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HESITANT FUZZY SUBGROUPS

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Abstaract − Hesitant fuzzy subgroup defined on a group G generalizes the idea of fuzzy subgroups. It mainly focuses on the multiplicity of values encountered when dealing with hesitant fuzzy sets. The preliminary concepts needed to build Hesitant fuzzy groups are discussed in this paper. Some results using the extension principle for hesitant fuzzy sets are discussed. The extension principle uses a new conjunction and disjunction operation on hesitant fuzzy sets. Various properties of composition and inverse operations on hesitant fuzzy sets are discussed before studying the structure of hesitant fuzzy groups. The concept of normal hesitant fuzzy subgroup of a group G is also studied in detail.

Keywords − *Hesitant fuzzy subgroups, hesitant fuzzy algebra, normal hesitant fuzzy subgroup.*

1 Introduction

The introduction of Fuzzy set theory by Lotfi A Zadeh [19] was a landmark which opened up new avenues for researchers. The use of membership functions to characterise a set meant that we can better simulate real world scenarios by mathematics. It brought along with it new challenges to researchers all over the world on how to introduce mathematical operations and structures into the new area.

A Rosenfeld introduced fuzzy groups [10] by applying fuzzy set theory to generalize some of the basic concepts of groups. Some mathematical structures which would intuitively seem to be fuzzy did not satisfy the stated definitions. Anthony and Sherwood [1] have redefined the fuzzy algebraic structures (giving examples) to meet such requirements. Seselja and Tepavcevic [11] have taken fuzzy subgroups to be mappings from a group to a partially ordered set and have studied them from a general point of view. Six kinds of fuzzy homomorphisms have been introduced in [7]. Chen and Gu [2] have dealt with fuzzy factor groups and the first fundamental theorem of isomorphism of fuzzy groups. Dixit et al. [6] have studied the union of fuzzy subgroups addressing some of the already existing queries in this area. Dib and

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Hassan [5] have defined the concept of a normal fuzzy subgroup in a fuzzy group. Mordeson and Bhutani [8] have illustrated all these concepts regarding fuzzy group theory. The concept of homologous fuzzy subgroups of a group G is introduced in [20]. A new kind of fuzzy group based on fuzzy binary operations is proposed in [18].

Hesitant fuzzy set was introduced by Vicenc Torra [12],[13] which further characterised an element by a set of membership values thereby decreasing the loss of information during fuzzification. Distance measures for Hesitant Fuzzy Sets (HFSs) have been investigated in [17]. Studies on hesitant fuzzy information aggregation techniques and their application in decision making can be seen in [15],[16]. Zhu et al. [21] have defined a hesitant fuzzy geometric Bonferroni mean as an extension of the geometric Bonferroni mean to hesitant fuzzy environment. Rodriguez et al. [9] have introduced hesitant fuzzy linguistic term sets which gives flexibility in applications involving hesitant situations under qualitative settings. Wei [14] has studied the Hesitant fuzzy prioritized operators and has illustrated their application in the hesitant fuzzy multiple attribute decision making problems in which the attributes are in different priority level.

This paper mainly introduces algebraic structures on the Hesitant fuzzy domain. Some of the preliminary definitions and results regarding hesitant fuzzy groups and homomorphisms between them have been discuseed in [4]. The paper begins by discussing basic results regarding Hesitant fuzzy set theory. It then moves on to study Hesitant fuzzy alpha-cut and the extension principle in Hesitant Fuzzy set domain. The next section discusses certain properties and results regarding Hesitant fuzzy subgroups. The last section examines the notion of a normal hesitant fuzzy subgroup and studies its general structure and properties.

2 Basic Concepts

This section introduces the basic concepts in Hesitant fuzzy set theory.

Definition 2.1 ([13]). Let X be a reference set then a Hesitant fuzzy set(HFS) on X is defined in terms of a function h that when applied to X returns a subset of $[0,1]$ $h : X \rightarrow P[0,1]$ where $P[0,1]$ denotes power set of $[0,1]$.

The set of all hesitant fuzzy subsets of X is called the hesitant fuzzy power set of X and is denoted by $HF(X)$.

The empty hesitant set, the full hesitant set, the set to represent complete ignorance for x and the nonsense set are defined as follows:
empty set : $h_0(x) = \{0\} \quad \forall\, x \in X$
full set : $h_X(x) = \{1\} \quad \forall\, x \in X$
complete ignorance $h(x) = [0,1]$
set for a nonsense $x : h(x) = \phi$

Given an hesitant fuzzy set h, its lower and upper bound are defined as follows :
$h^-(x) = \min h(x)$
$h^+(x) = \max h(x)$

For convenience we call $h(x)$ a hesitant fuzzy element (HFE) [15]. Let $l(h(x))$ be the number of values in $h(x)$.

Definition 2.2 ([15])**.** Score for a HFE , $s(h) = \frac{1}{l(h)} \sum_{\gamma \in h} \gamma$ is called the score function of h.

Note : If the HFE is infinite then $s(h(x)) = \frac{1}{2}(\inf(h(x)) + \sup(h(x)))$. .

Definition 2.3. [3] Let $h \in HF(X)$. Then the set $\bigcup_{x \in X} h(x)$ is called the image of h and is denoted by $h(X)$. The set $\{x | x \in X, s(h(x)) > 0\}$, is called the support of h and is denoted by h^* . h is called finite hesitant fuzzy set if h^* is a finite set , and an infinite hesitant fuzzy set otherwise.

Definition 2.4. [4] Let $Y \subseteq X$ and $A \subseteq [0, 1]$. We define $A_Y \in HF(X)$ as follows :

$$A_Y(x) = \begin{cases} A, & \text{for } x \in Y \\ \{0\}, & \text{for } x \in X \setminus Y \end{cases}$$

If Y is a singleton, say $\{y\}$, then $A_{\{y\}}$ is called a hesitant fuzzy point (or hesitant fuzzy singleton), and is denoted by y_A. Let $\{1\}_Y$ denote the characteristic function of Y. If S is a set of hesitant fuzzy singletons, then we let $foot(S) = \{y \in X | y_A \in S\}$.

