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Research Article

Some properties of Appell type degenerate Bell polynomials

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Abstract

In recent years, the degenerate versions of some polynomial families such as Bernoulli, Euler, Apostol and Bell polynomials have been intensively studied in the literature. Many new forms of Bell polynomials such as degenerate, partially degenerate and fully degenerate have attracted attention. The specific aim of this paper is to introduce a new family of general degenerate Bell type polynomials with the help of degenerate Appell polynomials and explore their properties, including explicit form, determinant representation, recurrence relation, lowering and raising operators and difference equation. Then, after discussing the special cases of Appell type degenerate Bell polynomial families, new polynomial families including Bernoulli and Euler polynomials are given. Furthermore, corresponding results are obtained for these new families. Lastly, new relations and summation formulas are obtained including Stirling numbers and Appell type degenerate Bell polynomials. Finally, we establish theorems that provide various families of multilinear and multilateral generating functions for the Appell type degenerate Bell polynomials.

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1. Introduction

In recent years, many studies show that the theory of special functions is one of the important branches of mathematics. For this reason, it has been studied extensively in the literature. Additionally, new hybrid type polynomial families are being investigated. Hybrid type polynomials also play an important role in many branches of mathematics. Appell polynomials and Bell polynomials have also attracted recent attention. They have a wide variety of applications including approximation theory, special functions and numerical analysis etc. (see [2, 12, 19, 21, 24, 34, 35, 37, 38, 41, 43, 45, 48] for details). Due to these applications, many generalizations have been examined in recent years. Some

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important polynomial families and their properties can be examined in the literature [3, 4, 8, 9, 13, 33, 39, 46, 47].

Costabile and Longo [12], Cheikh and Zaghouani [10] defined Δ_ω-Appell polynomials by the following generating function

$$\sum_{l=0}^{\infty} \mathcal{A}_l(u;\omega)\frac{t^l}{l!} = A(t)(1+\omega t)^{\frac{u}{\omega}} \tag{1.1}$$

where $A(t)$ is the power series of t given as

$$A(t) = A_{0,\omega} + \frac{t}{1!}A_{1,\omega} + \frac{t^2}{2!}A_{2,\omega} + ... + \frac{t^l}{l!}A_{l,\omega} + ...,\ A_{0,\omega} \neq 0.$$

In [10], Cheikh and Zaghouani gave the following equation for Δ_ω-Appell sequences

$$\Delta_\omega(\mathcal{A}_l(u;\omega)) = l\mathcal{A}_{l-1}(u;\omega),$$

where the operator Δ_ω [10] is given by

$$\Delta_\omega(\tau(u)) = \frac{\tau(u+\omega)-\tau(u)}{\omega},\ \ \omega \neq 0. \tag{1.2}$$

In the case $\omega \to 0$, it gives the usual Appell polynomials

$$A(t)e^{ut} = \sum_{l=0}^{\infty}\mathcal{A}_l(u)\frac{t^l}{l!}, \tag{1.3}$$

where $A(t)$ is the determining function with the formal power series

$$A(t) = \sum_{s=0}^{\infty} A_s\frac{t^s}{s!},\quad A_0 \neq 0 \tag{1.4}$$

[2, 42].

The generating function of the degenerate form of Bell polynomials $\mathcal{B}_l(u;\omega)$, which is the focus of significant disciplines of study including number theory and special functions, is defined through [31]

$$\left(1+\omega\left(e^t-1\right)\right)^{\frac{u}{\omega}} = \sum_{l=0}^{\infty}\mathcal{B}_l(u;\omega)\frac{t^l}{l!}, \tag{1.5}$$

where $\omega \to 0$, gives the Bell polynomials $\mathcal{B}_l(u)$ in the special case, defined via the generating function [7]

$$e^{u(e^t-1)} = \sum_{l=0}^{\infty}\mathcal{B}_l(u)\frac{t^l}{l!}. \tag{1.6}$$

For $u = 1$, $\mathcal{B}_l(1)$ are referred to as the Bell numbers. The relation between Stirling numbers of the second kind and degenerate Bell polynomials can be described as [31]

$$\mathcal{B}_l(u;\omega) = \sum_{s=0}^{l}(u)_s^\omega S_2(l,s)$$

and

$$(u)_0^\omega := 1,\quad (u)_l^\omega := u(u-\omega)...(u-(l-1)\omega),\ (l=1,2,...),$$

where $S_2(l,r)$ represent the Stirling numbers of the second kind, which are determined by the given relations [11, 43]

$$z^l = \sum_{r=0}^{l} S_2(l,r)\,z(z-1)\cdots(z-r+1), \tag{1.7}$$

with

$$\left(e^{z}-1\right)^{r}=r!\sum_{l=r}^{\infty}S_{2}\left(l,r\right)\frac{z^{l}}{l!}. \tag{1.8}$$

In recent years, the families for instance Bell polynomials, Bell type Bernoulli, Euler and Genocchi polynomials, Bell based Apostol-Bernoulli, Bell based Apostol type polynomials and families associated with Bell polynomials have been studied [16,17,22,23,28]. Recently, many forms of degenerate Bell polynomials have been defined. Partially degenerate Bell numbers and polynomials are one of them, and were introduced as a degenerate version of Bell numbers and polynomials. Later, fully degenerate Bell numbers and polynomials were defined as further degenerate of these polynomials. Recent studies have also shown the relationships of these polynomials to well-known polynomial families such as Bernoulli and Euler polynomials and numbers [14, 15, 27, 29, 30, 32].

We introduce the degenerate Bell based Appell polynomials (Appell type degenerate Bell polynomials) ${}_{\mathcal{B}}\mathcal{A}_{l}\left(u,v;\omega\right):={}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}$ as

$$A\left(t\right)\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^{t}-1\right)\right)^{\frac{v}{\omega}}=\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}\frac{t^{l}}{l!}, \tag{1.9}$$

where

$$A\left(t\right)=\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^{l}}{l!},\quad \alpha_{0,\omega}\neq 0. \tag{1.10}$$

By making certain choices for $A\left(t\right)$, it can be observed that there are some usual polynomials.

(1) The limit $\omega\rightarrow 0$, gives the Bell based Appell polynomials [40]

$$A\left(t\right)e^{ut}e^{v\left(e^{t}-1\right)}=\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_{l}\left(u,v\right)\frac{t^{l}}{l!},$$

also, when $A\left(t\right)=1$ and $u=0$, we have the Bell polynomials [7, 20] and taking $A\left(t\right)=1$, we have the two variable Bell polynomials [17].

(2) When $A\left(t\right)=1$, we get the two variable degenerate Bell polynomials $\mathcal{B}_{l}\left(u,v;\omega\right):=\mathcal{B}_{l}^{\omega}$

$$\left(1+\omega\left(e^{t}-1\right)\right)^{\frac{v}{\omega}}\left(1+\omega t\right)^{\frac{u}{\omega}}=\sum_{l=0}^{\infty}\mathcal{B}_{l}\left(u,v;\omega\right)\frac{t^{l}}{l!}. \tag{1.11}$$

The limit in case $\omega\rightarrow 0$ of (1.11), gives two variable Bell polynomials $\mathcal{B}_{l}\left(u,v\right)$ were defined the following generating function [17]

$$e^{ut}e^{v\left(e^{t}-1\right)}=\sum_{l=0}^{\infty}\mathcal{B}_{l}\left(u,v\right)\frac{t^{l}}{l!}.$$

(3) Setting $A\left(t\right)=\frac{t}{\left(1+\omega t\right)^{\frac{1}{\omega}}-1}$, we obtain the degenerate Bell-Bernoulli polynomials ${}_{\mathcal{B}}\mathcal{B}_{l}\left(v,v;\omega\right):={}_{\mathcal{B}}\mathcal{B}_{l}^{\omega}$

$$\frac{t}{\left(1+\omega t\right)^{\frac{1}{\omega}}-1}\left(1+\omega\left(e^{t}-1\right)\right)^{\frac{v}{\omega}}\left(1+\omega t\right)^{\frac{u}{\omega}}=\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{B}_{l}\left(v,v;\omega\right)\frac{t^{l}}{l!}. \tag{1.12}$$

Taking limit $\omega\rightarrow 0$ in (1.12), it gives the Bell based Bernoulli polynomials polynomials [17].

(4) The case $A\left(t\right)=\frac{2}{\left(1+\omega t\right)^{\frac{1}{\omega}}+1}$, gives the degenerate Bell-Euler polynomials ${}_{\mathcal{B}}\mathcal{E}_{l}\left(v,v;\omega\right):={}_{\mathcal{B}}\mathcal{E}_{l}^{\omega}$

$$\frac{2}{\left(1+\omega t\right)^{\frac{1}{\omega}}+1}\left(1+\omega\left(e^{t}-1\right)\right)^{\frac{v}{\omega}}\left(1+\omega t\right)^{\frac{u}{\omega}}=\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{E}_{l}\left(u,v;\omega\right)\frac{t^{l}}{l!}. \tag{1.13}$$

The limit in case $\omega \to 0$ of (1.13), gives the Bell based Euler polynomials polynomials [22].

The following describes the structure of this paper: In section 2 provides a detailed analysis of the Appell type degenerate Bell polynomials. It includes the explicit and determinant representations, recurrence relation, shift operators, difference equation, and addition formulas. As specific instances of the primary findings derived in section 2, new specifications of degenerate Bell-Bernoulli and Euler polynomials are presented in section 3. In section 4, we examine the relationship between Stirling numbers and Appell type degenerate Bell polynomials and obtain summation formulas by giving new relations. Finally, in this study, we give multilinear and multilateral generating function families for Appell type degenerate Bell polynomials in section 5.

2. Appell type degenerate Bell polynomials and their properties

In this part, we examine the explicit and determinant representation, lowering and raising operators, recurrence relation, difference equation and some summation formulas for the Appell type degenerate Bell polynomials.

