X. By normality of X, there exist disjoint open sets U and V such that $f^{-1}(F_1) \subseteq U$ and $f^{-1}(F_2) \subseteq V$. Then we obtain that $F_1 \subseteq f(U)$ and $F_2 \subseteq f(V)$ such that $f(U)$ and $f(V)$ are disjoint ω-open sets. Thus Y is ω-normal.

Theorem 4.12. If $f : X \to Y$ is a continuous, ω-open, ω-closed surjection and if X is regular, then Y is ω-regular.

Proof. Let $y \in Y$ and V be an open set in Y with $y \in V$. Take $y = f(x)$. Since f is continuous and X is regular, there exist an open set U such that $x \in U \subseteq Cl(U) \subseteq f^{-1}(V)$. Then $y \in f(U) \subseteq f(Cl(U)) \subseteq V$. By assumptions, $f(U)$ is ω-open and $f(Cl(U))$ is ω-closed set in Y. Therefore, we have $y \in f(U) \subseteq \omega Clf(U) \subseteq V$. This shows that Y is ω-regular.

Theorem 4.13. If $f : X \to Y$ is perfectly ω-irresolute, ω-open bijection and if X is regular, then Y is ω-regular.

Proof. It is similar to that of Theorem 4.11.

Definition 4.14. A space (X, τ) is called

(1) mildly compact [9] (resp. ω-compact [1]) if every clopen (resp. ω-open) cover of X has a finite subcover.

(2) mildly Lindelöf [9] if every cover of X by clopen sets has a countable subcover.

Theorem 4.15. Let $f : X \to Y$ be a perfectly ω-irresolute surjection. If X is mildly compact, then Y is ω-compact.

Proof. Let f be a perfectly ω-irresolute surjection and let X be a mildly compact space. If $\{V_i\}_{i\in I}$ is an ω-open cover of Y, by perfect ω-irresoluteness of f, $\{f^{-1}(V_i)\}_{i\in I}$ is a clopen cover of X and so there is a finite subset I_0 of I such that $X = \cup_{i\in I_0} f^{-1}(V_i)$. Therefore, we have $Y = \cup_{i\in I_0} V_i$ since f is surjective. Thus Y is ω-compact.

Theorem 4.16. [6]For a topological space (X, τ), (X, τ) Lindelöf if and only if (X, τ_ω) Lindelöf.

Theorem 4.17. Let $f : X \to Y$ be a perfectly ω-irresolute surjection. If X is mildly Lindelöf, then Y is Lindelöf.

Proof. It is similar to that of Theorem 4.15. We notice that a subspace A of a space X is mildly Lindelöf relative to X if for every cover $\{V_i : i \in I\}$ of A by clopen sets of X, there exists a countable subset I_0 of I such that $\{V_i : i \in I_0\}$ covers A.

Theorem 4.18. Let $f : X \to Y$ be an ω-closed surjection such that $f^{-1}(\{y\})$ is a mildly Lindelöf relative to X for each $y \in Y$. If Y is Lindelöf, then X is mildly Lindelöf.

Proof. Let $\{U_i : i \in I\}$ be an clopen cover of X. Since $f^{-1}(\{y\})$ is a mildly Lindelöf relative to X for each $y \in Y$, there exists a countable subset I_y of I such that $f^{-1}(\{y\}) \subseteq \cup\{U_i : i \in I_y\}$. Put $U_y = \cup\{U_i : i \in I_y\}$. Then since f is ω-closed, $V_y = Y - f(X - U_y)$ is an ω-open set containing y such that $f^{-1}(V_y) \subseteq U_y$. Again since $\{V_y : y \in Y\}$ is an ω-open cover of the Lindelöf space Y, by Theorem 4.16, there exist countable points of Y, says, $y_1, y_2, ..., y_n, ...$ such that $Y = \cup_{n\in\mathbb{N}} V_{y_n}$. Therefore, we have $X = f^{-1}(\cup_{n\in\mathbb{N}} V_{y_n}) = \cup_{n\in\mathbb{N}} f^{-1}(V_{y_n}) \subseteq \cup_{n\in\mathbb{N}} U_{y_n} = \cup_{n\in\mathbb{N}}(\cup\{U_i : i \in I_{y_n}\}) = \cup\{U_i : i \in I_{y_n}, n \in \mathbb{N}\}$. This completes the proof.

