http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 27.11.2015 *Year*: **2016**, *Number*: **10**, *Pages*: **54-65**
Published: 08.02.2016 *Original Article*[*]

HERMITE-HADAMARD'S INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE (s, m)-PREINVEX IN THE SECOND SENSE

Badreddine Meftah <badrimeftah@yahoo.fr>

Department of Mathematics, University of 8 Mai 1945 Guelma, Algeria.

Abstaract — In this paper, we introduce a new class of convex functions which is called (s, m)-preinvex functions in the second sense then we establish some new Hermite-Hadamard's inequalities whose modulus of the first derivatives are in this novel class.

Keywords — *Hermite-Hadamard inequality, Hölder's inequality, power mean inequality.*

1 Introduction

One of the most well-known inequalities in mathematics for convex functions is so called Hermite-Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where f is a real continuous convex function on the finite interval $[a, b]$. If the function f is concave, then (1) holds in the reverse direction (see [18]).

The Hermite-Hadamard inequality play an important role in nonlinear analysis and optimization. The above double inequality has attracted many researchers, various generalizations, refinements, extensions and variants of (1) have appeared in the literature, we can mention the works [1, 4, 5, 6, 8, 9, 13, 14, 15, 16, 17] and the references cited therein.

[*]*Edited by* Uzair Awan *and* Naim Çağman *(Editor-in-Chief).*

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson in [7], introduced a new class of generalized convex functions, called invex functions. In [2], the authors gave the concept of preinvex functions which is special case of invexity. Pini [19], Noor [11, 12], Yang and Li [23] and Weir [22], have studied the basic properties of preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

In [5], Dragomir and Agarwal established the following Hermite-Hadamard's inequalities for differentiable convex functions:

Theorem 1.1. [Theorem 2.2] Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{8}. \quad (2)$$

Theorem 1.2. [Theorem 2.3] Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and let $p > 1$. If the new mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2} \right)^{\frac{p-1}{p}}. \end{aligned} \quad (3)$$

In [17], Pearce and Pečarić generalized Theorem 2.3 from [5] as follows:

Theorem 1.3. [Theorem 1] Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and let $q \geq 1$. If the mapping $|f'|^{q}$ is convex on $[a, b]$, then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)}{4} \left(\frac{|f'(a)|^q+|f'(b)|^q}{2} \right)^{\frac{1}{q}}. \quad (4)$$

In [8], Kirmaci et al. gave a variant of Theorem 1.1 from [13] for functions whose first derivatives in absolute values are s-convex in the second sense as follows:

Theorem 1.4. [Theorem 1] Let $f : I \to \mathbb{R}$, $I \subset [0, \infty)$ be a differentiable mapping on I° such that $f' \in L([a, b])$, where $a, b \in I$, $a < b$, If $|f'|^{q}$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1)$ and $q \geq 1$, then

$$\begin{aligned} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| &\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}} \left(\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)} \right)^{\frac{1}{q}} \\ &\quad \times \left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}. \end{aligned} \quad (5)$$

In [1], Barani et al. obtained similar results given in [5] for differentiable preinvex functions as follows:

Theorem 1.5. [Theorem 2.1] Let $A \subseteq R$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on A then, for every $a, b \in A$ with $\eta(a, b) \neq 0$ the following inequality holds

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{|\eta(b,a)|}{8} \left[|f'(a)| + |f'(b)| \right]. \tag{6}$$

Theorem 1.6. [Theorem 2.2] Let $A \subseteq R$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. Assume that $p \in \mathbb{R}$ with $p > 0$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on A then, for every $a, b \in A$ with $\eta(a, b) \neq 0$ the following inequality holds

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{|\eta(b,a)|}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right)^{\frac{p-1}{p}}. \tag{7}$$

In [9], Latif and Shoaib established the following Hermite-Hadamard's inequalities for functions whose first derivatives in absolute values are m-preinvex as follows:

Theorem 1.7. [Theorem 4] Let $K \subseteq [0, b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f : K \to \mathbb{R}$ is differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|$ is m-preinvex on K, then the following inequalities holds