Theorem 2.1. *The explicit representation satisfied by the following degenerate Bell based Appell polynomials*

$$ {}_{\mathcal{B}}\mathcal{A}_l^{\omega} = \sum_{s=0}^{l}\sum_{r=0}^{s}\binom{l}{s}\binom{s}{r}\alpha_{l-s,\omega}\,(u)_{s-r}^{\omega}\,\mathcal{B}_r(v;\omega) \tag{2.1}$$

where

$$ (u)_s^{\omega} = \left(\frac{-u}{\omega}\right)_s (-\omega)^s $$

with

$$ (u)_s = u(u+1)(u+2)\ldots(u+s-1),\, s=1,2,\ldots \quad \text{and} \quad (u)_0 = 1. $$

Proof. The explicit representation of the degenerate Bell polynomials of the Appell type, derived from (1.5), (1.9) and (1.10), is provided as follows:

$$ \begin{aligned} \sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_l^{\omega}\frac{t^l}{l!} &= A(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} \\ &= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\left(\frac{-u}{\omega}\right)_s(-\omega)^s\frac{t^s}{s!}\right)\left(\sum_{r=0}^{\infty}\mathcal{B}_r(v;\omega)\frac{t^r}{r!}\right). \end{aligned} $$

If we write $s-r$ instead of s and then $l-s$ instead of l, it gives

$$ \sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_l^{\omega}\frac{t^l}{l!} = \sum_{l=0}^{\infty}\sum_{s=0}^{l}\sum_{r=0}^{s}\binom{l}{s}\binom{s}{r}\alpha_{l-s,\omega}\,(u)_{s-r}^{\omega}\,\mathcal{B}_r(v;\omega)\frac{t^l}{l!}. \tag{2.2}$$

If the coefficients of $\frac{t^l}{l!}$ are compared in the equation (2.2), it can be seen that the proof is completed. □

Theorem 2.2. *The Appell type degenerate Bell polynomials satisfy the following determinant representation*

$$ {}_{\mathcal{B}}\mathcal{A}_l^{\omega}=\frac{(-1)^l}{(\gamma_{0,\omega})^{l+1}}\begin{vmatrix} \mathcal{B}_0^{\omega} & \mathcal{B}_1^{\omega} & \cdots & \mathcal{B}_{l-1}^{\omega} & \mathcal{B}_l^{\omega} \\ \gamma_{0,\omega} & \gamma_{1,\omega} & \cdots & \gamma_{l-1,\omega} & \gamma_{l,\omega} \\ 0 & \gamma_{0,\omega} & \cdots & \binom{l-1}{1}\gamma_{l-2,\omega} & \binom{l}{1}\gamma_{l-1,\omega} \\ 0 & 0 & \cdots & \binom{l-1}{2}\gamma_{l-3,\omega} & \binom{l}{2}\gamma_{l-2,\omega} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{0,\omega} & \binom{l}{l-1}\gamma_{1,\omega} \end{vmatrix} \tag{2.3}$$

where $\gamma_{0,\omega},\gamma_{1,\omega},\gamma_{2,\omega},\ldots,\gamma_{l,\omega}$ *are the coefficients of the Maclaurin series of the function* $\frac{1}{A(t)}$ *and* $\mathcal{B}_j(u,v;\omega):=\mathcal{B}_j^{\omega}$ $(j=0,1,...)$ *are defined in* (1.11).

Proof. The application of series representation

$$[A(t)]^{-1}=\sum_{s=0}^{\infty}\gamma_{s,\omega}\frac{t^s}{s!}, \tag{2.4}$$

and the generating function (1.9), we get

$$(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}=\left(\sum_{s=0}^{\infty}\gamma_{s,\omega}\frac{t^s}{s!}\right)\left(\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_l^{\omega}\frac{t^l}{l!}\right). \tag{2.5}$$

Hence, we can write from (1.11)

$$\sum_{l=0}^{\infty}\mathcal{B}_l^{\omega}\frac{t^l}{l!}=\left(\sum_{s=0}^{\infty}\gamma_{s,\omega}\frac{t^s}{s!}\right)\left(\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_l^{\omega}\frac{t^l}{l!}\right). \tag{2.6}$$

If we write $l-s$ instead of l, we have

$$\sum_{l=0}^{\infty}\mathcal{B}_l^{\omega}\frac{t^l}{l!}=\sum_{l=0}^{\infty}\sum_{s=0}^{l}\binom{l}{s}\gamma_{s,\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega}\frac{t^l}{l!}. \tag{2.7}$$

When we compare the coefficients of $\frac{t^l}{l!}$ in the previous equation, we get

$$\mathcal{B}_l^{\omega}=\sum_{s=0}^{l}\binom{l}{s}\gamma_{s,\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega}. \tag{2.8}$$

Consequently, we present with the system of following equations:

$$\begin{aligned}
\mathcal{B}_0^{\omega}&=\gamma_{0,\omega}\ {}_{\mathcal{B}}\mathcal{A}_0^{\omega},\\
\mathcal{B}_1^{\omega}&=\gamma_{0,\omega}\ {}_{\mathcal{B}}\mathcal{A}_1^{\omega}+\gamma_{1,\omega}\ {}_{\mathcal{B}}\mathcal{A}_0^{\omega},\\
\mathcal{B}_2^{\omega}&=\gamma_{0,\omega}\ {}_{\mathcal{B}}\mathcal{A}_2^{\omega}+\binom{2}{1}\gamma_{1,\omega}\ {}_{\mathcal{B}}\mathcal{A}_1^{\omega}+\gamma_{2,\omega}\ {}_{\mathcal{B}}\mathcal{A}_0^{\omega},\\
&\vdots\\
\mathcal{B}_{l-1}^{\omega}&=\gamma_{0,,\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-1}^{\omega}+\binom{l-1}{1}\gamma_{1,\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-2}^{\omega}+\ldots+\gamma_{l-1,\omega}\ {}_{\mathcal{B}}\mathcal{A}_0^{\omega},\\
\mathcal{B}_l^{\omega}&=\gamma_{0,\omega}\ {}_{\mathcal{B}}\mathcal{A}_l^{\omega}+\binom{l}{1}\gamma_{1,\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-1}^{\omega}+\ldots+\gamma_{l,\omega}\ {}_{\mathcal{B}}\mathcal{A}_0^{\omega}.
\end{aligned}$$

Using Cramer's rule and then by taking the transpose in the last equation and using the lower triangular matrix property, we have

$$
{}_{\mathcal{B}}A_l^{\omega} = \frac{1}{(\gamma_{0,\omega})^{l+1}} \begin{vmatrix} \gamma_{0,\omega} & \gamma_{1,\omega} & \cdots & \gamma_{l-1,\omega} & \gamma_{l,\omega} \\ 0 & \gamma_{0,\omega} & \cdots & \binom{l-1}{1}\gamma_{l-2,\omega} & \binom{l}{1}\gamma_{l-1,\omega} \\ 0 & 0 & \cdots & \binom{l-1}{2}\gamma_{l-3,\omega} & \binom{l}{2}\gamma_{l-2,\omega} \\ 0 & 0 & \cdots & \binom{l-1}{3}\gamma_{l-4,\omega} & \binom{l}{3}\gamma_{l-3,\omega} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{0,\omega} & \binom{l}{l-1}\gamma_{1,\omega} \\ \mathcal{B}_0^{\omega} & \mathcal{B}_1^{\omega} & \cdots & \mathcal{B}_{l-1}^{\omega} & \mathcal{B}_l^{\omega} \end{vmatrix}.
$$

Finally, the proof is completed using elementary row operations. □

Theorem 2.3. *The Appell type degenerate Bell polynomials* ${}_{\mathcal{B}}\mathcal{A}_l^{\omega}$ *have following recurrence relation*

$$
\begin{aligned}
{}_{\mathcal{B}}\mathcal{A}_{l+1}^{\omega} = {} & (u+\beta_{0,\omega})\ {}_{\mathcal{B}}\mathcal{A}_l^{\omega} + \sum_{s=1}^{l}\binom{l}{s}\beta_{s,\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega} \\
& + u\sum_{s=1}^{l}\binom{l}{s} s!\,(-\omega)^s\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega} \\
& + v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\binom{l}{s}\binom{s}{r} m!\,(-\omega)^m S_2(s-r,m)\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega}
\end{aligned} \tag{2.9}
$$

where

$$
\frac{A'(t)}{A(t)} = \sum_{s=0}^{\infty}\beta_{s,\omega}\frac{t^s}{s!}. \tag{2.10}
$$

Proof. By applying the derivative operator with regard to t to both sides of equation (1.9), gives

$$
\begin{aligned}
\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{l+1}^{\omega}\frac{t^l}{l!} = {} & \frac{A'(t)}{A(t)}A(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} \\
& + u\frac{1}{(1+\omega t)}A(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} \\
& + ve^t\frac{1}{1+\omega(e^t-1)}A(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}.
\end{aligned} \tag{2.11}
$$

Using the equations (1.8), (1.9) and (2.10) and series form of

$$
\frac{1}{1+\omega t} = \sum_{s=0}^{\infty}(-\omega t)^s, \quad |\omega t| < 1,
$$

we have

$$\sum_{l=0}^{\infty} {}_{\mathbb{B}}\mathcal{A}_{l+1}^{\omega}\frac{t^l}{l!} = \left(\sum_{l=0}^{\infty} {}_{\mathbb{B}}\mathcal{A}_{l}^{\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\beta_{s,\omega}\frac{t^s}{s!}\right)+u\left(\sum_{l=0}^{\infty} {}_{\mathbb{B}}\mathcal{A}_{l}^{\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}(-\omega t)^s\right)$$
$$+v\left(\sum_{l=0}^{\infty} {}_{\mathbb{B}}\mathcal{A}_{l}^{\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\binom{s}{r}m!\,(-\omega)^m S_2(s-r,m)\frac{t^s}{s!}\right).$$

