http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 29.12.2015
Published: 21.03.2016
Year: **2016**, *Number*: **12**, *Pages*: **01-07**
Original Article[**]

DETERMINANTAL IDENTITIES FOR k LUCAS SEQUENCE

Ashok Dnyandeo Godase[1,*] <ashokgodse2012@gmail.com>
Machindra Baban Dhakne[2] <ijntindia@gmail.com>

[1]Department of Mathematics, V. P. College Vaijapur, 423701, Maharashtra, India
[2]Department of Mathematics, Dr. B. A. M. University Aurangabad, 431106, Maharashtra, India

Abstaract − In this paper, we defined new relationship between k Lucas sequences and determinants of their associated matrices, this approach is different and never tried in k Fibonacci sequence literature.

Keywords − *k-Fibonacci sequence, k-Lucas sequence, Recurrence relation.*

1 Introduction

The Fibonacci sequence is a source of many nice and interesting identities. Many identities have been documented in [9],[10],[11],[12],[16],[2],[3]. A similar interpretation exists for k Fibonacci and k Lucas numbers. Many of these identities have been documented in the work of Falcon and Plaza [6],[7],[8], where they are proved by algebraic means, many of another interesting algebraic identities are also proved in [1],[4]. In this paper determinantal techniques are used to obtain several k Lucas identities.

2 Preliminary

Definition 2.1. The $k-$Fibonacci sequence $\{F_{k,n}\}_{n=1}^{\infty}$ is defined as, $F_{k,n+1} = k \cdot F_{k,n} + F_{k,n-1}$, with $F_{k,0} = 0, F_{k,1} = 1$, for $n \geq 1$

Definition 2.2. The $k-$ Lucas sequence $\{L_{k,n}\}_{n=1}^{\infty}$ is defined as, $L_{k,n+1} = k \cdot L_{k,n} + L_{k,n-1}$, with $L_{k,0} = 2, L_{k,1} = k$, for $n \geq 1$

[**]*Edited by* Adem Şahin *and* Naim Çağman *(Editor-in-Chief).*
[*]*Corresponding Author.*

Characteristic equation of the initial recurrence relation is,

$$r^2 - k \cdot r - 1 = 0 \tag{1}$$

Characteristic roots are

$$r_1 = \frac{k + \sqrt{k^2+4}}{2} \tag{2}$$

and

$$r_2 = \frac{k - \sqrt{k^2+4}}{2} \tag{3}$$

Characteristic roots verify the properties

$$r_1 - r_2 = \sqrt{k^2+4} = \sqrt{\Delta} = \delta \tag{4}$$

$$r_1 + r_2 = k \tag{5}$$

$$r_1 . r_2 = -1 \tag{6}$$

Binet forms for $F_{k,n}$ and $L_{k,n}$ are

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{7}$$

and

$$L_{k,n} = r_1^n + r_2^n \tag{8}$$

2.1 First 11 k Fibonacci sequences as numbered in the Encyclopedia of Integer Sequences

$F_{k,n}$	**Classification**
$F_{1,n}$	$A000045$
$F_{2,n}$	$A000129$
$F_{3,n}$	$A006190$
$F_{4,n}$	$A001076$
$F_{5,n}$	$A052918$
$F_{6,n}$	$A005668$
$F_{7,n}$	$A054413$
$F_{8,n}$	$A041025$
$F_{9,n}$	$A099371$
$F_{10,n}$	$A041041$
$F_{11,n}$	$A049666$

3 Determinantal Identities

Theorem 3.1. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det \begin{bmatrix} L_{k,n+t}^2 + 4L_{k,n-i}^2 & L_{k,n+i+j} & L_{k,n+i+j} \\ L_{k,n+t} & 4L_{k,n+i}^2 + L_{k,n+i+j}^2 & L_{k,n+t} \\ L_{k,n+i} & 2L_{k,n+i} & \dfrac{L_{k,n+i+j}^2 + L_{k,n+t}^2}{2L_{k,n+i}} \end{bmatrix} = 8L_{k,n+i}L_{k,n+t}L_{k,n+i+j} \tag{9}$$