Definition 2.5 ([3])**.** Hesitant Equality : Let h_1 and h_2 be two hesitant fuzzy sets on X, then we say that h_1 is equal to h_2 (denoted $h_1 = h_2$) iff $h_1(x) = h_2(x)\ \forall x \in X$ and h_1 is hesitantly equal to h_2 (denoted $h_1 \approx h_2$) iff $s(h_1(x)) = s(h_2(x)\ \forall x \in X$.

Definition 2.6 ([3])**.** Hesitant subset :(h_1 hesitant subset of h_2) Let h_1 and h_2 be two hesitant fuzzy sets on X, then we say that h_1 is a hesitant subset of h_2 (denoted by $h_1 \preceq h_2$) iff $s(h_1(x)) \leq s(h_2(x))\ \forall x \in X$.

Definition 2.7 ([3])**.** Hesitant proper subset : $h_1 \prec h_2$ if $s(h_1(x)) \leq s(h_2(x) \forall x \in X$ and $s(h_1(x)) < s(h_2(x)$ for atleast one $x \in X$.

Definition 2.8 ([13])**.** Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, their union represented by $h_1 \bigcup h_2$ is defined as

$$\begin{aligned} (h_1 \bigcup h_2)(x) &= \left\{\gamma \in (h_1(x) \bigcup h_2(x) / \gamma \geq \max(h_1^-, h_2^-)\right\} \\ &= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}. \end{aligned}$$

Definition 2.9 ([13])**.** Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, their intersection represented by $h_1 \bigcap h_2$ is defined as

$$\begin{aligned} (h_1 \bigcap h_2)(x) &= \left\{\gamma \in (h_1(x) \bigcup h_2(x) / \gamma \leq min(h_1^+, h_2^+)\right\}. \\ &= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}. \end{aligned}$$

Definition 2.10. (max union and min intersection). Let H and G be two subsets of $[0,1]$. Then we define the max union of H and G denoted by $\bigcup_{\max}$, as
$H\bigcup_{\max}G = \bigcup\limits_{a\in H,b\in G} \max\{a,b\}$

The min intersection of H and G denoted by $\bigcap_{min}$ is defined as
$H\bigcap_{\min}G = \bigcap\limits_{a\in H,b\in G} \min\{a,b\}$

For any collection , $\{h_i|i\in I\}$, of hesitant fuzzy subsets of X, where I is a nonempty index set, then $\forall x\in X$

$$(\bigcup_{i\in I})h_i(x) = \bigcup_{i\in I}\cdot_{\max}h_i(x)$$
$$(\bigcap_{i\in I})h_i(x) = \bigcap_{i\in I}\cdot_{\min}h_i(x)$$

Note : The above definition applies to the case where we are taking the union of h_i 's the hesitant values for the different values of x in the same set in contrast to the definition of Torra where we are taking the union over two different sets.

Definition 2.11. [4] Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, we define a score based intersection of h_1 and h_2 (denoted by $h_1\widetilde{\wedge}h_2$)as

$$(h_1\widetilde{\wedge}h_2)(x) = \begin{cases} h_1(x) & \text{if } h_1(x)\prec h_2(x)\\ h_2(x) & \text{if } h_2(x)\prec h_1(x)\\ h_1(x)\bigcup h_2(x) & \text{if } h_1(x)\approx h_2(x)\end{cases}$$

and a score based union of h_1 and h_2 (denoted by $h_1\widetilde{\vee}h_2$)as

$$(h_1\widetilde{\vee}h_2)(x) = \begin{cases} h_1(x) & \text{if } h_1(x)\succ h_2(x)\\ h_2(x) & \text{if } h_2(x)\succ h_1(x)\\ h_1(x)\bigcup h_2(x) & \text{if } h_1(x)\approx h_2(x)\end{cases}$$

For a collection, $\{h_i|i\in I\}$ of hesitant fuzzy subsets of X, where I is a non empty index set, we have $\forall x\in X$

$$(\widetilde{\vee}_{i\in I}h_i)(x) = \widetilde{\vee}_{i\in I}h_i(x)$$

$$(\widetilde{\wedge}_{i\in I}h_i)(x) = \widetilde{\wedge}_{i\in I}h_i(x)$$

Definition 2.12. Let $h_A\in HF(X)$ and $\alpha\in[0,1]$ then the $\alpha-level$ cut set of hesitant fuzzy set h_A , denoted by $h_{A\alpha}$ is $h_{A\alpha}=\{x\in X|s(h_A(x))\geq\alpha\}$.
$h_{A\alpha+}=\{x\in X|s(h_A(x))>\alpha\}$. is called strong $\alpha-level$ cut set of h_A . We can define $h^+_{A\alpha}=\left\{x\in X|s(h^+_A(x))\geq\alpha\right\}$ and $h^-_{A\alpha}=\left\{x\in X|s(h^-_A(x))\geq\alpha\right\}$. Then clearly $h^-_{A\alpha}\subseteq h_{A\alpha}\subseteq h^+_{A\alpha}\ \forall\alpha\in[0,1]$

Note : When the set under consideration is clear we can denote $h_{A\alpha}$ by h_α , $h^+_{A\alpha}$ by h^+_α and $h^-_{A\alpha}$ by h^-_α.