Then, if we write $l-s$ instead of l, we obtain

$$\sum_{l=0}^{\infty} {}_{\mathbb{B}}\mathcal{A}_{l+1}^{\omega}\frac{t^l}{l!} = \sum_{l=0}^{\infty}\sum_{s=0}^{l}\binom{l}{s}\beta_{s,\omega}\; {}_{\mathbb{B}}\mathcal{A}_{l-s}^{\omega}\frac{t^l}{l!}+u\sum_{l=0}^{\infty}\sum_{s=0}^{l}\binom{l}{s}s!\,(-\omega)^s\; {}_{\mathbb{B}}\mathcal{A}_{l-s}^{\omega}\frac{t^l}{l!}$$
$$+v\sum_{l=0}^{\infty}\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\binom{l}{s}\binom{s}{r}m!\,(-\omega)^m S_2(s-r,m)\; {}_{\mathbb{B}}\mathcal{A}_{l-s}^{\omega}\frac{t^l}{l!}. \tag{2.12}$$

It can be seen from equation (2.12) that the proof is completed. □

Remark 2.4. Note that the following property is true when considering the generating function of Appell type degenerate Bell polynomials:

$${}_u\Delta_\omega\left({}_{\mathbb{B}}\mathcal{A}_{l}^{\omega}\right) = l\; {}_{\mathbb{B}}\mathcal{A}_{l-1}^{\omega}, \quad l=1,2,..., \tag{2.13}$$

where

$${}_u\Delta_\omega\left(f(u,v)\right)=\frac{f(u+\omega,v)-f(u,v)}{\omega}, \quad \omega\neq 0.$$

Theorem 2.5. *For the Appell type degenerate Bell polynomials ${}_{\mathbb{B}}\mathcal{A}_{l}^{\omega}$, we give the following lowering operators ${}_u\sigma_l^-$ and raising operators ${}_u\sigma_l^+$*

$${}_u\sigma_l^- := \frac{1}{l}\,{}_u\Delta_\omega, \tag{2.14}$$

$${}_u\sigma_l^+ := u+\beta_{0,\omega}+\sum_{s=1}^{l}\beta_{s,\omega}\frac{{}_u\Delta_\omega{}^s}{s!}+u\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^s$$
$$+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\frac{m!}{(s-r)!r!}(-\omega)^m S_2(s-r,m)\,{}_u\Delta_\omega^s, \tag{2.15}$$

respectively. The difference equation satisfied by the polynomials ${}_{\mathbb{B}}\mathcal{A}_{l}^{\omega}$ is given by

$$\Bigg[(u+\omega+\beta_{0,\omega})\,{}_u\Delta_\omega+\sum_{s=1}^{l}\beta_{s,\omega}\frac{{}_u\Delta_\omega{}^{s+1}}{s!}+(u+\omega)\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^{s+1}$$
$$+\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^s+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\frac{m!}{(s-r)!r!}(-\omega)^m S_2(s-r,m)\,{}_u\Delta_\omega^{s+1}-l\Bigg]\; {}_{\mathbb{B}}\mathcal{A}_{l}^{\omega}=0. \tag{2.16}$$

Proof. In light of the following difference operator relation

$${}_u\Delta_\omega\; {}_{\mathbb{B}}\mathcal{A}_{l}^{\omega}=l\; {}_{\mathbb{B}}\mathcal{A}_{l-1}^{\omega}, \tag{2.17}$$

we obtain

$$\frac{1}{l}\,{}_u\Delta_\omega\; {}_{\mathbb{B}}\mathcal{A}_{l}^{\omega}= {}_{\mathbb{B}}\mathcal{A}_{l-1}^{\omega}. \tag{2.18}$$

This clearly shows that the lowering operator is provided:

$${}_u\sigma_l^- := \frac{1}{l}\,{}_u\Delta_\omega.$$

In the recurrence relation (2.9), the term ${}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega}$ can be represented as follows in terms of the lowering operator:

$$\begin{aligned} {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega} &= \left[\sigma_{l-s+1}^{-}\sigma_{l-s+2}^{-}\cdots\sigma_{l}^{-}\right]\ {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega} \\ &= \frac{(l-s)!}{l!}\,{}_{u}\Delta_{\omega}^{s}\ {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}. \end{aligned} \tag{2.19}$$

It can be seen from the equation (2.9) and (2.19) that we have

$$\Bigg[u+\beta_{0,\omega}+\sum_{s=1}^{l}\beta_{s,\omega}\frac{{}_{u}\Delta_{\omega}{}^{s}}{s!}+u\sum_{s=1}^{l}(-\omega)^{s}\,{}_{u}\Delta_{\omega}^{s}$$
$$+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\frac{m!}{(s-r)!r!}(-\omega)^{m}S_{2}(s-r,m)\,{}_{u}\Delta_{\omega}^{s}\Bigg]\ {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}=\ {}_{\mathcal{B}}\mathcal{A}_{l+1}^{\omega},$$

so the operator ${}_{u}\sigma_{l}^{+}$ is derived. To obtain the difference equation, applying the factorization method

$$\sigma_{l+1}^{-}\left(\sigma_{l}^{+}\left(\ {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}\right)\right)=\ {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}, \tag{2.20}$$

with

$${}_{u}\Delta_{\omega}\left(f(u)\,g(u,v)\right)=f(u+\omega)\,{}_{u}\Delta_{\omega}\,g(u,v)+g(u,v)\,{}_{u}\Delta_{\omega}f(u),$$

we have

$$\Bigg[(u+\omega+\beta_{0,\omega})\,{}_{u}\Delta_{\omega}+\sum_{s=0}^{l}\beta_{s,\omega}\frac{{}_{u}\Delta_{\omega}{}^{s+1}}{s!}+(u+\omega)\sum_{s=1}^{l}(-\omega)^{s}\,{}_{u}\Delta_{\omega}^{s+1}$$
$$+\sum_{s=1}^{l}(-\omega)^{s}\,{}_{u}\Delta_{\omega}^{s}+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\frac{m!}{(s-r)!r!}(-\omega)^{m}S_{2}(s-r,m)\,{}_{u}\Delta_{\omega}^{s+1}-l\Bigg]\ {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}=0.$$

Whence the result. □

Theorem 2.6. *The Appell type degenerate Bell polynomials have the following summation formulas*

$${}_{\mathcal{B}}\mathcal{A}_{l}(u+z,v;\omega) = \sum_{s=0}^{l}\binom{l}{s}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}(z,v;\omega)\,(u)_{s}^{\omega}, \tag{2.21}$$

$${}_{\mathcal{B}}\mathcal{A}_{l}(u,v+z;\omega) = \sum_{s=0}^{l}\binom{l}{s}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}(u,v;\omega)\,\mathcal{B}_{s}(z;\omega), \tag{2.22}$$

$${}_{\mathcal{B}}\mathcal{A}_{l}(u+y,v+z;\omega) = \sum_{s=0}^{l}\binom{l}{s}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}(u,v;\omega)\,\mathcal{B}_{s}(y,z;\omega). \tag{2.23}$$

where $\mathcal{B}_{s}(z;\omega)$ *and* $\mathcal{B}_{s}(y,z;\omega)$ *are in the equations* (1.5) *and* (1.11), *respectively.*

Proof. By substituting $u+z$ for u in equation (1.9), the result is

$$\begin{aligned} \sum_{l=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_{l}(u+z,v;\omega)\frac{t^{l}}{l!} &= A(t)(1+\omega t)^{\frac{u+z}{\omega}}\left(1+\omega\left(e^{t}-1\right)\right)^{\frac{v}{\omega}} \\ &= \left(\sum_{l=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_{l}(z,v;\omega)\frac{t^{l}}{l!}\right)\left(\sum_{s=0}^{\infty}(u)_{s}^{\omega}\frac{t^{s}}{s!}\right). \end{aligned}$$

Applying the Cauchy product, we have

$$\sum_{l=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_{l}(u+z,v;\omega)\frac{t^{l}}{l!}=\sum_{l=0}^{\infty}\sum_{s=0}^{l}\binom{l}{s}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}(z,v;\omega)\,(u)_{s}^{\omega}\frac{t^{l}}{l!}. \tag{2.24}$$

If the coefficients of $\frac{t^l}{l!}$ are compared in the equation (2.24), it can be seen that the proof is completed. Equation (2.22) is derived through the substitution of $v+z$ for v, while equation (2.23) is validated through the substitution of $u+y$ for u and $v+z$ for v. □

Corollary 2.7. *The multiplication formula for degenerate Bell polynomials of the Appell type is as follows:*

$$ {}_{\mathcal{B}}\mathcal{A}_l\left(pu,qv;\omega\right) = \sum_{s=0}^{l}\binom{l}{s}\mathcal{B}_s\left(\left(p-1\right)u,\left(q-1\right)v;\omega\right)\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega},\quad p,q\in\mathbb{N}. \tag{2.25}$$

After taking $y=(p-1)\,u$ and $z=(q-1)\,v$ from equation (2.23), we get this result.

