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ON NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE FOURTH DERIVATIVE ABSOLUTE VALUES ARE QUASI-CONVEX WITH APPLICATIONS

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Abstaract — We establish some new inequalities of Hermite-Hadamard type for functions whose fourth derivatives absolute values are quasi-convex. Further, we give new identity.Using this new identity, we establish similar inequalities for left-hand side of Hermite-Hadamard result.Also, we present applications to special means.

Keywords — *Hermite-Hadamard type inequalities, Quasi-convex function, Power mean inequality.*

1 Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called convex function if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in I$ and $\lambda \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on graph of f with Q between P and R, then Q is on or below chord PR. There are many results associated with convex functions in the area of inequalities, but one of them is the classical Hermite-Hadamard inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

for all $a, b \in I$, with $a < b$.

Recently, numerous authors [1-7] developed and discussed Hermite-Hadamard's inequalities in terms of refinements, counter-parts, generalizations and new Hemitte-Hadamard's type inequalities.

The notion of quasi-convex function which is generalization of convex function is defined as:

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Definition 1.1. A function $F : [a, b] \to \mathbb{R}$ is called quasi-convex on $[a, b]$, if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \ \ \forall x, y \in [a, b].$$

Any convex function is quasi-convex but converse is not true in general(See for example [3]). D.A Ion [6] established inequalities of right hand side of Hermite-Hadamard's type inequality for functions whose derivatives in absolute values are quasi-convex functions. These inequalities appear in the following theorems:

Theorem 1.2. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on I° with $a, b \in I^\circ$ and $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then we have:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\max\{|f'(a)|, |f'(b)|\}.$$

Theorem 1.3. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on I° with $a, b \in I^\circ$ and $a < b$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$, then we have:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}(\max\{|f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}}\})^{\frac{p-1}{p}}.$$

Theorem 1.4. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a function, twice differentiable on I° with $a, b \in I^\circ$ and $a < b$. If $|f''|$ is quasi-convex on $[a, b]$, then we have:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\max\{|f''(a)|, |f''(b)|\}.$$

In paper [8], S.Qaisar, S.Hussain, C. He established new refined inequalities of right hand side of Hermite-Hadamard result for the class of functions whose third derivatives at certain powers are quasi-convex functions as follow:

Theorem 1.5. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be thrice differentiable mapping on I° such that $f''' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'''|$ is quasi-convex on $[a, b]$, then we have the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right| \leq \frac{(b-a)^3}{192}\max\{|f'''(a)|, |f'''(b)|\}.$$

Theorem 1.6. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be three time differentiable mapping on I° such that $f''' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'''|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$, and $p > 1$, then we have the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(\max\{|f'''(a)|^q, |f'''(b)|^q\})^{\frac{1}{q}}.$$

Theorem 1.7. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be thrice differentiable mapping on I° such that $f''' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'''|^q$ is quasi-convex on $[a, b]$, and $q \geq 1$, then we have following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{192}(\max\{|f'''(a)|^q, |f'''(b)|^q\})^{\frac{1}{q}}.$$

In this paper, we establish new refined inequalities of the right hand side of Hermite-Hadamard result for the class of functions whose fourth derivative at certain powers are quasi-convex functions. Further, we establish new identity using which, we establish new refined inequalities of left hand side of Hermit-Hadamard result for the same class of functions considered earlier.

2 Main Results

For establishing new inequalities of right hand side of Hermite-Hadamard result for the functions whose fourth derivative at certain powers are quasi-convex, we need the following identity:

Lemma 2.1. Let Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be four times differentiable mapping on I° such that $f^{(iv)} \in L[a,b]$, where $a, b \in I$ with $a < b$, then

$$\frac{1}{b-a}\int_a^b f(x)dx + \frac{b-a}{12}[f'(b) - f'(a)] - \frac{f(a)+f(b)}{2}$$
$$= \frac{(b-a)^4}{24}\int_0^1 (\lambda(1-\lambda))^2 f^{(iv)}(a\lambda + (1-\lambda)b)d\lambda$$

Theorem 2.2. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a four times differentiable mapping on I° such that $f^{(iv)} \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f^{iv}|$ is quasi-convex on $[a,b]$, then we have the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{720}\max\{|f^{(iv)}(a)|, |f^{(iv)}(b)|\}. \tag{1}$$