Lemma 2.13. Let $h_A,h_B\in HF(X)$, then the following assertions hold,

1. $h_A \preceq h_B, \alpha \in [0,1] \Rightarrow h_{A\alpha} \subseteq h_{B\alpha}$
2. $\alpha \leq \beta; \alpha, \beta, \in [0,1] \Rightarrow h_{A\beta} \subseteq h_{A\alpha}$
3. $h_A = h_B \Leftrightarrow h_{A\alpha} = h_{B\alpha}$

Definition 2.14. Let I be a nonempty index set and let $\{X_i | I \in I\}$ be a collection of nonempty sets. Let X denote the Cartesian product of the X_i's, namely,

$$X = \prod_{i\in I} X_i = \{(x_i)_{i\in I} | x_i \in X_i, i \in I\}$$

Let $h_i \in HF(X_i)$ for all $i \in I$. Define the fuzzy subset h of X by $h(x) = \widetilde{\wedge}_{i\in I} h_i(x_i) \quad \forall x = (x_i)_{I\in I} \in X$. Then h is called the complete direct product of the h_i's and is denoted by $h = \widetilde{\prod}_{i\in I} h_i$.

Definition 2.15. [4] (Extension Principle). Let f be a function from X into Y, and let $h_1 \in HF(X)$ and $h_2 \in HF(Y)$. Define the hesitant fuzzy subsets $f(h_1) \in HF(Y)$ and $f^{-1}(h_2) \in HF(Y)$ by $\forall y \in Y$,

$$f(h_1)(y) = \begin{cases} \widetilde{\vee}\,\{h_1(x) | x \in X, f(x) = y\}\,; \text{ if } f^{-1}(y) \neq \phi, \\ \{0\}\,, \text{ otherwise} \end{cases}$$

and $\forall x \in x, f^{-1}(h_2)(x) = h_2(f(x))$., Then $f(h_1)$ is called the image of h_1 under f and $f^{-1}(h_2)$ is called the pre image of h_2 under f.

Theorem 2.16. Let f be a function from X into Y and g a function from Y into Z. Then the following assertions hold.

1. $h_1 \preceq h_2 \Rightarrow f(h_1) \preceq f(h_2) \quad \forall h_1, h_2 \in HF(X)$
2. $h_1 \preceq h_2 \Rightarrow f^{-1}(h_1) \preceq f^{-1}(h_2) \quad \forall h_1, h_2 \in HF(Y)$
3. $f^{-1}(f(h_1)) \succeq h_1 \quad \forall h_1 \in HF(X)$. In particular if f is a injection, then $f^{-1}(f(h_1)) = h_1 \quad \forall h_1 \in HF(X)$.
4. $f(f^{-1}(h_2)) \preceq h_2 \quad \forall h_2 \in HF(X)$. In particular if f is a surjection, then $f(f^{-1}(h_2)) = h_2 \quad \forall h_2 \in HF(Y)$.
5. $f(h_1) \preceq h_2 \Leftrightarrow h_1 \preceq f^{-1}(h_2) \quad \forall h_1 \in HF(X)$ and $h_2 \in HF(Y)$.
6. $g(f(h_1)) = (g \circ f)(h_1) \quad \forall h_1 \in HF(X)$ and $f^{-1}(g^{-1}(h_3)) = (g \circ f)^{-1}(h_3) \quad \forall h_3 \in HF(Z)$.

Proof. 1.

$$\begin{aligned} h_1 \preceq h_2 &\Rightarrow s(h_1(x)) \leq s(h_2(x)) \quad \forall x \in X \qquad (1) \\ f(h_1)(y) &= \widetilde{\vee}\,\{h_1(x) | x \in X, f(x) = y\} \\ &\preceq \widetilde{\vee}\,\{h_2(x) | x \in X, f(x) = y\} \qquad \text{from (1)} \\ &= f(h_2(y)) \end{aligned}$$

2. $f^{-1}(h_1)(x) = h_1(f(x)) \preceq h_2(f(x)) = f^{-1}(h_2)(x)$

3.

$$\begin{aligned} f^{-1}(f(h_1))(x) &= f(h_1)(f(x)) \\ &= \widetilde{\vee}\{h_1(x')|x' \in X, f(x') = f(x)\} \\ &\succeq h_1 \quad \forall x \in X \end{aligned}$$

In particular if f is an injection then
$f^{-1}(f(h_1))(x) = \widetilde{\vee}\{h_1(x')|x' \in X, f(x') = f(x)\} = h_1(x)$

4. Let $h_2 \in HF(Y)$ then

$$\begin{aligned} f(f^{-1}(h_2))(y) &= \widetilde{\vee}\left\{f^{-1}(h_2(x))|x \in X, f(x) = y\right\} \\ &= \widetilde{\vee}\{h_2(f(x))|x \in X, f(x) = y\} \\ &= \begin{cases} h_2(y), & \text{for } y \in f(X) \\ \{0\}, & \text{otherwise} \end{cases} \\ &\preceq h_2(y) \quad \forall y \in Y \end{aligned}$$

Thus $f(f^{-1}(h_2))(y) = h_2(y) \quad \forall y \in Y$ if f is a surjection.

5. $f(h_1) \preceq h_2 \Rightarrow f^{-1}(f(h_1)) \preceq f(h_2) \quad \forall y \in Y$ from (2)
$\Rightarrow h_1 \preceq f^{-1}(f(h_1)) \preceq f(h_2)$ from (3)
$\Rightarrow h_1 \preceq f(h_2)$
Conversely, $h_1 \preceq f^{-1}(h_2)$ from (1)
$\Rightarrow f(h)_1 \preceq f(f^{-1}(h_2))$
$\Rightarrow f(h_1) \preceq f(f^{-1}(h_2)) \preceq h_2$ from (4)
$\Rightarrow f(h_1) \preceq h_2$

6. Consider any $h \in HF(X)$ and any $z \in Z$. Then

$$\begin{aligned} g(f(h))(z) &= \widetilde{\vee}\{f(h)(y)|y \in Y, g(y) = z\} \\ &= \widetilde{\vee}\left\{\widetilde{\vee}\{h(x)|x \in X, f(x) = Y\}|y \in Y, g(y) = z\right\} \\ &= \widetilde{\vee}\{h(x)|x \in X, (g \circ f)(x) = z\} \\ &= (g \circ f)(h)(z) \quad \forall z \in Z \end{aligned}$$

Further, $\forall h' \in HF(Z)$ and $\forall x \in X$,

$$\begin{aligned} ((g \circ f)^{-1}(h')(x)) &= h'(g(f(x))) \\ &= g^{-1}(h')(f(x)) \\ &= f^{-1}(g^{-1}(h'))(x) \end{aligned}$$

□

3 Hesitant Fuzzy Subgroups

This section discusses the concept of composition in the case of hesitant fuzzy sets. Certain results regarding Hesitant fuzzy subgroups are discussed. Let G denote an arbitrary group with a multiplicative binary operation and identity e.