Theorem 2.8. *The polynomials ${}_{\mathcal{B}}\mathcal{A}_l^{\omega}$ satisfied the following relation*

$$\sum_{s=0}^{l}\binom{l}{s}\delta_{l-s,\omega}\ {}_{\mathcal{B}}\mathcal{A}_s\left(2u,2v;\omega\right)=\sum_{s=0}^{l}\binom{l}{s}\ {}_{\mathcal{B}}\mathcal{R}_s^{\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega},$$

where

$$\sum_{s=0}^{\infty}{}_{\mathcal{B}}\mathcal{R}_s^{\omega}\frac{t^s}{s!}=B\left(t\right)\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}},\quad B\left(t\right)=\sum_{s=0}^{\infty}\delta_{s,\omega}\frac{t^s}{s!}.$$

Proof. With the following generating functions in mind:

$$A\left(t\right)\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}=\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_l^{\omega}\frac{t^l}{l!}, \tag{2.26}$$

$$B\left(t\right)\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}=\sum_{s=0}^{\infty}{}_{\mathcal{B}}\mathcal{R}_s^{\omega}\frac{t^s}{s!}. \tag{2.27}$$

From (2.26) and (2.27), we obtain

$$B\left(t\right)\left[A\left(t\right)\left(1+\omega t\right)^{\frac{2u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{2v}{\omega}}\right]=\left(\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_l^{\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}{}_{\mathcal{B}}\mathcal{R}_s^{\omega}\frac{t^s}{s!}\right).$$

Using equations (1.9), (1.10) and applying the Cauchy product rule, we have

$$\sum_{l=0}^{\infty}\sum_{s=0}^{l}\binom{l}{s}\delta_{l-s,\omega}\ {}_{\mathcal{B}}\mathcal{A}_s\left(2u,2v;\omega\right)\frac{t^l}{l!}=\sum_{l=0}^{\infty}\sum_{s=0}^{l}\binom{l}{s}\ {}_{\mathcal{B}}\mathcal{R}_s^{\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega}\frac{t^l}{l!}.$$

Hence we get the result. □

Corollary 2.9. *In the Theorem 2.8, the following properties are provided according to the special cases of $A(t)$ and $B(t)$.*

- *Taking $B(t)=A(t)$, we have*

$$\sum_{s=0}^{l}\binom{l}{s}\alpha_{l-s,\omega}\ {}_{\mathcal{B}}\mathcal{A}_s\left(2u,2v;\omega\right)=\sum_{s=0}^{l}\binom{l}{s}\ {}_{\mathcal{B}}\mathcal{A}_s^{\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega}.$$

When $\omega\to 0$, the above equation is also provided the following Bell based Appell polynomials ${}_{\mathcal{B}}\mathcal{A}_s(u,v)$ [40]

$$\sum_{s=0}^{l}\binom{l}{s}\alpha_{l-s}\ {}_{\mathcal{B}}\mathcal{A}_s\left(2u,2v\right)=\sum_{s=0}^{l}\binom{l}{s}\ {}_{\mathcal{B}}\mathcal{A}_s\left(u,v\right)\ {}_{\mathcal{B}}\mathcal{A}_{l-s}\left(u,v\right).$$

- *Taking $A(t)=\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}$ and $B(t)=\frac{2}{(1+\omega t)^{\frac{1}{\omega}}+1}$, we have from (1.12) and (1.13)*

$$\sum_{s=0}^{l}\binom{l}{s}\mathcal{E}_{l-s,\omega}\ {}_{\mathcal{B}}\mathcal{B}_s\left(2u,2v;\omega\right)=\sum_{s=0}^{l}\binom{l}{s}\ {}_{\mathcal{B}}\mathcal{B}_{l-s}^{\omega}\ {}_{\mathcal{B}}\mathcal{E}_s^{\omega},$$

in which $\mathcal{E}_{l,\omega}$ are as follows

$$\frac{2}{(1+\omega t)^{\frac{1}{\omega}}+1}=\sum_{l=0}^{\infty}\mathcal{E}_{l,\omega}\frac{t^l}{l!}.$$

When $\omega \to 0$, the above equation is also provided the following Bell based Bernoulli polynomials ${}_{\mathcal{B}}\mathcal{B}_l(u,v)$ and Bell based Euler polynomials ${}_{\mathcal{B}}\mathcal{E}_l(u,v)$ [17, 22, 40]

$$\sum_{s=0}^{l}\binom{l}{s}\mathcal{E}_{l-s}\,\mathcal{B}_s(2u,2v)=\sum_{s=0}^{l}\binom{l}{s}\,{}_{\mathcal{B}}\mathcal{B}_{l-s}(u,v)\,\,{}_{\mathcal{B}}\mathcal{E}_s(u,v),$$

in which $\mathcal{E}_l$ are as follows

$$\frac{2}{e^t+1}=\sum_{l=0}^{\infty}\mathcal{E}_l\frac{t^l}{l!}.$$

3. Examples of the Appell type degenerate Bell polynomial families and their properties

In this part, we examine some special cases of the determining function $A(t)$ and define degenerate Bell-Bernoulli polynomials and degenerate Bell-Euler polynomials. As an application of the main results, we give the recurrence relation, determinant representation, shift operators and difference equation for these polynomials.

3.1. The degenerate Bell-Bernoulli polynomials

The generating function for the degenerate Bell-Bernoulli polynomials ${}_{\mathcal{B}}\mathcal{B}_l(u,v;\omega):={}_{\mathcal{B}}\mathcal{B}_l^{\omega}$ is as follows:

$$\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}=\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{B}_l^{\omega}\frac{t^l}{l!}.$$

The first three degenerate Bell-Bernoulli polynomials are as follows:

$$\begin{aligned}{}_{\mathcal{B}}\mathcal{B}_0^{\omega}&=1,\\ {}_{\mathcal{B}}\mathcal{B}_1^{\omega}&=u+v+\frac{\omega-1}{2},\\ {}_{\mathcal{B}}\mathcal{B}_2^{\omega}&=(u+v)^2-u+\frac{1-\omega^2}{6}.\end{aligned}$$

Corollary 3.1. *The degenerate Bell-Bernoulli polynomials ${}_{\mathcal{B}}\mathcal{B}_l^{\omega}$ satisfy the following recurrence relation*

$$\begin{aligned}{}_{\mathcal{B}}\mathcal{B}_{l+1}^{\omega}&=(u-1)\,{}_{\mathcal{B}}\mathcal{B}_l^{\omega}-\sum_{s=1}^{l}\binom{l}{s}s!\,(-\omega)^s\,{}_{\mathcal{B}}\mathcal{B}_{l-s}^{\omega}\\ &-\sum_{s=0}^{l}\sum_{r=0}^{s+1}\binom{l}{s}\frac{s!}{(s+1-r)!}(-\omega)^r\,\mathcal{B}_{s+1-r,\omega}\,{}_{\mathcal{B}}\mathcal{B}_{l-s}^{\omega}\\ &+u\sum_{s=1}^{l}\binom{l}{s}s!\,(-\omega)^s\,{}_{\mathcal{B}}\mathcal{B}_{l-s}^{\omega}\\ &+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\binom{l}{s}\binom{s}{r}m!\,(-\omega)^m\,S_2(s-r,m)\,{}_{\mathcal{B}}\mathcal{B}_{l-s}^{\omega}\end{aligned}$$

where $\mathcal{B}_{s,\omega}$ are called degenerate Bernoulli numbers [5, 6]

$$\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}=\sum_{s=0}^{\infty}\mathcal{B}_{s,\omega}\frac{t^s}{s!}.$$

Corollary 3.2. *The polynomials ${}_{\mathcal{B}}\mathcal{B}_l^{\omega}$ have the following lowering operators ${}_u\sigma_l^-$ and raising operators ${}_u\sigma_l^+$*

$$\begin{aligned}
{}_u\sigma_l^- &:= \frac{1}{l}\,{}_u\Delta_\omega,\\
{}_u\sigma_l^+ &:= u-1-\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^s-\sum_{s=0}^{l}\sum_{r=0}^{s+1}\frac{(-\omega)^r}{(s+1-r)!}\mathcal{B}_{s+1-r,\omega}\,{}_u\Delta_\omega^s\\
&\quad +u\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^s+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\frac{m!}{(s-r)!r!}(-\omega)^m S_2(s-r,m)\,{}_u\Delta_\omega^s,
\end{aligned}$$

respectively. The difference equation satisfied by the ${}_{\mathcal{B}}\mathcal{B}_l^{\omega}$ is given by

$$\begin{aligned}
&\Bigg[(u+\omega-1)\,{}_u\Delta_\omega-\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^{s+1}-\sum_{s=0}^{l}\sum_{r=0}^{s+1}\frac{(-\omega)^r}{(s+1-r)!}\mathcal{B}_{s+1-r,\omega}\,{}_u\Delta_\omega^{s+1}\\
&+\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^s+(u+\omega)\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^{s+1}\\
&+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\frac{m!}{(s-r)!r!}(-\omega)^m S_2(s-r,m)\,{}_u\Delta_\omega^{s+1}-l\Bigg]\,{}_{\mathcal{B}}\mathcal{B}_l^{\omega}=0.
\end{aligned}$$

Corollary 3.3. *The polynomials ${}_{\mathcal{B}}\mathcal{B}_l^{\omega}$ have the following determinant representation*

$${}_{\mathcal{B}}\mathcal{B}_l^{\omega}=(-1)^l\begin{vmatrix}
\mathcal{B}_0^{\omega} & \mathcal{B}_1^{\omega} & \mathcal{B}_2^{\omega} & \cdots & \mathcal{B}_{l-1}^{\omega} & \mathcal{B}_l^{\omega}\\
1 & \frac{(1)_2^{\omega}}{2} & \frac{(1)_3^{\omega}}{3} & \cdots & \frac{(1)_l^{\omega}}{l} & \frac{(1)_{l+1}^{\omega}}{l+1}\\
0 & 1 & \binom{2}{1}\frac{(1)_2^{\omega}}{2} & \cdots & \binom{l-1}{1}\frac{(1)_{l-1}^{\omega}}{l-1} & \binom{l}{1}\frac{(1)_l^{\omega}}{l}\\
0 & 0 & 1 & \cdots & \binom{l-1}{2}\frac{(1)_{l-2}^{\omega}}{l-2} & \binom{l}{2}\frac{(1)_{l-1}^{\omega}}{l-1}\\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 1 & \binom{l}{l-1}\frac{(1)_2^{\omega}}{2}
\end{vmatrix}.$$

The second and third degenerate Bell-Bernoulli polynomials ${}_{\mathcal{B}}\mathcal{B}_l(u,v;\omega)$ are given below with 3D surface plots. Degenerate Bell-Bernoulli polynomials with ${}_{\mathcal{B}}\mathcal{B}_2(u,v;0.1)$ and ${}_{\mathcal{B}}\mathcal{B}_3(u,v;0.1)$ for $\omega=0.1$ in Figure 1 and with ${}_{\mathcal{B}}\mathcal{B}_2(u,v;0)$ and ${}_{\mathcal{B}}\mathcal{B}_3(u,v;0)$ polynomials shown in Figure 2 for $\omega=0$ given below.