Corollary 4.19. Let $f : X \to Y$ be an perfectly ω-irresolute and ω-closed surjection such that $f^{-1}(\{y\})$ is a mildly Lindelöf relative to X for each $y \in Y$. Then X is mildly Lindelöf if and only if Y is Lindelöf.

5 Graphs of Perfectly ω-irresolute Functions

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 5.1. The graph $G(f)$ of a function $f : X \to Y$ is said to be quasi-ω-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist a clopen set U containing x and an ω-open set V containing y such that $(U \times V) \cap G(f) = \varnothing$.

The proof of the following lemma is clear.

Lemma 5.2. The graph $G(f)$ of a function $f : X \to Y$ is quasi-ω-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist a clopen set U containing x and an ω-open set V containing y such that $f(U) \cap V = \varnothing$.

Theorem 5.3. If $f : X \to Y$ is perfectly ω-irresolute and Y is ω-T_2, then $G(f)$ is quasi-ω-closed.

Proof. Let $(x, y) \notin G(f)$, then $y \neq f(x)$. Since Y is ω-T_2, there exist disjoint ω-open sets V_1 and V_2 containing $f(x)$ and y, respectively. Again since f is perfectly ω-irresolute, $f^{-1}(V_1)$ is clopen set containing x. If we choose $U = f^{-1}(V_1)$, then we have $f(U) \cap V_2 = \varnothing$ and hence $G(f)$ is quasi-ω-closed. A subset A of a space X is said to be mildly compact (resp. ω-compact) relative to X if for every cover $\{V_i : i \in I\}$ of A by clopen (resp. ω-open) sets of X, there exists a finite subset I_0 of I such that $A \subseteq \cup\{V_\omega : \omega \in I_0\}$.

Theorem 5.4. If a function $f : (X, \tau) \to (Y, \sigma)$ has quasi-ω-closed graph, then the followings are true.

(1) $f(E)$ is ω-closed in Y for every subset E which is mildly compact relative to X.

(2) $f^{-1}(K)$ is quasi-closed in X for every subset K which is ω-compact relative to Y.

Proof. (1) Let E be mildly compact relative to X and $y \notin f(E)$. Then we have $(x, y) \in (X \times Y) - G(f)$ for each $x \in E$ and by Lemma 5.2, there exist a clopen set U_x and ω-open set V_x containing x and y respectively, such that $f(U_x) \cap V_x = \varnothing$. Since the family of $\{U_x : x \in E\}$ is a cover of E by clopen sets of X, there exists a finite number of points, say, $x_1, x_2, ..., x_n$ of E such that $E \subseteq \cup\{U_{x_i} : i = 1, 2, ..., n\}$. Set $V = \cap\{V_{x_i} : i = 1, 2, ..., n\}$, then V is an ω-open set containing y and $f(E) \cap V \subseteq (\cup\{f(U_{x_i}) : i = 1, 2, ..., n\}) \cap V = \varnothing$. Therefore, we have, $y \notin \omega Cl(f(E))$. This shows that $f(E)$ is ω-closed in Y.

(2) It is similar.

Theorem 5.5. Let $f : (X, \tau) \to (Y, \sigma)$ have the quasi-ω-closed graph. If f is injective, then (X, τ_q) is T_1.

Proof. Let $x_1,\ x_2 \in X$ and $x_1 \neq x_2$. Then, we have $f(x_1) \neq f(x_2)$ and so $(x_1, f(x_2)) \in (X \times Y) - G(f)$. By quasi-$\omega$-closedness of graph $G(f)$, there exist a clopen set U and an ω-open set V containing $x_1$and $f(x_2)$ respectively, such that $f(U) \cap V = \varnothing$, and hence $U \cap f^{-1}(V) = \varnothing$. Since $x_2 \in f^{-1}(V)$, clopen sets U and $X - U$ are desired sets. This completes the proof.