Definition 3.1. [4] We define the binary operation $\circ$ on $HF(G)$ and the unary operation $^{-1}$ on $HF(G)$ as follows : $\forall h_1, h_2 \in HF(G)$ and $\forall x \in G$,
$(h_1 \circ h_2)(x) = \widetilde{\vee}\left\{h_1(x)\widetilde{\wedge}h_2(x)|y, z \in G, yz = x\right\}$ and $h_1^{-1}(x) = h(x^{-1})$

We call $h_1 \circ h_2$ the product of h_1 and h_2, and h_1^{-1} the inverse of h_1.

Lemma 3.2. Let $h, h', h_i \in HF(G), i \in I$. Let $A = \widetilde{\vee}\{h(x)|x \in G\}$. Then the following assertions hold:

1. $(h \circ h')(x) = \widetilde{\vee}_{y\in G}(h(y)\widetilde{\wedge}h'(y^{-1}x))$
$= \widetilde{\vee}_{y\in G}(h(xy^{-1})\widetilde{\wedge}h'(y)) \quad \forall x \in G$
2. $(A_y \circ h)(x) = h(y^{-1}x) \quad \forall x, y \in G$
3. $(h \circ A_y)(x) = h(xy^{-1}) \quad \forall x, y \in G$
4. $(h^{-1})^{-1} = h$
5. $h \preceq h^{-1} \Leftrightarrow h^{-1} \preceq h$
$\Leftrightarrow h \approx h^{-1}$
$\Leftrightarrow h(x) \preceq h(x^{-1}) \quad \forall x \in G$
6. $h \preceq h' \Leftrightarrow h^{-1} \preceq (h')^{-1}$
7. $(\widetilde{\vee}_{i\in I}h_i)^{-1} = \widetilde{\vee}_{i\in I}h_i^{-1}$
8. $(\widetilde{\wedge}_{i\in I}h_i)^{-1} = \widetilde{\wedge}_{i\in I}h_i^{-1}$
9. $(h \circ h')^{-1} = h^{-1} \circ (h')^{-1}$

Definition 3.3. [4] Let $h \in HF(G)$. Then h is called a hesitant fuzzy subgroup of G if

(i) $h(xy) \succeq h(x)\widetilde{\wedge}h(y) \quad \forall x, y \in G$ and

(ii) $h(x^{-1}) \succeq h(x) \quad \forall x \in G$

Denote by $HFG(G)$, the set of all Hesitant Fuzzy subgroups of G . If $h \in HFG(G)$, then let $h_* = \{x \in G|h(x) = h(e)\}$. From (i) of the above definition we have $h(x^n) \succeq h(x) \quad \forall x \in G,$ where $n \in \mathbb{N}$.

Example 3.4. Let $G = \{e, a, b, c\}$ be the Klein's 4-group. $h : G \to [0, 1]$ G be a hesitant fuzzy set with $h(e) = \{1\}, h(a) = \left\{\frac{5}{12}, \frac{11}{12}\right\}$, $h(b) = \left\{\frac{9}{20}, \frac{3}{4}, \frac{4}{5}\right\}, h(c) = \left\{\frac{2}{3}\right\}$. Then h is a hesitant fuzzy subgroup of G .

Lemma 3.5. $h \in HF(G)$ is a hesitant fuzzy subgroup iff $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y) \quad \forall x, y \in G$

Proof. $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y^{-1}) \succeq h(x)\widetilde{\wedge}h(y) \quad (\because h(y) \preceq h(y^{-1}))$
Conversely, To prove (ii)
$h(x) = h(xe) = h(xx^{-1}x) \succeq h(x)\widetilde{\wedge}h(xx^{-1}) \succeq h(x)\widetilde{\wedge}(h(x)\widetilde{\wedge}h(x^{-1}))$
$\succeq h(x)\widetilde{\wedge}h(x^{-1})$
$\Rightarrow h(x) \succeq h(x^{-1})$
Now to prove that $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y)$
$h(xy) \succeq h(x)\widetilde{\wedge}h(y^{-1}) \succeq h(x)\widetilde{\wedge}h(y); (\because h(y) \preceq h(y^{-1}))$ □

Lemma 3.6. Let $h \in HFG(G)$. Then $\forall x \in G$,

(i) $h(e) \succeq h(x)$

(ii) $h(x) \approx h(x^{-1})$

Proof. Let $x \in G$

(i) $h(e) = h(xx^{-1}) \succeq h(x)\widetilde{\wedge}h(x^{-1}) \succeq h(x)\widetilde{\wedge}h(x) = h(x)$

(ii) $h(x) = h((x^{-1}))^{-1} \succeq h(x^{-1})$
Hence $h(x) \approx h(x^{-1})$

□

Lemma 3.7. If $h \in HFG(G)$ and if $x, y \in G$ with $h(x) \succeq h(y)$, then

$$h(xy) \approx h(x)\widetilde{\wedge}h(y)$$

Proof. Let $h(x) \succeq h(y)$.
Then $h(y) = h(x^{-1}xy) \succeq h(x^{-1})\widetilde{\wedge}h(xy) \approx h(x)\widetilde{\wedge}h(xy)$
$h(y) \succeq h(x)\widetilde{\wedge}h(xy)$ and since $h(x) \succeq h(y)$ it follows that
$h(y) \succeq h(xy) \succeq h(x)\widetilde{\wedge}h(y) \approx h(y)$
From this it follows that
$h(xy) \succeq h(x)\widetilde{\wedge}h(y)$ and $h(x)\widetilde{\wedge}h(y) \approx h(y) \succeq h(xy)$
Hence the result □

Lemma 3.8. If h is a hesitant fuzzy subgroup of G then $H = \{x \in X : h(x) = \{1\}\}$ is either empty or is a subgroup of G.