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{\eta(b,a)}{8} \left[|f'(a)| + m \left| f'(\frac{b}{m}) \right| \right]. \tag{8}$$

Theorem 1.8. [Theorem 5] Let $K \subseteq [0, b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f : K \to \mathbb{R}$ is differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|^q$ is m-preinvex on K for $q > 1$, then we have the following inequalities holds

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^q + m\left|f'(\frac{b}{m})\right|^q}{2} \right)^{\frac{1}{q}}, \quad (9)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.9. [Theorem 6] Let $K \subseteq [0, b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is differentiable mapping on K such that $f' \in L([a, a + \eta(b,a)])$. If $|f'|^q$ is m-preinvex on K for $q \geq 1$, then we have the following inequalities holds

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{\eta(b,a)}{4} \left(\frac{|f'(a)|^q + m\left|f'(\frac{b}{m})\right|^q}{2} \right)^{\frac{1}{q}}. \quad (10)$$

Motivated by the above results, in this paper we first define a novel class of convex functions called (s,m)-preinvexity in the second sense then we establish some new Hermite-Hadamard's inequalities based on this new definition.

2 Preliminaries

In this section we recall some concepts of convexity that are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 2.1. [18] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.2. [3] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense, if the following inequality

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

holds, for all $x, y \in I$ and $t \in [0,1]$.

Definition 2.3. [20] A function $f : [0, b^*] \to \mathbb{R}$, is said to be m-convex function where $m \in [0, 1]$ and $b^* > 0$, if

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y),$$

holds for all $x, y \in [0, b^*]$ and $t \in [0, 1]$.

Definition 2.4. [6] A nonnegative function $f : [0, b^*] \to \mathbb{R}$, is said to be (s, m)-convex function in the second sense where $s, m \in (0, 1]$ and $b^* > 0$, if

$$f(tx + m(1 - t)y) \leq t^s f(x) + m(1 - t)^s f(y),$$

holds for all $x, y \in [0, b^*]$ and $t \in [0, 1]$.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions.

Definition 2.5. [22] A set K is said to be invex at x with respect to η, if

$$x + t\eta(y, x) \in K,$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 2.6. [22] A function f on the invex set K is said to be preinvex with respect to η, if

$$f(x + t\eta(y, x)) \leq (1 - t) f(x) + tf(y),$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.7. [13, 21] A nonnegative function f on the invex set $K \subseteq [0, \infty)$ is said to be s-preinvex in the second sense with respect to η, if

$$f(x + t\eta(y, x)) \leq (1 - t)^s f(x) + t^s f(y),$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.8. [9] Let K be an invex set with $K \subseteq [0, b^*]$, $b^* > 0$. A function $f : K \to \mathbb{R}$ is said to be m-preinvex function with respect to η, where $m \in (0, 1]$ and $b^* > 0$, if

$$f(x + t\eta(y, x)) \leq (1 - t) f(x) + mtf(\frac{y}{m}),$$

holds for all $x, y \in K$, $t \in [0, 1]$.

Lemma 2.9. [10] The Hypergeometric function is defined as follows:

$${}_2F_1(a, b; c; z) = \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} dt,$$

for $t \in [0, 1]$ and $\text{Re}c > \text{Re}b > 0$ and $|\arg(1 - z)| < \pi$.

Lemma 2.10. [1] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable function such that $f' \in L([a, a + \eta(b, a)])$, then the following equality holds

$$\begin{aligned} &\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx - \frac{f(a) + f(a + \eta(b,a))}{2} \\ = &\frac{\eta(b,a)}{2} \int_0^1 (1 - 2t) f'(a + t\eta(b,a))dt. \end{aligned}$$

3 Main Results

We will start with the following definition.

Definition 3.1. Let K be an invex set with $K \subseteq [0, b^*]$, $b^* > 0$. A nonnegative function $f : K \to \mathbb{R}$ is said to be (s, m)-preinvex function in the second sense with respect to η, where $(s, m) \in (0, 1]^2$, if

$$f(x + t\eta(y, x)) \le (1 - t)^s f(x) + mt^s f(\frac{y}{m}),$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Remark 3.2. Definition 3.1 recapture all definitions cited above with the exception of Definition 2.5 for well-chosen values of $\eta(.,.)$, s and m.