Proof. Let

$$\aleph_1 = det \begin{bmatrix} L_{k,n+t}^2 + 4L_{k,n-i}^2 & L_{k,n+i+j} & L_{k,n+i+j} \\ L_{k,n+t} & 4L_{k,n+i}^2 + L_{k,n+i+j}^2 & L_{k,n+t} \\ L_{k,n+i} & 2L_{k,n+i} & \dfrac{L_{k,n+i+j}^2 + L_{k,n+t}^2}{2L_{k,n+i}} \end{bmatrix} \tag{10}$$

Assume that

$$L_{k,n+t} = \phi$$
$$L_{k,n+i} = \varphi$$

Then

$$L_{k,n+i+j} = k\varphi + \phi$$

Now,

$$\aleph_1 = det \begin{bmatrix} \dfrac{\phi^2 + \varphi^2}{k\varphi + \phi} & k\varphi + \phi & k\varphi + \phi \\ \phi & \frac{\varphi^2 + (k\varphi+\phi)^2}{\phi} & \phi \\ \varphi & \varphi & \dfrac{\phi^2 + (k\varphi + \phi)^2}{\varphi} \end{bmatrix}$$

Making the row operations $\dfrac{1}{(k\varphi + \phi)}[(k\varphi + \phi)R_1]$, $\dfrac{1}{\phi}[\phi R_2]$, $\dfrac{1}{\varphi}[\varphi R_3]$, gives

$$\aleph_1 = \frac{1}{\phi\varphi(k\varphi + \phi)} det \begin{bmatrix} \phi^2 + \varphi^2 & (k\varphi + \phi)^2 & (k\varphi + \phi)^2 \\ \phi^2 & \varphi^2 + (k\varphi + \phi)^2 & \phi^2 \\ \varphi^2 & \varphi^2 & \phi^2 + (k\varphi + \phi)^2 \end{bmatrix} \tag{11}$$

making row operations $R_1 + R_2 + R_3 \to R_1$, $R_3 - R_1 \to R_3$ and $R_2 - R_1 \to R_2$, gives

$$\aleph_1 = \frac{1}{\phi\varphi(k\varphi+\phi)} det \begin{bmatrix} \phi^2+\varphi^2 & \varphi^2+(k\varphi+\phi)^2 & \phi^2+(k\varphi+\phi)^2 \\ -\varphi^2 & 0 & -(k\varphi+\phi)^2 \\ \phi^2 & -(k\varphi+\phi)^2 & 0 \end{bmatrix}$$

Expanding we get

$$\aleph_1 = 8\phi\varphi(k\varphi+\phi)$$

Putting

$$L_{k,n+t} = \phi$$
$$L_{k,n+i} = \varphi$$
$$L_{k,n+i+j} = k\varphi+\phi$$

Gives

$$\aleph_1 = 8L_{k,n+i}L_{k,n+t}L_{k,n+i+j}$$

□

Theorem 3.2. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det \begin{bmatrix} L^2_{k,n+t} & 2L_{k,n+i}L_{k,n+i+j} & L_{k,n+t}L_{k,n+i+j} + L_{k,n+i+j} \\ L^2_{k,n+t} + 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} + 2L_{k,n+i}L_{k,n+i+j} & L^2_{k,n+i+j} \end{bmatrix} \quad (12)$$

$= [4L_{k,n+i}L_{k,n+i+j}]^2$

Proof. Let

$$\aleph_2 = det \begin{bmatrix} L^2_{k,n+t} & 2L_{k,n+i}L_{k,n+i+j} & L_{k,n+t}L_{k,n+i+j} + L_{k,n+i+j} \\ L^2_{k,n+t} + 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & 4L^2_{k,n+i} + 2L_{k,n+i}L_{k,n+i+j} & L^2_{k,n+i+j} \end{bmatrix} \quad (13)$$

Assume that

$$L_{k,n+t} = \phi$$
$$L_{k,n+i} = \varphi$$

Then

$$\aleph_2 = det \begin{bmatrix} \phi^2 & \varphi(k\varphi+\phi) & \phi(k\varphi+\phi)+(k\varphi+\phi)^2 \\ \phi^2+\phi\varphi & \varphi^2 & \phi(k\varphi+\phi) \\ \phi\varphi & \varphi^2+\varphi(k\varphi+\phi) & (k\varphi+\phi)^2 \end{bmatrix}$$