Proof. If $x, y \in H$ then
$h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y^{-1}) \approx h(x)\widetilde{\wedge}h(y) \approx \{1\}\widetilde{\wedge}\{1\} = \{1\}$ Therefore $xy^{-1} \in H$. Hence H is a subgroup of G. □

Lemma 3.9. If h is a hesitant fuzzy subgroup of a group G and if there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} s(h(x_n)) = 1$ then $s(h(e)) = 1$ where e is the identity in G.

Proof. Let $x \in G$. Then
$s(h(e)) = s(h(xx^{-1})) \geq s(h(x))\widetilde{\wedge}s(h(x^{-1})) = s(h(x))$
Therefore for each n, $s(h(e)) \geq s(h(x_n))$
Since $1 \geq s(h(e)) \geq \lim_{n\to\infty} s(h(x_n)) = 1 \Rightarrow s(h(e)) = 1$ □

Note : If $s(h(x)) = 1$ then clearly $h(x) = \{1\}$.

Lemma 3.10. Let h be a hesitant fuzzy subgroup of a group G . If $h(xy^{-1}) = \{1\}$ then $h(x) \approx h(y)$.

Proof. $h(x) = h((xy^{-1})y) \succeq h(xy^{-1})\widetilde{\wedge}h(y) \approx \{1\}\,\widetilde{\wedge}h(y) = h(y)$
$\Rightarrow h(x) \succeq h(y)$
Similarly we can prove that $h(y) \succeq h(x)$.
Hence we can conclude that $h(x) \approx h(y)$. □

Lemma 3.11. Let h be a hesitant fuzzy set on a group G. If $h(e) = \{1\}$ and $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y) \quad \forall x, y \in G$ then h is a hesitant fuzzy subgroup of G.

Proof. $h(y^{-1}) = h(ey^{-1}) \succeq h(e)\widetilde{\wedge}h(y) = \{1\}\,\widetilde{\wedge}h(y) = h(y)$.
Similarly $h(y) \succeq h(y^{-1}) \Rightarrow h(y) \approx h(y^{-1})$
$h(xy) = h(x(y^{-1})^{-1}) \succeq h(x)\widetilde{\wedge}h(y^{-1}) \approx h(x)\widetilde{\wedge}h(y)$
Hence h is a hesitant fuzzy subgroup of G. □

Lemma 3.12. Let $h \in HF(G)$. If h is a hesitant fuzzy subgroup of G then h_α is a subgroup of G $\forall \alpha \in [0, s(h(e))]$.

Proof. Suppose $h \in HF(G)$.
Let $\alpha \in [0, s(h(e))]$. Since $h(e) \succeq h(x)$ $\forall x \in G, e \in h_a$. Thus $h_\alpha \neq \phi$
Let $x, y \in h_\alpha$. Then $s(h(x)) \geq \alpha, s(h(y)) \geq \alpha$. Since h is a hesitant fuzzy subgroup $h(xy^{-1}) \succeq h(x)\widetilde{\wedge}h(y) \geq \alpha\widetilde{\wedge}\alpha = \alpha \Rightarrow s(h(xy^{-1})) \geq \alpha$.
Hence $xy^{-1} \in h_\alpha$ and so h_α is a subgroup of G.
Converse, given $h(xy) \succeq h(x)\widetilde{\wedge}h(y)$,
Suppose h_α is a subgroup of G $\forall \alpha \in [0, s(h(e))]$, then $\forall \alpha \in h(G)$ we must have $e \in h_\alpha$ and so it follows that $s(h(e)) \geq \alpha$. Let $x, y \in G$ and let $s(h(x)) = \alpha$ and $s(h(y)) = \beta$. Let $\gamma = \alpha \wedge \beta$. Then $x, y \in h_\gamma$ and $\gamma \leq s(h(e))$. By hypothesis h_γ is a subgroup of G and so $xy^{-1} \in h_\gamma$. Hence $s(xy^{-1}) \geq \gamma = \alpha \wedge \beta = s(h(x)) \wedge s(h(y))$.
Thus h is a HF subgroup of G. □

Theorem 3.13. $h \in HFG(G)$ iff h satisfies the following conditions :

1. $h \circ h \preceq h$
2. $h^{-1} \preceq h$

Proof.

$$\begin{aligned}(h \circ h)(x) &= \widetilde{\vee}\left\{h(y)\widetilde{\wedge}h(z) | y, z \in G, yz = x\right\} \\ &= h(y_*)\widetilde{\wedge}h(z_*) \text{ for some } y_*z_* \in G \text{ such that } y_*z_* = x \\ &\preceq h(x) \quad (\because h(x) = h(yz)) \succeq h(y)\widetilde{\wedge}h(z))\end{aligned}$$

Converse, let $h \circ h \preceq h$
$\Rightarrow \widetilde{\vee}\left\{h(y)\widetilde{\wedge}h(z) | y, z \in G, yz = x\right\} \preceq h(x)$
$\Rightarrow h(y)\widetilde{\wedge}h(z) \preceq h(x) \quad \forall t, z \in G$ such that $yz = x$
$\Rightarrow h(y)\widetilde{\wedge}h(z) \preceq h(yz)$, Hence proved. □

Theorem 3.14. Let $h_1, h_2 \in HFG(G)$. Then $h_1 \circ h_2 \in HFG(G)$ iff $h_1 \circ h_2 \approx h_2 \circ h_1$.