(a) The polynomials ${}_{\mathcal{B}}\mathcal{B}_2\left(u,v;0.1\right)$. **(b)** The polynomials ${}_{\mathcal{B}}\mathcal{B}_3\left(u,v;0.1\right)$.

Figure 1. Degenerate Bell-Bernoulli polynomials for $\omega = 0.1$.

(a) The polynomials ${}_{\mathcal{B}}\mathcal{B}_2\left(u,v;0\right)$. **(b)** The polynomials ${}_{\mathcal{B}}\mathcal{B}_3\left(u,v;0\right)$.

Figure 2. Degenerate Bell-Bernoulli polynomials for $\omega = 0$.

3.2. The degenerate Bell-Euler polynomials

The generating function for the degenerate Bell-Euler polynomials ${}_{\mathcal{B}}\mathcal{E}_l\left(u,v;\omega\right) := {}_{\mathcal{B}}\mathcal{E}_l^{\omega}$ is as follows:

$$\frac{2}{\left(1+\omega t\right)^{\frac{1}{\omega}}+1}\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} = \sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{E}_l^{\omega}\frac{t^l}{l!}.$$

The first three degenerate Bell-Euler polynomials are as follows:

$$\begin{aligned} {}_{\mathcal{B}}\mathcal{E}_0^{\omega} &= 1, \\ {}_{\mathcal{B}}\mathcal{E}_1^{\omega} &= u+v-\frac{1}{2}, \\ {}_{\mathcal{B}}\mathcal{E}_2^{\omega} &= \left(u+v\right)^2-u-\omega\left(u+v\right)+\frac{\omega}{2}. \end{aligned}$$

Corollary 3.4. *The polynomials* ${}_{\mathcal{B}}\mathcal{E}_l^{\omega}$ *satisfy the following recurrence relation*

$$\begin{aligned} {}_{\mathcal{B}}\mathcal{E}_{l+1}^{\omega} &= (u-1)\ {}_{\mathcal{B}}\mathcal{E}_l^{\omega} - \sum_{s=1}^{l}\binom{l}{s}s!\,(-\omega)^s\ {}_{\mathcal{B}}\mathcal{E}_{l-s}^{\omega} \\ &+\frac{1}{2}\sum_{s=0}^{l}\sum_{r=0}^{s}\binom{l}{s}\binom{s}{r}r!\,(-\omega)^r\,\mathcal{E}_{s-r,\omega}\ {}_{\mathcal{B}}\mathcal{E}_{l-s}^{\omega} \\ &+u\sum_{s=1}^{l}\binom{l}{s}s!\,(-\omega)^s\ {}_{\mathcal{B}}\mathcal{E}_{l-s}^{\omega} \\ &+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\binom{l}{s}\binom{s}{r}m!\,(-\omega)^m\,S_2(s-r,m)\ {}_{\mathcal{B}}\mathcal{E}_{l-s}^{\omega} \end{aligned}$$

where $\mathcal{E}_{s,\omega}$ *are called degenerate Euler numbers [6]*

$$\frac{2}{(1+\omega t)^{\frac{1}{\omega}}+1} = \sum_{s=0}^{\infty}\mathcal{E}_{s,\omega}\frac{t^s}{s!}.$$

Corollary 3.5. *For the degenerate Bell-Euler polynomials* ${}_{\mathcal{B}}\mathcal{E}_l^{\omega}$, *we have the following lowering operators* ${}_u\sigma_l^-$ *and raising operators* ${}_u\sigma_l^+$

$$\begin{aligned} {}_u\sigma_l^- &:= \frac{1}{l}\,{}_u\Delta_\omega, \\ {}_u\sigma_l^+ &:= u-1-\sum_{s=1}^{l}(-\omega)^s\ {}_u\Delta_\omega^s+\frac{1}{2}\sum_{s=0}^{l}\sum_{r=0}^{s}\frac{(-\omega)^r}{(s-r)!}\mathcal{E}_{s-r,\omega}\ {}_u\Delta_\omega^s \\ &\quad +u\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^s+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\frac{m!}{(s-r)!r!}(-\omega)^m\,S_2(s-r,m)\,{}_u\Delta_\omega^s, \end{aligned}$$

respectively. The difference equation satisfied by the ${}_{\mathcal{B}}\mathcal{E}_l^{\omega}$ *is given by*

$$\begin{aligned} &\Bigg[(u+\omega-1)\,{}_u\Delta_\omega-\sum_{s=1}^{l}(-\omega)^s\ {}_u\Delta_\omega^{s+1}+\frac{1}{2}\sum_{s=0}^{l}\sum_{r=0}^{s}\frac{(-\omega)^r}{(s-r)!}\mathcal{E}_{s-r,\omega}\ {}_u\Delta_\omega^{s+1} \\ &+(u+\omega)\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^{s+1}+\sum_{s=1}^{l}(-\omega)^s\,{}_u\Delta_\omega^s \\ &+v\sum_{s=0}^{l}\sum_{r=0}^{s}\sum_{m=0}^{s-r}\frac{m!}{(s-r)!r!}(-\omega)^m\,S_2(s-r,m)\,{}_u\Delta_\omega^{s+1}-l\Bigg]\ {}_{\mathcal{B}}\mathcal{E}_l^{\omega}=0. \end{aligned}$$

Corollary 3.6. *The degenerate Bell-Euler polynomials* ${}_{\mathcal{B}}\mathcal{E}_l^{\omega}$ *have the following determinant representation*

$${}_{\mathcal{B}}\mathcal{E}_l^{\omega} = (-1)^l\begin{vmatrix} \mathcal{B}_0^{\omega} & \mathcal{B}_1^{\omega} & \mathcal{B}_2^{\omega} & \cdots & \mathcal{B}_{l-1}^{\omega} & \mathcal{B}_l^{\omega} \\ 1 & \frac{1}{2} & \frac{1}{2}(1)_2^{\omega} & \cdots & \frac{1}{2}(1)_{l-1}^{\omega} & \frac{1}{2}(1)_l^{\omega} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{l-1}{1}\frac{1}{2}(1)_{l-2}^{\omega} & \binom{l}{1}\frac{1}{2}(1)_{l-1}^{\omega} \\ 0 & 0 & 1 & \cdots & \binom{l-1}{2}\frac{1}{2}(1)_{l-3}^{\omega} & \binom{l}{2}\frac{1}{2}(1)_{l-2}^{\omega} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{l}{l-1}\frac{1}{2} \end{vmatrix}.$$

The second and third degenerate Bell-Euler polynomials ${}_{\mathcal{B}}\mathcal{E}_l(u, v; \omega)$ are given below with $3D$ surface plots. Degenerate Bell-Euler polynomials with ${}_{\mathcal{B}}\mathcal{E}_2(u, v; 0.1)$ and ${}_{\mathcal{B}}\mathcal{E}_3(u, v; 0.1)$ for $\omega = 0.1$ in Figure 3 and with ${}_{\mathcal{B}}\mathcal{E}_2(u, v; 0)$ and ${}_{\mathcal{B}}\mathcal{E}_3(u, v; 0)$ polynomials shown in Figure 4 for $\omega = 0$ given below.

(a) The polynomials ${}_{\mathcal{B}}\mathcal{E}_2(u, v; 0.1)$. **(b)** The polynomials ${}_{\mathcal{B}}\mathcal{E}_3(u, v; 0.1)$.

Figure 3. Degenerate Bell-Euler polynomials for $\omega = 0.1$

(a) The polynomials ${}_{\mathcal{B}}\mathcal{E}_2(u, v; 0)$. **(b)** The polynomials ${}_{\mathcal{B}}\mathcal{E}_3(u, v; 0)$.

Figure 4. Degenerate Bell-Euler polynomials for $\omega = 0$.

4. Some relations and properties between Stirling numbers and Appell type degenerate Bell polynomials

In present section, we investigate the connections between the Stirling numbers and Appell type degenerate Bell polynomials.