Making the row operations $R_2 \to R_2 - (R_1 + R_3))$, gives

$$\aleph_1 = \frac{1}{\phi\varphi(k\varphi+\phi)} det \begin{bmatrix} \phi & (k\varphi+\phi) & \phi + (k\varphi+\phi) \\ 0 & -2(k\varphi+\phi) & -2(k\varphi+\phi) \\ \varphi & \varphi + (k\varphi+\phi) & (k\varphi+\phi) \end{bmatrix} \tag{14}$$

making Column operations $C_2 \to C_2 - C_3$ and expanding gives

$$\aleph_2 = 4\left[2\phi\varphi(k\varphi+\phi)\right]^2$$

Putting

$$L_{k,n+t} = \phi$$
$$L_{k,n+i} = \varphi$$
$$L_{k,n+i+j} = k\varphi + \phi$$

Gives

$$\aleph_2 = \left[4L_{k,n+i}L_{k,n+i+j}\right]^2$$

□

Corollary 3.3. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det \begin{bmatrix} -L_{k,n+t}^2 & 2L_{k,n+i}L_{k,n+t} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & -4L_{k,n+i}^2 & 2L_{k,n+i}L_{k,n+i+j} \\ L_{k,n+t}L_{k,n+i+j} & 2L_{k,n+i}L_{k,n+i+j} & -L_{k,n+i+j}^2 \end{bmatrix} = \left[4L_{k,n+i}L_{k,n+t}L_{k,n+i+j}\right]^2 \tag{15}$$

Corollary 3.4. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det \begin{bmatrix} 4L_{k,n+i}^2 + L_{k,n+i+j}^2 & 2L_{k,n+i}L_{k,n+t} & L_{k,n+t}L_{k,n+i+j} \\ 2L_{k,n+i}L_{k,n+t} & L_{k,n+t}^2 & 2L_{k,n+i}L_{k,n+i+j} \\ L_{k,n+t}L_{k,n+i+j} & 2L_{k,n+i}L_{k,n+i+j} & 4L_{k,n+i}^2 + L_{k,n+t}^2 \end{bmatrix} = \left[4L_{k,n+i}L_{k,n+t}L_{k,n+i+j}\right]^2 \tag{16}$$

Corollary 3.5. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det \begin{bmatrix} 2L_{k,n+i+j} + 2L_{k,n+i} + L_{k,n+t} & L_{k,n+t} & 2L_{k,n+i} \\ L_{k,n+i+j} & 2L_{k,n+t} + 2L_{k,n+i} + L_{k,n+i+j} & 2L_{k,n+i} \\ L_{k,n+i+j} & 2L_{k,n+t} & 4L_{k,n+i} + L_{k,n+t} + L_{k,n+i+j} \end{bmatrix}$$

$$= 2\left[2L_{k,n+i} + L_{k,n+t} + L_{k,n+i+j}\right]^3$$

Corollary 3.6. If n, i, j, t, m are positive integers with $0 < t < i$, $i + 1 < m$, $j = 1$, then

$$det \begin{bmatrix} 1 + L_{k,n+t} & 1 & 1 \\ 1 & 1 + 2L_{k,n+i} & 1 \\ 1 & 1 & 1 + L_{k,n+i+j} \end{bmatrix}$$

$$= \{2L_{k,n+i}L_{k,n+t}L_{k,n+i+j}\}\{\frac{1}{L_{k,n+t}} + \frac{1}{2L_{k,n+i}} + \frac{1}{L_{k,n+i+j}} + 1\}$$
$$\{2L_{k,n+i}L_{k,n+t}L_{k,n+i+j} + 2L_{k,n+i}L_{k,n+i+j} + L_{k,n+t}L_{k,n+t}L_{k,n+i+j} + 2L_{k,n+i}L_{k,n+t}\}$$

4 Conclusion

In this paper we described determinantal identities for k Lucas sequence; same identities can be derived for k Fibonacci sequence.

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