Proof. Using Lemma 2.1 and quasi-convexity of $|f^{(iv)}|$, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{24}\int_0^1 (\lambda(1-\lambda))^2 \big|f^{(iv)}(a\lambda + (1-\lambda)b)\big|d\lambda$$
$$\leq \frac{(b-a)^4}{24}\max\{|f^{(iv)}(a)|, |f^{(iv)}(a)|\}\int_0^1 (\lambda(1-\lambda))^2 d\lambda$$
$$= \frac{(b-a)^4}{720}\max\{|f^{(iv)}(a)|, |f^{(iv)}(a)|\}$$

the proof is completed. □

Theorem 2.3. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a four times differentiable mapping on I° such that $f^{(iv)} \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f^{(iv)}|^{\frac{p}{p-1}}$ is quasi-convex on $[a,b]$, and $p > 1$, then we have the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right|$$

$$\leq \frac{(b-a)^4}{24}\beta^{\frac{1}{p}}(2p+1,2p+1)(\max\{|f^{(iv)}(a)|^q,|f^{(iv)}(b)|^q\})^{\frac{1}{q}}, \tag{2}$$

where $q=\frac{p}{p-1}$.

Proof. Using Lemma 2.1, Holder's inequality and quasi-convexity of $|f^{(iv)}|^{\frac{p}{p-1}}$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\frac{b-a}{12}[f'(b)-f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{24}\int_0^1(\lambda(1-\lambda))^2\big|f^{(iv)}(a\lambda+(1-\lambda)b)\big|d\lambda$$
$$\leq \frac{(b-a)^4}{24}\left(\int_0^1(\lambda(1-\lambda))^{2p}d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\big|f^{(iv)}(a\lambda+(1-\lambda)b)|^q\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^4}{24}\left(\int_0^1(\lambda(1-\lambda))^{2p}d\lambda\right)^{\frac{1}{p}}(\max\{|f^{(iv)}(a)|^q,|f^{(iv)}(b)|^q\})^{\frac{1}{q}}$$

It is easy to note that

$$\beta(2p+1,2p+1)=\int_0^1(\lambda(1-\lambda))^{2p}d\lambda$$

which completes the proof. □

Theorem 2.4. Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a four times differentiable mapping on I° such that $f^{(iv)}\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f^{iv}|^q$ is quasi-convex on $[a,b]$, and $q\geq 1$, then we have following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\frac{b-a}{12}[f'(b)-f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{720}(\max\{|f^{(iv)}(a)|^q,|f^{((iv))}(b)|^q\})^{\frac{1}{q}}. \tag{3}$$

Proof. Using Lemma 2.1, power mean inequality and quasi-convexity of $|f^{(iv)}|^q$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\frac{b-a}{12}[f'(b)-f'(a)]\right|$$
$$\leq \frac{(b-a)^4}{24}\int_0^1(\lambda(1-\lambda))^2\big|f^{(iv)}(a\lambda+(1-\lambda)b)\big|d\lambda$$
$$\leq \frac{(b-a)^4}{24}\left(\int_0^1(\lambda(1-\lambda))^2d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1(\lambda(1-\lambda))^2\big|f^{(iv)}(a\lambda+(1-\lambda)b)|^q\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^4}{24}\left(\frac{1}{30}\right)^{1-\frac{1}{q}}\left(\frac{1}{30}\max\{|f^{(iv)}(a)|^q,|f^{(iv)}(b)|^q\}\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)^4}{720}(\max\{|f^{(iv)}(a)|^q,|f^{(iv)}(b)|^q\})^{\frac{1}{q}}.$$

which completes the proof. □

Now, to develop new refined inequalities of left hand side of Hermite-Hadamard result for the class of functions whose third derivatives at certain powers are quasi-convex, we need the following identity:

Lemma 2.5. Let Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be three times differentiable mapping on I° such that $f^{(''')} \in L[a, b]$, where $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx + \frac{b-a}{24}[f'(b) - f'(a)]$$

$$= \frac{(b-a)^3}{24}\left[\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)f'''(\lambda a + (1-\lambda)b)d\lambda\right.$$

$$\left. - \int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)f'''(\lambda b + (1-\lambda)a)d\lambda\right]$$