We recall that the generating function for the Stirling numbers of the first kind $S_1(l, s)$ [43] is as follows:

$$\{\log(1+t)\}^s = s! \sum_{l=s}^{\infty} S_1(l, s) \frac{t^l}{l!}, \quad (|t| < 1). \tag{4.1}$$

Degenerate versions of the Stirling numbers of the first kind $S_{1,\omega}(l, s)$ and Stirling numbers of the second kind $S_{2,\omega}(l, s)$ are introduced as follows, respectively [25, 26]:

$$< u >_l = \sum_{i=0}^{l} S_{1,\omega}(l, i) (u)_i^{\omega}, \quad (u)_l^{\omega} = \sum_{i=0}^{l} S_{2,\omega}(l, i) < u >_i, \tag{4.2}$$

where

$$< u >_0 := 1, \quad < u >_l = u(u-1)\ldots(u-l+1), \quad (l = 1, 2, \ldots)$$

and

$$(u)_0^\omega := 1, \quad (u)_l^\omega = u\left(u-\omega\right)...\left(u-\left(l-1\right)\omega\right), \quad \left(l=1,2,...\right).$$

The degenerate Stirling numbers of the first and second kinds can be expressed using the following generating functions [25, 26]

$$\frac{1}{s!}\left(\log_\omega\left(1+t\right)\right)^s = \sum_{l=s}^{\infty} S_{1,\omega}\left(l,s\right)\frac{t^l}{l!}, \tag{4.3}$$

$$\frac{1}{s!}\left(\left(1+\omega t\right)^{\frac{1}{\omega}}-1\right)^s = \sum_{l=s}^{\infty} S_{2,\omega}\left(l,s\right)\frac{t^l}{l!}, \tag{4.4}$$

where the definition of the degenerate logarithm function is

$$\log_\omega\left(1+t\right) = \frac{1}{\omega}\left(\left(1+t\right)^\omega-1\right).$$

Theorem 4.1. *The following explicit representation provided by the Appell-type degenerate Bell polynomials is given in the form of Stirling numbers of second kind*

$$_{\mathcal{B}}\mathcal{A}_l^\omega = \sum_{s=0}^{l}\sum_{m=0}^{s}\sum_{r=0}^{m}\binom{l}{s}\binom{s}{m}\left(v\right)_r^\omega\left(u\right)_{s-m}^\omega S_2\left(m,r\right)\alpha_{l-s,\omega}. \tag{4.5}$$

Proof. From (1.9), we observe that

$$\begin{aligned}
\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_l^\omega\frac{t^l}{l!} &= A\left(t\right)\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} \\
&= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\left(u\right)_s^\omega\frac{t^s}{s!}\right)\left(\sum_{r=0}^{\infty}\left(v\right)_r^\omega\frac{\left(e^t-1\right)^r}{r!}\right) \\
&= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\left(u\right)_s^\omega\frac{t^s}{s!}\right)\left(\sum_{r=0}^{\infty}\left(v\right)_r^\omega\left(\sum_{m=r}^{\infty}S_2\left(m,r\right)\frac{t^m}{m!}\right)\right) \\
&= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\left(u\right)_s^\omega\frac{t^s}{s!}\right)\left(\sum_{m=0}^{\infty}\sum_{r=0}^{m}\left(v\right)_r^\omega S_2\left(m,r\right)\frac{t^m}{m!}\right) \\
&= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\sum_{m=0}^{s}\sum_{r=0}^{m}\binom{s}{m}\left(u\right)_{s-m}^\omega\left(v\right)_r^\omega S_2\left(m,r\right)\frac{t^m}{m!}\right) \\
&= \sum_{l=0}^{\infty}\sum_{s=0}^{l}\sum_{m=0}^{s}\sum_{r=0}^{m}\binom{l}{s}\binom{s}{m}\left(u\right)_{s-m}^\omega\left(v\right)_r^\omega S_2\left(m,r\right)\alpha_{l-s,\omega}\frac{t^l}{l!}.
\end{aligned}$$

If the coefficients of $\frac{t^l}{l!}$ are compared in the above equation, it can be seen that the proof is completed. □

Corollary 4.2. *When* $A\left(t\right) = \frac{t}{\left(1+\omega t\right)^{\frac{1}{t}}-1}$ *and* $A\left(t\right) = \frac{2}{\left(1+\omega t\right)^{\frac{1}{t}}+1}$ *in Theorem 4.1, we have the following explicit representations of degenerate Bell-Bernoulli and Bell-Euler polynomials*

$$_{\mathcal{B}}\mathcal{B}_l^\omega = \sum_{s=0}^{l}\sum_{m=0}^{s}\sum_{r=0}^{m}\binom{l}{s}\binom{s}{m}\left(v\right)_r^\omega\left(u\right)_{s-m}^\omega S_2\left(m,r\right)\mathcal{B}_{l-s,\omega}, \tag{4.6}$$

$$_{\mathcal{B}}\mathcal{E}_l^\omega = \sum_{s=0}^{l}\sum_{m=0}^{s}\sum_{r=0}^{m}\binom{l}{s}\binom{s}{m}\left(v\right)_r^\omega\left(u\right)_{s-m}^\omega S_2\left(m,r\right)\mathcal{E}_{l-s,\omega}, \tag{4.7}$$

respectively.

Theorem 4.3. *The following explicit representation provided by the Appell-type degenerate Bell polynomials is given in the form of Stirling numbers of first and second kinds*

$$ {}_{\mathcal{B}}\mathcal{A}_l^{\omega} = \sum_{s=0}^{l}\sum_{n=0}^{s}\sum_{r=0}^{n}\sum_{m=0}^{r}\binom{l}{s}\binom{s}{n}(u)_{s-n}^{\omega}(v)^m(\omega)^{r-m}\alpha_{l-s,\omega}S_1(r,m)S_2(n,r). \quad (4.8) $$

Proof. From (1.8), (1.9) and (4.1), we observe that

$$ \begin{aligned} \sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_l^{\omega}\frac{t^l}{l!} &= A(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} \\ &= A(t)(1+\omega t)^{\frac{u}{\omega}}e^{\frac{v}{\omega}\log\left(1+\omega\left(e^t-1\right)\right)} \\ &= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}(u)_s^{\omega}\frac{t^s}{s!}\right)\left(\sum_{m=0}^{\infty}\left(\frac{v}{\omega}\right)^m\frac{\left(\log\left(1+\omega\left(e^t-1\right)\right)\right)^m}{m!}\right) \\ &= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}(u)_s^{\omega}\frac{t^s}{s!}\right)\left(\sum_{m=0}^{\infty}\left(\frac{v}{\omega}\right)^m\sum_{r=m}^{\infty}S_1(r,m)(\omega)^r\frac{\left(e^t-1\right)^r}{r!}\right) \\ &= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}(u)_s^{\omega}\frac{t^s}{s!}\right)\left(\sum_{r=0}^{\infty}\sum_{m=0}^{r}(v)^m(\omega)^{r-m}S_1(r,m)\sum_{n=r}^{\infty}S_2(n,r)\frac{t^n}{n!}\right) \\ &= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}(u)_s^{\omega}\frac{t^s}{s!}\right)\left(\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{m=0}^{r}(v)^m(\omega)^{r-m}S_1(r,m)S_2(n,r)\frac{t^n}{n!}\right) \\ &= \left(\sum_{l=0}^{\infty}\alpha_{l,\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\sum_{n=0}^{s}\sum_{r=0}^{n}\sum_{m=0}^{r}\binom{s}{n}(u)_{s-n}^{\omega}(v)^m(\omega)^{r-m}S_1(r,m)S_2(n,r)\frac{t^s}{s!}\right) \\ &= \sum_{l=0}^{\infty}\sum_{s=0}^{l}\sum_{n=0}^{s}\sum_{r=0}^{n}\sum_{m=0}^{r}\binom{l}{s}\binom{s}{n}(u)_{s-n}^{\omega}(v)^m(\omega)^{r-m}\alpha_{l-s,\omega}S_1(r,m)S_2(n,r)\frac{t^l}{l!}. \end{aligned} $$

If the coefficients of $\frac{t^l}{l!}$ are compared in the resulting equation, it can be seen that the proof is completed. □

Corollary 4.4. *When* $A(t)=\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}$ *and* $A(t)=\frac{2}{(1+\omega t)^{\frac{1}{\omega}}+1}$ *in Theorem 4.3, we have the following explicit representations of degenerate Bell-Bernoulli and Bell-Euler polynomials*

$$ \begin{aligned} {}_{\mathcal{B}}\mathcal{B}_l^{\omega} &= \sum_{s=0}^{l}\sum_{n=0}^{s}\sum_{r=0}^{n}\sum_{m=0}^{r}\binom{l}{s}\binom{s}{n}(u)_{s-n}^{\omega}(v)^m(\omega)^{r-m}S_1(r,m)S_2(n,r)\mathcal{B}_{l-s,\omega}, \quad (4.9) \\ {}_{\mathcal{B}}\mathcal{E}_l^{\omega} &= \sum_{s=0}^{l}\sum_{n=0}^{s}\sum_{r=0}^{n}\sum_{m=0}^{r}\binom{l}{s}\binom{s}{n}(u)_{s-n}^{\omega}(v)^m(\omega)^{r-m}S_1(r,m)S_2(n,r)\mathcal{E}_{l-s,\omega}, \quad (4.10) \end{aligned} $$

respectively.

Theorem 4.5. *The following relation is satisfied by Appell type degenerate Bell polynomials*

$$ \begin{aligned} {}_{\mathcal{B}}\mathcal{A}_{l+1}^{\omega} &= \sum_{s=0}^{l}\binom{l}{s}\beta_{s,\omega}\,{}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega} + u\sum_{s=0}^{l}\binom{l}{s}s!(-\omega)^s\,{}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega} \\ &\quad + v\sum_{k=0}^{l}\sum_{r=0}^{k}\sum_{s=0}^{r}\binom{l}{k}\binom{k}{r}(v-\omega)_s^{\omega}S_2(r,s)\mathcal{A}_{l-k}(u;\omega). \quad (4.11) \end{aligned} $$

Proof. By applying the derivative operator with regard to t to both sides of equation (1.9), we get

$$\begin{aligned}\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{l+1}^{\omega}\frac{t^l}{l!} &= \frac{A'(t)}{A(t)}A(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} \\ &+u\frac{1}{(1+\omega t)}A(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} \\ &+ve^tA(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v-\omega}{\omega}}. \end{aligned} \tag{4.12}$$