Now, we can state our results.

Theorem 3.3. Let $K \subseteq [0, b^*]$, $b^* > 0$ be an invex subset with respect to η, $a, b \in K^\circ$ (interior of K) with $\eta(b, a) > 0$. Let $f : K \to (0, \infty)$ be differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|$ is (s, m)-preinvex in the second sense on K for some fixed $s, m \in (0, 1]$, then the following inequality:

$$\begin{aligned} &\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ \le &\frac{\eta(b,a)}{2} \frac{s + \left(\frac{1}{2}\right)^s}{(s+1)(s+2)} \left[|f'(a)| + m \left| f'(\frac{b}{m}) \right| \right], \end{aligned} \quad (11)$$

holds for all $x \in [a, a + \eta(b, a)]$.

Proof. From Lemma 2.10 and properties of modulus, we have

$$\begin{aligned} &\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ \le &\frac{\eta(b,a)}{2} \int_0^1 |1 - 2t| \, |f'(a + t\eta(b,a))| \, dt, \end{aligned} \quad (12)$$

since $|f'|$ is (s,m)-preinvex function in the second sense, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left[(1-t)^s|f'(a)|+mt^s\left|f'(\frac{b}{m})\right|\right]dt$$

$$= \frac{\eta(b,a)}{2}\left(|f'(a)|\int_0^1 |1-2t|(1-t)^s dt\right.$$

$$\left.+m\left|f'(\frac{b}{m})\right|\int_0^1 |1-2t|t^s dt\right)$$

$$= \frac{\eta(b,a)}{2}\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\left(|f'(a)|+m\left|f'(\frac{b}{m})\right|\right), \quad (13)$$

where we use the facts that

$$\int_0^1 |1-2t|(1-t)^s dt=\int_0^1 |1-2t|t^s dt=\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}. \quad (14)$$

The proof is completed. □

Remark 3.4. In Theorem 3.3, if we choose $s=1, (s,m)=(1,1)$ or $(\eta(b,a),s,m)=(b-a,1,1)$, then (11) reduce to (8), (6) and (2) respectively.

Theorem 3.5. Let $K\subseteq[0,b^*], b^*>0$ be an invex subset with respect to η, $a,b\in K^\circ$ (interior of K) with $\eta(b,a)>0$. Let $f:K\to(0,\infty)$ be differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$ and let $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|f'|^q$ is (s,m)-preinvex in the second sense on K for some fixed $s,m\in(0,1]$, then the following inequality:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left(\left(|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right)\right)^{\frac{1}{q}}, \quad (15)$$

holds for all $x\in[a,a+\eta(b,a)]$.

Proof. From Lemma 2.10, properties of modulus, Hölder's inequality and (s,m)-preinvexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\
\leq\ & \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2(\ p+1)^{\frac{1}{p}}}\left(\int_0^1 |f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{\eta(b,a)}{2(\ p+1)^{\frac{1}{p}}}\left(|f'(a)|^q\int_0^1 (1-t)^s\,dt+m\left|f'(\frac{b}{m})\right|^q\int_0^1 t^s dt\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2(s+1)^{\frac{1}{q}}(\ p+1)^{\frac{1}{p}}}\left(|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where we have use the facts that

$$\int_0^1 (1-t)^s\,dt=\int_0^1 t^s dt=\frac{1}{s+1},$$

and

$$\int_0^1 |1-2t|^p\,dt=\frac{1}{p+1}.$$

The proof is achieved. □

Remark 3.6. In Theorem 3.5, if we choose $s=1,(s,m)=(1,1)$ or $(\eta(b,a),s,m)=(b-a,1,1)$, then (15) reduce to (9), (7) and (3) respectively.

Theorem 3.7. Let $K\subseteq[0,b^*]$, $b^*>0$ be an invex subset with respect to η, $a,b\in K^\circ$ (interior of K) with $\eta(b,a)>0$. Let $f:K\to(0,\infty)$ be differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$ and let $q\geq 1$. If $|f'|^q$ is (s,m)-preinvex in the second sense on K for some fixed $s,m\in(0,1]$, then the following inequality:

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\
\leq\ & \frac{\eta(b,a)}{2^{\,2-\frac{1}{q}}}\left(\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left[|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right]^{\frac{1}{q}}, \qquad (16)
\end{aligned}$$

holds for all $x\in[a,a+\eta(b,a)]$.