Proof. Integrating by parts, we have

$$\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)f'''(\lambda a + (1-\lambda)b)d\lambda$$

$$= \frac{1}{b-a}\int_0^{\frac{1}{2}} (1-12\lambda^2)f''(\lambda a + (1-\lambda)b)d\lambda$$

$$= \frac{2}{(b-a)^2}f'\left(\frac{a+b}{2}\right) + \frac{f'(b)}{(b-a)^2} - \frac{24}{(b-a)^2}\int_0^{\frac{1}{2}} \lambda f'(\lambda a + (1-\lambda)b)d\lambda$$

$$= \frac{2}{(b-a)^2}f'\left(\frac{a+b}{2}\right) + \frac{f'(b)}{(b-a)^2} + \frac{12}{(b-a)^3}f\left(\frac{a+b}{2}\right) + \frac{24}{(a-b)^4}\int_b^{\frac{a+b}{2}} f(x)dx$$

and

$$\int_0^{\frac{1}{2}} \lambda(1-2\lambda)(1+2\lambda)f'''(\lambda b + (1-\lambda)a)d\lambda$$

$$= \frac{-1}{b-a}\int_0^{\frac{1}{2}} (1-12\lambda^2)f''(\lambda b + (1-\lambda)a)d\lambda$$

$$= \frac{2}{(b-a)^2}f'\left(\frac{a+b}{2}\right) + \frac{f'(a)}{(b-a)^2} - \frac{24}{(b-a)^2}\int_0^{\frac{1}{2}} \lambda f'(\lambda b + (1-\lambda)a)d\lambda$$

$$= \frac{24}{(b-a)^2}f'\left(\frac{a+b}{2}\right) + \frac{f'(a)}{(b-a)^2} - \frac{12}{(b-a)^3}f\left(\frac{a+b}{2}\right) + \frac{2}{(b-a)^4}\int_a^{\frac{a+b}{2}} f(x)dx$$

this ends the proof. □

Theorem 2.6. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three time differentiable mapping on I° such that $f''' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'''|$ is quasi-convex on $[a, b]$, then we have following inequality:

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx + \frac{b-a}{24}[f'(b) - f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{192}\left(\max\{|f'''(a)|, |f'''(b)|\}\right). \quad (4)$$

Proof. Using Lemma 2.5 and quasi-convexity of $|f'''|$, we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{24}\left[\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda a+(1-\lambda)b)|d\lambda\right.$$

$$\left.+\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda b+(1-\lambda)a)|d\lambda\right]$$

$$\leq \frac{(b-a)^3}{24}\Big(\max\{|f'''(a)|,|f'''(b)|\}\Big)$$

$$\times\left[\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda+\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\right]$$

$$=\frac{(b-a)^3}{192}\Big(\max\{|f'''(a)|,|f'''(b)|\}\Big)$$

this complete the proof. □

Theorem 2.7. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three time differentiable mapping on I° such that $f''' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'''|^{\frac{p}{p-1}}$ is quasi-convex on $[a,b]$, and $p > 1$, then we have following inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}, \tag{5}$$

where $q=\frac{p}{p-1}$.

Proof. Using Lemma 2.5, Holder's inequality and quasi-convexity of $|f'''|^{\frac{p}{p-1}}$, we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{24}\left[\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda a+(1-\lambda)b)|d\lambda\right.$$

$$\left.+\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda b+(1-\lambda)a)|d\lambda\right]$$

$$\leq \frac{(b-a)^3}{24}\left[\left(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)^p(1+2\lambda)^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\lambda|f'''(\lambda a+(1-\lambda)b)|^q d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)^p(1+2\lambda)^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\lambda|f'''(\lambda b+(1-\lambda)a)|^q d\lambda\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)^3}{24}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}\Bigg[\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)^p(1+2\lambda)^p d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}}\lambda d\lambda\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)^p(1+2\lambda)^p d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}}\lambda d\lambda\Big)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(b-a)^3}{96}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}$$

□

Theorem 2.8. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a three time differentiable mapping on I° such that $f''' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'''|^q$ is quasi-convex on $[a,b]$, and $q \geq 1$, then we have following inequality:

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{192}\big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\big)^{\frac{1}{q}}. \tag{6}$$

Proof. Using Lemma 2.5, power mean inequality and quasi-convexity of $|f'''|^q$, we get