Proof. Suppose $h_1 \circ h_2 \in HFG(G)$.
Then $h_1 \circ h_2 \approx h_1^{-1} \circ h_2^{-1} \approx (h_2 \circ h_1)^{-1} \approx h_2 \circ h_1$.
Conversely suppose that $h_1 \circ h_2 \approx h_2 \circ h_1$ then
$(h_1 \circ h_2)^{-1} \approx (h_2 \circ h_1)^{-1} \approx h_1^{-1} \circ h_2^{-1} \preceq h_1 \circ h_2$ and
$(h_1 \circ h_2) \circ (h_1 \circ h_2) = h_1 \circ (h_2 \circ h_1) \circ h_2 \approx h_1 \circ (h_1 \circ h_2) \circ h_2$
$= (h_1 \circ h_1) \circ (h_2 \circ h_2) \preceq h_1 \circ h_2$
Hence by Theorem 3.13, $h_1 \circ h_2 \in HFG(G)$. □

Theorem 3.15. For $i \in I$. Let $h_i \in HFG(G)$. Then $\widetilde{\wedge}_{i\in I} h_i \in HFG(G)$.

Proof. Let $x, y \in G$. then
$(\widetilde{\wedge}_{i\in I} h_i)(xy^{-1}) = \widetilde{\wedge}\{h_i(xy^{-1})|i \in I\} \widetilde{\wedge} \{h_i(x)\widetilde{\wedge} h_i(y)|i \in I\}$
$\approx (\widetilde{\wedge}\{h_i(x)|i \in I\})\widetilde{\wedge}(\widetilde{\wedge}\{h_i(y)|i \in I\}) \approx (\widetilde{\wedge}_{i\in I} h_i)(x)\widetilde{\wedge}(\widetilde{\wedge}_{i\in I} h_i)(y)$. □

4 Normal Hesitant Fuzzy Subgroups, Homomorphisms and Isomorphisms

This section discusses the notion of a Normal Hesitant fuzzy subgroup. Normal Hesitant fuzzy subgroup is an important concept when it comes to the study of Hesitant fuzzy group theory. This section moves on to discuss various results regarding them.

Definition 4.1. [4] Let G be a group . A hesitant fuzzy subgroup h of a group G is called normal if $h(x) \approx h(y^{-1}xy) \quad \forall x, y \in G$. Let $NHF(G)$ denote the set of all normal hesitant fuzzy subgroups of G.

Theorem 4.2. Let $h \in HF(G)$ then the following conditions are equivalent

1. $h(y) \approx h(xyx^{-1})$
2. $h \circ h' \approx h' \circ h \quad \forall h' \in HF(G)$

Proof. Let $x \in G$. We have $h(y) \approx h(xyx^{-1})$. Then
$(h \circ h')(x) \approx \widetilde{\vee}_{y\in G}(h(xy^{-1})\widetilde{\wedge} h'(y)) \quad \forall x \in G$
$\approx \widetilde{\vee}_{y\in G}(h(x^{-1}xy^{-1}x)\widetilde{\wedge} h'(y)) \approx \widetilde{\vee}_{y\in G}(h(y^{-1}x)\widetilde{\wedge} h'(y))$
$\approx \widetilde{\vee}_{y\in G}(h'(y)\widetilde{\wedge} h(y^{-1}x)) \approx (h' \circ h)(x)$
Hence $h \circ h' \approx h' \circ h$
(2) ⇒ (1): We have $h \circ h' \approx h' \circ h \quad \forall h' \in HF(G)$
$\{1\}_{y^{-1}} \circ h \approx h \circ \{1\}_{y^{-1}} \quad y \in G$
$(\{1\}_{y^{-1}} \circ h)(x) \approx \widetilde{\vee}_{z\in G}(\{1\}_{y^{-1}}(z)\widetilde{\wedge} h(z^{-1}x))$
$\approx \{1\}\widetilde{\wedge} h(yx)$ since $\{1\}_{y^{-1}}(z) = \begin{cases} \{1\} & \text{for } x = y \\ \{0\} & \text{otherwise} \end{cases}$
$\approx h(yx)$
Similarly $(h \circ \{1\}_{y^{-1}})(x) \approx h(xy)$
Now we have that $(\{1\}_{y^{-1}} \circ h)(x) \approx (h \circ \{1\}_{y^{-1}})(x) \quad \forall x \in G$
Thus $h(yx) \approx h(xy)$ □

Note : If $h_1, h_2 \in HFG(G)$ and there exists $u \in G$ such that $h_1(x) = h_2(uxu^{-1}) \quad \forall x \in G$, , then h_1 and h_2 are called conjugate hesitant fuzzy subgroups (with respect to u) and we write $h_1 = h_2^u$ where $h_2^u(x) = h_1(uxu^{-1}) \quad \forall x \in G$.

Theorem 4.3. Let $h \in HF(G)$. Then $h \in NHF(G)$ if and only if h_α is a normal subgroup of $G \quad \forall \alpha \in [0, s(h(e))]$.