Proof. From Lemma 2.10, properties of modulus and power mean inequality, we have

$$\begin{aligned}
&\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(x)dx \right| \\
\leq\ & \frac{\eta(b,a)}{2} \left(\int_0^1 |1-2t|\, dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2t| \left| f'(a+t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2^{\,2-\frac{1}{q}}} \left(\int_0^1 |1-2t| \left| f'(a+t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}}, \qquad (17)
\end{aligned}$$

since $|f'|^q$ is (s,m)-preinvex function in the second sense, we have

$$\begin{aligned}
&\int_0^1 |1-2t| \left| f'(a+t\eta(b,a)) \right|^q dt \\
\leq\ & |f'(a)|^q \int_0^1 |1-2t|\,(1-t)^s + m \left| f'(\frac{b}{m}) \right|^q \int_0^1 t^s\, |1-2t|\, dt \\
=\ & \frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)} \left[|f'(a)|^q + m \left| f'(\frac{b}{m}) \right|^q \right], \qquad (18)
\end{aligned}$$

where we have use (14). Substituting (18) into (17), we get the desired inequality in (16). □

Remark 3.8. In Theorem 3.7, if we choose $s = 1$, $(\eta(b,a), s, m) = (b-a, s, 1)$ and $(\eta(b,a), s, m) = (b-a, 1, 1)$, then (16) reduce to (10), (5) and (4) respectively.

Theorem 3.9. Suppose that all the assumptions of Theorem 3.7 are satisfied. Then the following inequality:

$$\begin{aligned}
&\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(x)dx \right| \leq \frac{\eta(b,a)}{2} \\
&\times \left(\frac{1}{2^{s+1}} B(s+1,q+1) + \frac{{}_2F_1\left(1,-s;q+2;\frac{1}{2}\right)}{q+2} \right)^{\frac{1}{q}} \left(|f'(a)|^q + m \left| f'(\frac{b}{m}) \right|^q \right)^{\frac{1}{q}}, \qquad (19)
\end{aligned}$$

holds for all $x \in [a, a+\eta(b,a)]$ where ${}_2F_1(.,.;.;.)$ is the Hypergeometric function and $B(.,.)$ is the beta function.

Proof. From Lemma 2.10, properties of modulus and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)} f(x)dx\right| \\
\leq\ & \frac{\eta(b,a)}{2}\left(\int\limits_0^1 dt\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 |1-2t|^q\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2}\left(\int\limits_0^1 |1-2t|^q\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}, \qquad (20)
\end{aligned}$$

using (s,m)-preinvexity of $|f'|^q$, (20) gives

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)} f(x)dx\right| \\
\leq\ & \frac{\eta(b,a)}{2}\left(\int\limits_0^1 |1-2t|^q\left((1-t)^s\,|f'(a)|^q+mt^s\left|f'(\frac{b}{m})\right|^q\right)dt\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2}\left(|f'(a)|^q\left(\int\limits_0^{\frac{1}{2}}(1-t)^s(1-2t)^q\,dt+\int\limits_{\frac{1}{2}}^1(1-t)^s(2t-1)^q\,dt\right)\right. \\
&\left.+m\left|f'(\frac{b}{m})\right|^q\left(\int\limits_0^{\frac{1}{2}}t^s(1-2t)^q\,dt+\int\limits_{\frac{1}{2}}^1 t^s(2t-1)^q\,dt\right)\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2}\left(|f'(a)|^q\left(\frac{1}{2}\int\limits_0^1(1-t)^q\left(1-\frac{1}{2}t\right)^s dt+\frac{1}{2^{s+1}}\int\limits_0^1 t^s(1-t)^q\,dt\right)\right. \\
&\left.+m\left|f'(\frac{b}{m})\right|^q\left(\frac{1}{2^{s+1}}\int\limits_0^1 t^s(1-t)^q\,dt+\frac{1}{2}\int\limits_0^1(1-t)^q\left(1-\frac{1}{2}t\right)^s dt\right)\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(b,a)}{2}\left(\frac{1}{2^{s+1}}B(s+1,q+1)+\frac{{}_2F_1\left(1,-s;q+2;\frac{1}{2}\right)}{2q+4}\right)^{\frac{1}{q}} \\
&\times\left(|f'(a)|^q+m\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Which is the desired result. □

References

[1] A. Barani, A. G. Ghazanfari and S. S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl. 2012, 2012:247, 9 pp.