Using the equations (1.8), (1.9) and (2.10), we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{l+1}^{\omega}\frac{t^l}{l!} &= \left(\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\beta_{s,\omega}\frac{t^s}{s!}\right)+u\left(\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}(-\omega t)^s\right) \\ &+v\left(\sum_{l=0}^{\infty}\mathcal{A}_l(u;\omega)\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\right)\left(\sum_{s=0}^{\infty}(v-\omega)_s^{\omega}\frac{\left(e^t-1\right)^s}{s!}\right) \\ &= \left(\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}\beta_{s,\omega}\frac{t^s}{s!}\right)+u\left(\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{l}^{\omega}\frac{t^l}{l!}\right)\left(\sum_{s=0}^{\infty}(-\omega t)^s\right) \\ &+v\sum_{l=0}^{\infty}\sum_{k=0}^{l}\sum_{r=0}^{k}\sum_{s=0}^{r}\binom{l}{k}\binom{k}{r}(v-\omega)_s^{\omega}S_2(r,s)\mathcal{A}_{l-k}(u;\omega)\frac{t^l}{l!}. \end{aligned} \tag{4.13}$$

Using the Cauchy product, we get

$$\begin{aligned}\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{l+1}^{\omega}\frac{t^l}{l!} &= \sum_{l=0}^{\infty}\sum_{s=0}^{l}\binom{l}{s}\beta_{s,\omega}\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega}\frac{t^l}{l!}+u\sum_{l=0}^{\infty}\sum_{s=0}^{l}\binom{l}{s}s!(-\omega)^s\ {}_{\mathcal{B}}\mathcal{A}_{l-s}^{\omega}\frac{t^l}{l!} \\ &+v\sum_{l=0}^{\infty}\sum_{k=0}^{l}\sum_{r=0}^{k}\sum_{s=0}^{r}\binom{l}{k}\binom{k}{r}(v-\omega)_s^{\omega}S_2(r,s)\mathcal{A}_{l-k}(u;\omega)\frac{t^l}{l!}. \end{aligned} \tag{4.14}$$

If the coefficients of $\frac{t^l}{l!}$ are compared in the equation (4.14), it can be seen that the proof is completed. □

Corollary 4.6. *When* $A(t)=\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}$ *and* $A(t)=\frac{2}{(1+\omega t)^{\frac{1}{\omega}}+1}$ *in Theorem 4.5, we have the following explicit representations of degenerate Bell-Bernoulli and Bell-Euler polynomials*

$$\begin{aligned}{}_{\mathcal{B}}\mathcal{B}_{l+1}^{\omega} &= -\sum_{s=0}^{l}\binom{l}{s}s!(-\omega)^s\ {}_{\mathcal{B}}\mathcal{B}_{l-s}^{\omega}-\sum_{s=0}^{l}\sum_{r=0}^{s+1}\binom{l}{s}\frac{s!}{(s+1-r)!}(-\omega)^r\mathcal{B}_{s+1-r,\omega}\ {}_{\mathcal{B}}\mathcal{B}_{l-s}^{\omega} \\ &+u\sum_{s=0}^{l}\binom{l}{s}s!(-\omega)^s\,{}_{\mathcal{B}}\mathcal{B}_{l-s}^{\omega}+v\sum_{k=0}^{l}\sum_{r=0}^{k}\sum_{s=0}^{r}\binom{l}{k}\binom{k}{r}(v-\omega)_s^{\omega}S_2(r,s)\mathcal{A}_{l-k}(u;\omega), \end{aligned} \tag{4.15}$$

$$\begin{aligned}{}_{\mathcal{B}}\mathcal{E}_{l+1}^{\omega} &= -\sum_{s=0}^{l}\binom{l}{s}s!(-\omega)^s\ {}_{\mathcal{B}}\mathcal{E}_{l-s}^{\omega}+\frac{1}{2}\sum_{s=0}^{l}\sum_{r=0}^{s}\binom{l}{s}\binom{s}{r}r!(-\omega)^r\mathcal{E}_{s-r,\omega}\ {}_{\mathcal{B}}\mathcal{E}_{l-s}^{\omega} \\ &+u\sum_{s=0}^{l}\binom{l}{s}s!(-\omega)^s\,{}_{\mathcal{B}}\mathcal{E}_{l-s}^{\omega}+v\sum_{k=0}^{l}\sum_{r=0}^{k}\sum_{s=0}^{r}\binom{l}{k}\binom{k}{r}(v-\omega)_s^{\omega}S_2(r,s)\mathcal{A}_{l-k}(u;\omega), \end{aligned} \tag{4.16}$$

respectively.

For $r \in \mathbb{N}$, we now examine the degenerate Bell-Euler polynomials of order r, ${}_{\mathcal{B}}\mathcal{E}_l^{(r)}(u,v;\omega)$, which have the following definition according to the generating function:

$$\left(\frac{2}{(1+\omega t)^{\frac{1}{\omega}}+1}\right)^r (1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}=\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{E}_l^{(r)}(u,v;\omega)\frac{t^l}{l!}. \tag{4.17}$$

Theorem 4.7. *Degenerate Bell-Euler polynomials of order r have the following relation with degenerate Stirling numbers of second kind and two-variable degenerate Bell polynomials*

$${}_{\mathcal{B}}\mathcal{E}_l^{(r)}(u,v;\omega)=\sum_{m=0}^{l}\sum_{s=0}^{m}\binom{l}{m}\binom{r+s-1}{s}\left(-\frac{1}{2}\right)^s s!\,S_{2,\omega}(m,s)\,\mathcal{B}_{l-m}(u,v;\omega).$$

Proof. Considering the generating function given by (4.17), we write

$$\begin{aligned}
\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{E}_l^{(r)}(u,v;\omega)\frac{t^l}{l!} &= \left(\frac{2}{(1+\omega t)^{\frac{1}{\omega}}+1}\right)^r (1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}\\
&= \left(\frac{(1+\omega t)^{\frac{1}{\omega}}-1}{2}+1\right)^{-r}\sum_{l=0}^{\infty}\mathcal{B}_l(u,v;\omega)\frac{t^l}{l!}\\
&= \sum_{s=0}^{\infty}\binom{r+s-1}{s}\left(-\frac{1}{2}\right)^s\left((1+\omega t)^{\frac{1}{\omega}}-1\right)^s\sum_{l=0}^{\infty}\mathcal{B}_l(u,v;\omega)\frac{t^l}{l!}\\
&= \sum_{s=0}^{\infty}\binom{r+s-1}{s}\left(-\frac{1}{2}\right)^s s!\sum_{m=s}^{\infty}S_{2,\omega}(m,s)\frac{t^m}{m!}\sum_{l=0}^{\infty}\mathcal{B}_l(u,v;\omega)\frac{t^l}{l!}\\
&= \sum_{m=0}^{\infty}\sum_{s=0}^{m}\binom{r+s-1}{s}\left(-\frac{1}{2}\right)^s s!S_{2,\omega}(m,s)\frac{t^m}{m!}\sum_{l=0}^{\infty}\mathcal{B}_l(u,v;\omega)\frac{t^l}{l!}\\
&= \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{s=0}^{m}\binom{l}{m}\binom{r+s-1}{s}\left(-\frac{1}{2}\right)^s s!S_{2,\omega}(m,s)\mathcal{B}_{l-m}(u,v;\omega)\frac{t^l}{l!}.
\end{aligned} \tag{4.18}$$

If the coefficients of $\frac{t^l}{l!}$ are compared in the equation (4.18), it can be seen that the proof is completed. □

5. Constructing multilinear and multilateral generating functions for Appell type degenerate Bell polynomials

In present section, we obtain multilinear and multilateral generation functions for Appell type degenerate Bell polynomials of the parametric type by means of some methods that have been previously used in the literature (see also [1, 18, 36, 44]).

Theorem 5.1. *Assume that $\Omega_\mu^\omega(x_1,x_2,...,x_s)$ is an identically non-vanishing function of s complex variables $x_1,x_2,...,x_s$, $(s\in\mathbb{N})$ and of complex order μ. Also let the following generating function:*

$$\Lambda_{\mu,\vartheta}^{\omega}(x_1,x_2,...,x_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\vartheta k}^{\omega}(x_1,x_2,...,x_s)\,z^k \tag{5.1}$$

$$(a_k\neq 0,\quad \mu,\vartheta\in\mathbb{C}).$$

Let us assume that for $\Theta_{n,m,\mu,\vartheta}(u,v;\omega;x_1,x_2,...,x_s;\rho)$ *given by*

$$\Theta_{n,m,\mu,\vartheta}(u,v;\omega;x_1,x_2,...,x_s;\rho)$$

$$:= \sum_{k=0}^{[n/m]} a_k \frac{{}_{\mathcal{B}}\mathcal{A}_{n-mk}(u,v;\omega)}{(n-mk)!} \Omega^{\omega}_{\mu+\vartheta k}(x_1,x_2,...,x_s)\rho^k, \tag{5.2}$$

it is asserted that

$$\sum_{n=0}^{\infty} \Theta_{n,m,\mu,\vartheta}\left(u,v;\omega;x_1,x_2,...,x_s;\frac{\eta}{t^m}\right)t^n = A(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} \times \Lambda^{\omega}_{\mu,\vartheta}(x_1,x_2,...,x_s;\eta) \quad (m \in \mathbb{N}). \tag{5.3}$$

Proof. Let S stand for the first member of Theorem 5.1's statement (5.3) for the sake of convenience. Substituting the polynomials

$$\Theta_{n,m,\mu,\vartheta}\left(u,v;\omega;x_1,...,x_s;\frac{\eta}{t^m}\right)$$

from the definition (5.2) into the left-hand side of (5.3) yields the following result:

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} a_k \frac{{}_{\mathcal{B}}\mathcal{A}_{n-mk}(u,v;\omega)}{(n-mk)!} \Omega^{\omega}_{\mu+\vartheta k}(x_1,x_2,...,x_s)\eta^k t^{n-mk} \tag{5.4}$$

By inverting the order of summation in equation (5.4) and substituting n for $n+mk$, the result can be expressed as

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k \frac{{}_{\mathcal{B}}\mathcal{A}_{n}(u,v;\omega)}{n!} \Omega^{\omega}_{\mu+\vartheta k}(x_1,x_2,...,x_s)\eta^k t^{n} \\ &= \left(\sum_{n=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{n}(u,v;\omega)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty} a_k\, \Omega^{\omega}_{\mu+\vartheta k}(x_1,x_2,...,x_s)\eta^k\right) \\ &= A(t)(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}} \Lambda^{\omega}_{\mu,\vartheta}(x_1,x_2,...,x_s;\eta). \end{aligned}$$

which evidently completes the proof of Theorem 5.1. □

Consider the multivariable function as follows

$$\Omega^{\omega}_{\mu+\vartheta k}(x_1,x_2,...,x_s)$$

we can provide more applications of Theorem 5.1 in terms of simpler functions of one or more variables. For instance, if we taking

$$s=1,\ x_1=x \quad \text{and} \quad \Omega^{\omega}_{\mu+\vartheta k}(x) = B_{\mu+\vartheta k}(x;\omega)$$

in Theorem 5.1, where the degenerate Bernoulli polynomials [5,6]

$$\sum_{s=0}^{\infty} B_s(x;\omega)\frac{t^s}{s!} = \frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}(1+\omega t)^{\frac{x}{\omega}}.$$

The following result yields a class of bilateral generating function for Appell type degenerate Bell polynomials and degenerate Bernoulli polynomials.