Proof. Let $h \in NHF(G)$ and $\forall \alpha \in [0, s(h(e))]$ since $h \in HFG(G)$, h_α is a subgroup of G. If $x \in G$ and $y \in h_\alpha$ it follows that $s(h(xyx^{-1})) = s(h(y)) \geq \alpha$ by definition of a normal hesitant fuzzy subgroup. Thus $xyx^{-1} \in h_\alpha$. Hence h_α is a normal subgroup of G.
Conversely, Assume that h_α is a normal subgroup of $G \quad \forall \alpha \in [0, s(h(e))]$ Then we have that $h \in HFG(G)$ by lemma [3.12] .
Let $x, y \in G$ and $\alpha = s(h(y))$. Then $y \in h_\alpha$ and so $xyx^{-1} \in h_\alpha$. Hence $s(h(xyx^{-1})) \geq \alpha = s(h(y))$ i.e., we have that $s(h(xyx^{-1})) \geq s(h(y))$ Now put $y = x^{-1}yx$ and we have $s(h(x^{-1}yx)) \leq s(h(xx^{-1}yxx^{-1})) = s(h(y))$. So we have $h(y) \approx h(xyx^{-1}) \Rightarrow h \in NHF(G)$. □

Theorem 4.4. Let $h \in NHF(G)$. Then h_* and h^* are normal subgroups of G.

Proof. $h \in HFG(G)$. It follows from lemma[3.12] that h_* and h^* are subgroups of G .
Let $x \in G$ and $y \in h_*$. $h(xyx^{-1}) \approx h(y) \approx h(e)$
$\Rightarrow xyx^{-1} \in h_*$. Hence h_* is a normal subgroup of G. Now let $x \in G$ and $y \in h_*$. Since $h(xyx^{-1}) \approx h(y) \succ \{0\}$ and $xyx^{-1} \in h^*$. Therefore h^* is a normal subgroup of G. □

Note : The converse need not be true.
Example : Let G be a group and H be a subgroup of G which is not normal. Define

$$h \in HF(G) \text{ as } h(x) = \begin{cases} \left\{\frac{3}{4}\right\}; x = e \\ \left\{\frac{7}{12}, \frac{1}{3}, \frac{1}{12}\right\}; x \in H\backslash\{e\} \\ \left\{\frac{1}{2}, \frac{1}{4}\right\}; x \in G\backslash H \end{cases}$$

Then $h \in HFG(G)$. We have $h_{\frac{1}{3}} = H$ which is not normal in G. But $h_* = \{e\}$ and $h^* = G$ are normal in G.

Theorem 4.5. Let $h \in HFG(G)$. Then $\widetilde{\wedge}_{u \in G} h^u \in HFG(G)$ and is the largest normal hesitant fuzzy group of G that is contained in h.

Proof. We have $h \in HFG(G)$
$h^u(xy) = h(uxyu^{-1}) = h(uxu^{-1}uyu^{-1}) \succeq h(uxu^{-1})\widetilde{\wedge} h(uyu^{-1})$
$\Rightarrow h^u(xy) \succeq h^u(x)\widetilde{\wedge} h^u(y)$
$\Rightarrow h^u \in HFG(G) \quad \forall u \in G$
$\Rightarrow \widetilde{\wedge}_{u \in G} h^u \in HFG(G)$

For all $x \in G$ we have $\{h^u | u \in G\} = \{h^{ux} | u \in G\}$ because each $u = vx$ for some $v \in G$. Thus $\widetilde{\wedge}_{u \in G} h^u(xyx^{-1}) = \widetilde{\wedge}_{u \in G} h(uxyx^{-1}u^{-1}) = \widetilde{\wedge}_{u \in G} h(uxy(ux)^{-1}) = \widetilde{\wedge}_{u \in G} h^{ux}(y)$.

Hence by the definition of normal hesitant fuzzy group we have $\widetilde{\wedge}_{u \in G} h^u \in NHF(G)$.
Let $h_1 \in NHF(G)$ with $h_1 \preceq h$ Then $h_1 \preceq h \Rightarrow s(h_1(x)) \leq s(h(x)) \quad \forall x \in G$
$\Rightarrow s(h_1(uxu^{-1})) \leq s(h(uxu^{-1}))$
$\Rightarrow h_1^u(x) \preceq h^u(x) \quad \forall x \in G$
$\Rightarrow h_1^u \preceq h^u$
Since $h_1 \in NHF(G)$ we have $h_1(uxu^{-1}) \approx h_1(u)$.

$\Rightarrow h_1 \approx h_1^u \preceq h^u \quad \forall u \in G$
$h_1 \preceq \widetilde{\wedge}_{u \in G} h^u$
Hence the result. □

Definition 4.6. [4] Let $h \in HFG(G)$ and $x \in G$. The hesitant fuzzy subset $h(e)_{\{x\}} \circ h$ is referred to as the left coset of h with respect to x and is written $x\widetilde{h}$ and is referred to as the right coset of h with respect to x and is written $\widetilde{h}x$.
Now,

$$\begin{aligned}(h(e)_x \circ h)(a) &= \widetilde{\vee}\left\{h(e)_x(y)\widetilde{\wedge}h(z)|y,z \in G; yz = a\right\}\\ &= \widetilde{\vee}\begin{cases}h(e)\widetilde{\wedge}h(z) \text{ for } xz = a\\ \{0\}\,\widetilde{\wedge}h(z) \text{ for } xz \neq a\end{cases}\\ &= h(e)\widetilde{\wedge}h(x^{-1}a)\\ &= h(x^{-1}a)\end{aligned}$$

Therefore $x\widetilde{h}(a) = h(x^{-1}a)$ and $\widetilde{h}x(a) = h(ax^{-1})$.

Note : We write $\widetilde{h}$ in the notation of a coset in place of the hesitant fuzzy set h so as to differentiate between the element x and the hesitant fuzzy set h. We have that if $h \in NHF(G)$ then $x\widetilde{h} = \widetilde{h}x$. Thus we call $x\widetilde{h}$ a coset of h (dropping the notion of left or right coset).

Theorem 4.7. Let $h \in HFG(G)$. Then $\forall x, y \in G$ we have $x\widetilde{h} = y\widetilde{h} \Leftrightarrow xh_* = yh_*$.