[2] A. Ben-Israel and B. Mond, *What is invexity?,* J. Austral. Math. Soc. Ser. B 28 (1986), no. 1, 1–9.

[3] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, (German) Publ. Inst. Math. (Beograd) (N.S.) 23(37) (1978), 13–20.

[4] S. S. Dragomir and N. M. Ionescu, *On some inequalities for convex-dominated functions*, Anal. Numer. Theor. Approx. 19 (1990), no. 1, 21–27.

[5] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. 11 (1998), no. 5, 91–95.

[6] N. Eftekhari, *Some remarks on* (s,m)*-convexity in the second sense*, J. Math. Inequal. 8 (2014), no. 3, 489–495.

[7] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. 80 (1981), no. 2, 545–550.

[8] U. S. Kirmaci, M. K. Bakula, M. E. Özdemir and J. E. Pečarić, *Hadamard-type inequalities for* s*-convex functions,* Appl. Math. Comput. 193 (2007), no. 1, 26–35.

[9] M. A. Latif and M. Shoaib, *Hermite-Hadamard type integral inequalities for differentiable* m*-preinvex and* (α, m)*-preinvex functions,* J. Egyptian Math. Soc. 23 (2015), no. 2, 236–241.

[10] Y. L. Luke and Y. L. Luke, *The special functions and their approximations*, Vol. 1. New York: Academic press, 1969.

[11] M. A. Noor, *Variational-like inequalities*, Optimization 30 (1994), no. 4, 323–330.

[12] M. A. Noor, *Invex equilibrium problems*, J. Math. Anal. Appl. 302 (2005), no. 2, 463–475.

[13] M. A. Noor, K. I. Noor, M. U. Awan, J. Li, *On Hermite-Hadamard type inequalities for* h*-preinvex functions*, Filomat. 28(7), 1463-1474, (2014).

[14] M. A. Noor, K. I. Noor, M. U. Awan, S. Khan, *Hermite-Hadamard inequalities for* s*-Godunova-Levin preinvex functions*, Journal of Advanced Mathematical Studies. 7(2), 12-19, (2014).

[15] M. A. Noor, K. I. Noor, M. U. Awan, S. Khan, *Hermite-Hadamard inequalities for differentiable h_φ-preinvex functions*, Arabian Journal of Mathematics. 4, 63-76, (2015).

[16] M. A. Noor, K. I. Noor, M. U. Awan, F. Qi, *Hermite-Hadamard inequalities for logarithmic h-preinvex functions*, Cogent Mathematics. 2, (2015).

[17] C.E.M. Pearce and J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulæ*, Appl. Math. Lett. 13 (2000), no. 2, 51–55.

[18] J. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

[19] R. Pini, *Invexity and generalized convexity*, Optimization 22 (1991), no. 4, 513–525.

[20] G. H. Toader, *Some generalizations of the convexity*, Proceedings of the colloquium on approximation and optimization (Cluj-Napoca, 1985), 329–338.

[21] Y. Wang, S-H. Wang and F. Qi, *Simpson type integral inequalities in which the power of the absolute value of the first derivative of the integrand is s-preinvex*, Facta Univ. Ser. Math. Inform. 28 (2013), no. 2, 151–159.

[22] T. Weir and B. Mond, *Pre-invex functions in multiple objective optimization*, J. Math. Anal. Appl. 136 (1988), no. 1, 29–38.

[23] X. -M. Yang and D. Li, *On properties of preinvex functions*, J. Math. Anal. Appl. 256 (2001), no. 1, 229–241.