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{b-a}{24}[f'(b)-f'(a)]\right|$$

$$\leq \frac{(b-a)^3}{24}\Bigg[\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda a+(1-\lambda)b)|d\lambda$$

$$+\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda b+(1-\lambda)a)|d\lambda\Bigg]$$

$$\leq \frac{(b-a)^3}{24}\Bigg[\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda a+(1-\lambda)b)|^q d\lambda|\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)|f'''(\lambda b+(1-\lambda)a)|^q d\lambda\Big)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(b-a)^3}{24}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}$$

$$\times\Bigg[\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda|\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}}\lambda(1-2\lambda)(1+2\lambda)d\lambda\Big)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(b-a)^3}{192}\Big(\max\{|f'''(a)|^q,|f'''(b)|^q\}\Big)^{\frac{1}{q}}$$

the proof is so completed. □

3 Application to Some Special Means

We now consider the application of our theorem to the special means.
For positive numbers $a > 0$ and $b > 0$, define $A(a,b) = \frac{a+b}{2}$ and

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right], & p \neq -1, 0 \\ \frac{b-a}{\ln b - \ln a}, & p = -1 \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0 \end{cases}$$

We know that A and L_p respectively are called the arithmetic and generalized logarithmic means of two positive numbers a and b. By applying Hermite-Hadamard type inequalities established in Section 2, we are in a position to construct some inequalities for special means A and L_P. Consider the following function:

$$f(x) = \frac{x^{\alpha+4}}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \tag{7}$$

for $0 < \alpha \leq 1$ and $x > 0$. Since $f^{(iv)}(x) = x^\alpha$ and $(\lambda x + (1-\lambda)y)^\alpha \leq \lambda^\alpha x^\alpha + (1-\lambda)^\alpha y^\alpha$ for all $x, y > 0$ and $\lambda \in [0,1]$, then $f^{(iv)}(x) = x^\alpha$ is α-convex function on $\mathbb{R}^+$ and

$$\frac{f(a)+f(b)}{2} = \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} A(a^{\alpha+4}, b^{\alpha+4}),$$

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} L_{\alpha+4}(a,b),$$

$$f'(b) - f'(a) = \frac{b-a}{(\alpha+1)(\alpha+2)} L_{\alpha+2}(a,b)$$

Theorem 3.1. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left| 12A(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b) \right|$$

$$\leq \frac{(b-a)^4}{60}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\max\{|a^\alpha|, |b^\alpha|\}$$

Proof. The assertion follows from inequality (1) applied to mapping (7). □

Theorem 3.2. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^4}{2}\beta^{\frac{1}{p}}(2p+1, 2p+1)(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\max\{|a^{\alpha}|^q, |b^{\alpha}|^q\})^{\frac{1}{q}}.$$

Proof. The assertion follows from inequality (2) applied to the mapping (7). □

Theorem 3.3. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^4}{60}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\max\{|a^{\alpha}|^q, |b^{\alpha}|^q\})^{\frac{1}{q}}.$$

Proof. The assertion follows from inequality (3) applied to the mapping (7). □

Theorem 3.4. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A^{\alpha+4}(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^3}{16}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\max\{|a^{\alpha}|, |b^{\alpha}|\}).$$

Proof. The assertion follows from inequality (4) applied to the mapping (7). □

Theorem 3.5. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A^{\alpha+4}(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^3}{8}(\frac{1}{p+1})^{\frac{1}{p}}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\big(\max\{|a^{\alpha}|^q, |b^{\alpha}|^q\}\big)^{\frac{1}{q}}$$

Proof. The assertion follows from inequality (5) applied to the mapping (7). □

Theorem 3.6. For positive numbers a and b such that $b > a$ and $0 < \alpha \leq 1$, we have

$$\left|12A^{\alpha+4}(a^{\alpha+4}, b^{\alpha+4}) - 12L_{\alpha+4}(a,b) - (b-a)^2(\alpha+3)(\alpha+4)(\alpha+4)L_{\alpha+2}(a,b)\right|$$

$$\leq \frac{(b-a)^3}{16}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\big(\max\{|a^{\alpha}|^q, |b^{\alpha}|^q\}\big)^{\frac{1}{q}}.$$

Proof. The assertion follows from inequality (6) applied to the mapping (7). □

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