Corollary 5.2. *If*

$$\Lambda^{\omega}_{\mu,\vartheta}(x;z) := \sum_{k=0}^{\infty} a_k B_{\mu+\vartheta k}(x;\omega) z^k \tag{5.5}$$

$$(a_k \neq 0, \quad \mu,\vartheta \in \mathbb{N}_0),$$

and

$$\mathcal{K}_{n,m,\mu,\vartheta}\left(u,v,x;\omega;\rho\right):=\sum_{k=0}^{[n/m]}a_k\frac{{}_{\mathcal{B}}\mathcal{A}_{n-mk}\left(u,v;\omega\right)}{(n-mk)!}B_{\mu+\vartheta k}\left(x;\omega\right)\rho^k, \tag{5.6}$$

then

$$\sum_{n=0}^{\infty}\mathcal{K}_{n,m,\mu,\vartheta}\left(u,v,x;\omega;\frac{\eta}{t^m}\right)t^n=A\left(t\right)\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}\Lambda^{\omega}_{\mu,\vartheta}\left(x;\eta\right),\quad\left(m\in\mathbb{N}\right). \tag{5.7}$$

Remark 5.3. Let $a_k=\frac{1}{k!}$, $\mu=0$ and $\vartheta=1$. We find that

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{{}_{\mathcal{B}}\mathcal{A}_{n-mk}\left(u,v;\omega\right)}{(n-mk)!}\frac{B_k\left(x;\omega\right)}{k!}\eta^k t^{n-mk}\\&=\left(\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_n\left(u,v;\omega\right)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}B_k\left(x;\omega\right)\frac{\eta^k}{k!}\right)\\&=A\left(t\right)\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}\left(1+\omega\eta\right)^{\frac{x}{\omega}}\frac{\eta}{\left(1+\omega\eta\right)^{\frac{1}{\omega}}-1}.\end{aligned} \tag{5.8}$$

By choosing

$$s=1,\ x_1=x\quad\text{and}\quad\Omega^{\omega}_{\mu+\vartheta k}\left(x\right)=\mathcal{E}_{\mu+\vartheta k}\left(x;\omega\right)$$

in Theorem 5.1, where the degenerate Euler polynomials [6]

$$\sum_{s=0}^{\infty}\mathcal{E}_s\left(x;\omega\right)\frac{t^s}{s!}=\frac{2}{\left(1+\omega t\right)^{\frac{1}{\omega}}+1}\left(1+\omega t\right)^{\frac{x}{\omega}}.$$

The following conclusion is obtained, which gives a class of bilateral generating function for the degenerate Bell polynomials and the Appell type degenerate Euler polynomials.

Corollary 5.4. *If*

$$\Lambda^{\omega}_{\mu,\vartheta}\left(x;z\right):=\sum_{k=0}^{\infty}a_k\mathcal{E}_{\mu+\vartheta k}\left(x;\omega\right)z^k \tag{5.9}$$

$$\left(a_k\neq 0,\quad \mu,\vartheta\in\mathbb{N}_0\right),$$

and

$$\mathcal{M}_{n,m,\mu,\vartheta}\left(u,v,x;\omega;\rho\right):=\sum_{k=0}^{[n/m]}a_k\frac{{}_{\mathcal{B}}\mathcal{A}_{n-mk}\left(u,v;\omega\right)}{(n-mk)!}\mathcal{E}_{\mu+\vartheta k}\left(x;\omega\right)\rho^k, \tag{5.10}$$

then

$$\sum_{n=0}^{\infty}\mathcal{M}_{n,m,\mu,\vartheta}\left(u,v,x;\omega;\frac{\eta}{t^m}\right)t^n=A\left(t\right)\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}\Lambda^{\omega}_{\mu,\vartheta}\left(x;\eta\right),\quad\left(m\in\mathbb{N}\right). \tag{5.11}$$

Remark 5.5. Let $a_k=\frac{1}{k!}$, $\mu=0$ and $\vartheta=1$. We find that

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{{}_{\mathcal{B}}\mathcal{A}_{n-mk}\left(u,v;\omega\right)}{(n-mk)!}\frac{\mathcal{E}_k\left(x;\omega\right)}{k!}\eta^k t^{n-mk}\\&=\left(\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_n\left(u,v;\omega\right)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\mathcal{E}_k\left(x;\omega\right)\frac{\eta^k}{k!}\right)\\&=A\left(t\right)\left(1+\omega t\right)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}\left(1+\omega\eta\right)^{\frac{x}{\omega}}\frac{2}{\left(1+\omega\eta\right)^{\frac{1}{\omega}}+1}.\end{aligned} \tag{5.12}$$

Finally, using the generating function of Appell type degenerate Bell polynomials, in Theorem 5.1, we determine

$$s=2,\ x_1=x,\ x_2=y \quad \text{and} \quad \Omega^{\omega}_{\mu+\vartheta k}(x,y)={}_{\mathcal{B}}\mathcal{A}_{\mu+\vartheta k}(x,y;\omega).$$

Then, the following conclusion is obtained, which gives a class of bilinear generating function for for these polynomials.

Corollary 5.6. *If*

$$\Lambda^{\omega}_{\mu,\vartheta}(x,y;z):=\sum_{k=0}^{\infty}a_k\,{}_{\mathcal{B}}\mathcal{A}_{\mu+\vartheta k}(x,y;\omega)\,z^k \tag{5.13}$$

$$(a_k\neq 0,\quad \mu,\vartheta\in\mathbb{N}_0),$$

and

$$\mathcal{R}_{n,m,\mu,\vartheta}(u,v,x;\omega;\rho):=\sum_{k=0}^{[n/m]}a_k\frac{{}_{\mathcal{B}}\mathcal{A}_{n-mk}(u,v;\omega)}{(n-mk)!}\,{}_{\mathcal{B}}\mathcal{A}_{\mu+\vartheta k}(x,y;\omega)\,\rho^k, \tag{5.14}$$

then

$$\sum_{n=0}^{\infty}\mathcal{R}_{n,m,\mu,\vartheta}\left(u,v,x;\omega;\frac{\eta}{t^m}\right)t^n=A(t)\,(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}\Lambda^{\omega}_{\mu,\vartheta}(x,y;\eta),$$

$$(m\in\mathbb{N}). \tag{5.15}$$

Remark 5.7. Let $a_k=\frac{1}{k!}$, $\mu=0$ and $\vartheta=1$. We find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{{}_{\mathcal{B}}\mathcal{A}_{n-mk}(u,v;\omega)}{(n-mk)!}\frac{{}_{\mathcal{B}}\mathcal{A}_{k}(x,y;\omega)}{k!}\eta^k t^{n-mk}\\
&=\left(\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_{n}(u,v;\omega)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_{k}(x,y;\omega)\frac{\eta^k}{k!}\right)\\
&=A(t)\,(1+\omega t)^{\frac{u}{\omega}}\left(1+\omega\left(e^t-1\right)\right)^{\frac{v}{\omega}}A(\eta)\,(1+\omega\eta)^{\frac{x}{\omega}}\left(1+\omega\left(e^{\eta}-1\right)\right)^{\frac{y}{\omega}}.
\end{aligned} \tag{5.16}$$

Remark 5.8. For the appropriate choice of coefficients a_k, when expressed as different choices of the multivariable function $\Omega^{\omega}_{\mu+\vartheta k}(x_1,x_2,...,x_s)$, the result given by Theorem 5.1 can be used to obtain various families of multilinear and multilateral generating functions for Appell type degenerate Bell polynomials.

6. Conclusion

In our current research paper, we have identified a new generating function of Appell type degenerate Bell polynomials, and we give these properties that include recurrence relation, determinant representation, lowering and raising operators, difference equations, and some summation formula. We then show the properties for degenerate Bell-Bernoulli polynomials and degenerate Bell-Euler polynomials. Lastly, we examine the new degenerate Bell based special polynomials by examining the special cases and specifying the results. Finally, in this study, we derive multilinear and multilateral generating function families for Appell type degenerate Bell polynomials.

Moreover, in this study, recurrence relation, determinant representation, lowering and raising operators, difference equations, and some summation formulas given for degenerate Bell based Appell polynomials [40] are reduced to Bell based Appell polynomials in case of $\omega\to 0$. The properties given in the special cases section are reduced to Bell based Bernoulli and Bell based Euler polynomials if $\omega\to 0$.

In addition, the twice-iterated form of Appell type degenerate Bell polynomials and their different degenerate forms can be defined for upcoming studies.

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