Theorem 5.6. If $f : X \to Y$ is an injection with quasi-ω-closed graph, then X is ultra Hausdorff.

Proof. Let x_1 and x_2 be distinct points in X. Then $f(x_1) \neq f(x_2)$ and so $(x_1, f(x_2)) \notin G(f)$. Therefore, there exist a clopen set U and an ω-open set V such that $(x_1, f(x_2)) \in U \times V$ and $U \cap f^{-1}(V) = \varnothing$. Hence we have disjoint clopen sets U and $X \backslash U$ containing x_1 and x_2 respectively. This shows that X is ultra Hausdorff.

Theorem 5.7. Let $f : X \to Y$ have the quasi-ω-closed graph. If f is a surjective ω-open function, then Y is ω-T_2.

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective, $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) - G(f)$. By quasi-$\omega$-closedness of graph $G(f)$, there exist a clopen set U and an ω-open set V such that $(x, y_2) \in (U \times V)$ and $f(U) \cap V = \varnothing$. Since f is ω-open, then $f(U)$ is ω-open such that $f(x) = y_1 \in f(U)$. This shows that Y is ω-T_2.

Definition 5.8. A topological space X is said to be hyperconnected [10] if every pair nonempty open sets of X has nonempty intersection.

Theorem 5.9. Let X be hyperconnected. If $f : X \to Y$ is a perfectly ω-irresolute function with quasi-ω-closed graph, then f is constant.

Proof. Suppose that f is not constant. Then there exist two point x_1 and x_2 of X such that $f(x_1) \neq f(x_2)$. Then we have $(x_1, f(x_2)) \notin G(f)$. Since $G(f)$ is quasi-ω-closed, there exist a clopen set U and an ω-open set V such that $(x_1, f(x_2)) \in U \times V$ and $f(U) \cap V = \varnothing$. Therefore, we have $U \cap f^{-1}(V) = \emptyset$. This is a contradiction with the hyperconnectedness of X since $f^{-1}(V)$ is non-empty open set in X.

6 Acknowledgement

This work is supported by the Scientific Research Project Fund of Cumhuriyet University under the project number F-437.

References

[1] A. Al-OMARI, M.S.M. NOORANI, *Contra-ω-continuous and almost contra-ω-continuous*, Int. J. Math. Math. Sci. (2007) ID 40469.

[2] A. Al-OMARI, T. NOIRI, M.S.M. NOORANI, *Weak and strong forms of ω-continuous functions*, Int. J. Math. Math. Sci. (2009) ID 174042.

[3] K. Al-ZOUBI, B. Al-NASHEF, *The topology of ω-open subsets*, Al-Manarah 9(2) (2003) 169-179.

[4] K. Al-ZOUBI, *Semi ω-continuous functions*, Abhath Al-Yarmouk 12(1) (2003) 119-131.

[5] H.Z. HDEIB, *ω-closed mappings,* Revista Colombiana de Matem´aticas 16(1-2) (1982) 65-78.

[6] H.Z. HDEIB, *ω-continuous functions*, Dirasat Journal 16(2) (1989) 136-142.

[7] T. NOIRI, A. Al-OMARI, M.S.M. NOORANI, *Slightly ω-continuous functions*, Fasciculi Mathematici 41 (2009) 97-106.

[8] T. NOIRI, *Super continuity and some strong forms of continuity*, Indian J. Pure Appl. Math. 15 (3) (1984) 241-250.

[9] R. STAUM, *The algebra of bounded continuous functions into a non archimedian field*, Pacific J. Math. 50 (1974) 169-185.

[10] L.A. STEEN, J.A. SEEBACH, *Counterexamples in topology*, Holt, Rinehart and Winston New york 1970.