Proof. Let us take $x\widetilde{h} = y\widetilde{h}$. Then $h(e)_{\{x\}} \circ h \approx h(e)_{\{y\}} \circ h$
$\Rightarrow h(x^{-1}z) \approx h(y^{-1}z) \quad \forall z \in G$
Substituting $z = y \Rightarrow h(x^{-1}y) \approx h(y^{-1}y) \approx h(e)$.
$\Rightarrow x^{-1}y \in h_*$
Hence $xh_* = yh_*$.
Conversely Let us take $xh_* = yh_*$.
Then $x^{-1}y \in h_*$ and $y^{-1}x \in h_* \Rightarrow h(x^{-1}y) = h(e)$ and $h(y^{-1}x) = h(e)$ Hence $h(x^{-1}z) = h(x^{-1}y \cdot y^{-1}z) \succeq h(x^{-1}y)\widetilde{\wedge}h(y^{-1}z) \approx h(e)\widetilde{\wedge}h(y^{-1}z)$
$\approx h(y^{-1}z) \quad \forall z \in G$
Similarly $h(y^{-1}z) \succeq h(x^{-1}z) \quad \forall z \in G$
Therefore $h(x^{-1}z) \approx h(y^{-1}z) \quad \forall z \in G$
$\Rightarrow x\widetilde{h} = y\widetilde{h}$ □

Theorem 4.8. Let $h \in NHF(G)$ and $x, y \in G$. If $x\widetilde{h} = y\widetilde{h}$, then $h(x) \approx h(y)$

Proof. Suppose that $x\widetilde{h} = y\widetilde{h}$.
Then by Theorem[4.7] we have that $x^{-1}y \in h_*$ and $y^{-1}x \in h_*$.
$h \in NHF(G) \approx h(x) \approx h(y^{-1}xy) \succeq h(y^{-1}x)\widetilde{\wedge}h(y) \approx h(e)\widetilde{\wedge}h(y) \approx h(y)$ Similarly $h(y) \succeq h(x)$.
Therefore $h(x) \approx h(y)$. □

Theorem 4.9. Let $h \in NHF(G)$. Set $G/h = \left\{x\widetilde{h}|x \in G\right\}$ Then

1. $(x\widetilde{h} \circ (y\widetilde{h})) = (xy)\widetilde{h} \quad \forall x, y \in G$

2. $(G/h, \circ)$ is a group

Proof. 1. $\forall x, y \in G$ we have

$$\begin{aligned}
(x\widetilde{h} \circ (y\widetilde{h})) &= (h(e)_{\{x\}} \circ h) \circ (h(e)_{\{y\}} \circ h) \text{ by definition} \\
&= h(e)_{\{x\}} \circ (h \circ (h(e)_{\{y\}}) \circ h \\
&= h(e)_{\{x\}} \circ (h \circ h) \circ h(e)_{\{y\}} \\
&= (h(e)_{\{x\}} \circ h(e)_{\{y\}}) \circ h \\
&= h(e)_{\{xy\}} \circ h \\
&= (xy)\widetilde{h}
\end{aligned}$$

2. By 1), G/h is closed under the operation $\circ$. Also $\circ$ satisfies the associative law.
Now $h \circ (x\widetilde{h}) = (e\widetilde{h}) \circ (x\widetilde{h}) = (ex)\widetilde{h} = x\widetilde{h} \quad \forall x \in G$.
Hence the identity element is $(e\widetilde{h})$
and $(x^{-1}\widetilde{h}) \circ (x\widetilde{h}) = (x^{-1}x)\widetilde{h} = e\widetilde{h} = h \quad \forall x \in G$
Hence the inverse element of $x\widetilde{h}$ is $x^{-1}\widetilde{h}$.
Therefore $(G/h, \circ)$ is a group.

□

Definition 4.10. [4] The group $(G/h, \circ)$ defined in the above theorem, where $G/h = \left\{ x\widetilde{h} | x \in G \right\}$, is called the quotient group or the factor group of G relative to the normal hesitant fuzzy subgroup h.

Theorem 4.11. Let $h \in HFG(G)$ and let N be a normal subgroup of G. Define $(h/N) \in HF(G/N)$ as $(h/N)(xN) = \widetilde{\vee} \{h(z)|z \in xN\} \quad \forall x \in G$ Then $(h/N) \in HFG(G/N)$.

Proof. Now

$$\begin{aligned}
(h/N)((xN)^{-1}) &= (h/N)(x^{-1}N) \\
&= \widetilde{\vee} \left\{h(z)|z \in x^{-1}N\right\} \\
&= \widetilde{\vee} \left\{h(y^{-1})|y^{-1} \in x^{-1}N\right\} \\
&= \widetilde{\vee} \{h(y)|y \in xN\} \\
&= (h/N)(xN) \quad \forall x \in G
\end{aligned}$$

$$\begin{aligned}
(h/N)(xNyN) &= \widetilde{\vee} \{h(z)|z \in xyN\} \\
&= \widetilde{\vee} \{h(uv)|u \in xN, v \in yN\} \\
&\succeq \widetilde{\vee} \left\{h(u)\widetilde{\wedge}h(v)|u \in xN, v \in yN\right\} \\
&= (\widetilde{\vee} \{h(u)|u \in xN\})\widetilde{\wedge}(\widetilde{\vee} \{h(v)|, v \in yN\}) \\
&= (h/N)(xN)\widetilde{\wedge}(h/N)(yN) \quad \forall x, y \in G
\end{aligned}$$

Hence $(h/N) \in HFG(G/N)$ □

5 Conclusion

In this paper certain properties and results regarding hesitant fuzzy groups are studied. The general structure of normal hesitant fuzzy subgroups are discussed. Some conditions for hesitant fuzzy subgroups to be normal are established. The notion of a quotient group relative to a normal hesitant fuzzy subgroup is studied.

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