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SYMMETRIC IDENTITIES INVOLVING CARLITZ'S-TYPE TWISTED (*h*,*q*)-TANGENT-TYPE POLYNOMIALS UNDER S_5

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Abstract – In [11], Ryoo introduced the Carlitz's-type twisted *(h,q)*-Tangent numbers and polynomials. In this paper, we consider some new symmetric identities involving Ryoo's Carlitz's-type twisted *(h,q)* Tangent-type polynomials arising from the fermionic *p*-adic invariant integral on $\mathbb{Z}_p$ under S_5 termed symmetric group of degree five.

Keywords – *Symmetric identities; Carlitz's-type twisted (h,q)-Tangent-type polynomials; Fermionic p-adic invariant integral on* $\mathbb{Z}_p$*; Invariant under* S_5.

1 Introduction

In the complex plane, the Euler polynomials are defined by

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}, \quad (|t| < \pi).$$

When $x=0$, then we get $E_n(0) := E_n$ is called the n-th Euler numbers, see [5], [7], [14]. As well-known that the Tangent numbers T_{2n-1} $(n \geq 1)$ are defined as the coefficients of the Taylor expansion of $\tan x$:

$$\tan x = \sum_{n=1}^{\infty} \frac{T_{2n-1}}{(2n-1)!} x^{2n-1} \quad (\text{see } [10,14]).$$

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Kim *et al.* [10] obtained the following relation between Tangent numbers and Euler numbers:

$$E_{2n-1} = (-1)^n \frac{T_{2n-1}}{2^{2n-1}}. \tag{1.1}$$

Ryoo [14] introduced Tangent-type polynomial $\mathcal{T}_n(x)$ which is different from original definition, as follows:

$$\sum_{n=0}^{\infty} \mathcal{T}_n(x) \frac{t^n}{n!} = \frac{2}{e^{2t}+1} e^{xt}, \qquad (|t| < \frac{\pi}{2}). \tag{1.2}$$

Letting $x = 0$ in the Eq. (1.2) reduces to $\mathcal{T}_n(0) := \mathcal{T}_n$ that is called n-th Tangent-type number (see, e.g., [11], [14]).

Ryoo's Tangent polynomial holds the following equality (see [14])

$$E_{2n-1} = \frac{\mathcal{T}_{2n-1}}{2^{2n-1}}. \tag{1.3}$$

Note that the Eq. (1.3) is different from the Eq. (1.1). Further we have

$$\mathcal{T}_{2n-1} = (-1)^n T_{2n-1}. \tag{1.4}$$

Because of (1.4), we call $\mathcal{T}_n(x)$ and $\mathcal{T}_n$ as Tangent-type polynomials and Tangent-type numbers, respectively.

Let p be chosen as a fixed odd prime number. Along this paper $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote topological closure of $\mathbb{Z}$, the field of rational numbers, topological closure of $\mathbb{Q}$ and the field of p-adic completion of an algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

For d an odd positive number with $(d, p) = 1$, let

$$X := X_d = \varprojlim_{n} \mathbb{Z}/dp^N\mathbb{Z} \text{ and } X_1 = \mathbb{Z}_p$$

and

$$t + dp^N \mathbb{Z}_p = \{x \in X \mid x \equiv t \left(\text{mod } dp^N\right)\}$$

where $t \in \mathbb{Z}$ lies in $0 \leq t < dp^N$. See, for more details, $[1-11]$.

The normalized absolute value according to the theory of p-adic analysis is given by $|p|_p = p^{-1}$. The notation "q" can be considered as an indeterminate, a complex number

$q \in \mathbb{C}$ with $|q| < 1$, or a p-adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-1/(p-1)}$ and $q^x = \exp(x \log q)$ for $|x|_p \le 1$. It is always clear in the content of the paper.

For any x, let us introduce the following notation (see [1-14])

$$[x]_q = \frac{1-q^x}{1-q} \quad (q \neq 1) \tag{1.5}$$

known as q-number of x. Note that as $q \to 1$, the notation $[x]_q$ reduces to the x. For

$$f \in UD(\mathbb{Z}_p) = \left\{ f \,\middle|\, g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \right\},$$

Kim [7] defined the p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{1.6}$$

From Eq. (1.6), we get

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{k=0}^{n-1} (-1)^{n-k-1} f(k)$$

where $f_n(x)$ means $f(x+n)$. For more details about the p-adic invariant integral on $\mathbb{Z}_p$, see the references, e.g., [5], [7], [11], [12], [13], [14].

Let $h \in \mathbb{Z}$ and $T_p = \bigcup_{N \ge 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \left\{ w : w^{p^N} = 1 \right\}$ is the cyclic group of order p^N. For $w \in T_p$, we indicate by $\phi_w : \mathbb{Z}_p \to C_p$ the locally constant function $x \to w^x$. For $q \in C_p$ with $|1-q|_p < 1$ and $w \in T_p$, the h-extension of Carlitz's-type twisted q-Tangent-type polynomials are defined by the following p-adic invariant integral on $\mathbb{Z}_p$, with respect to μ_{-1}, in [11]:

$$\int_{\mathbb{Z}_p} w^y q^{hy} [2y + x]_q^n \, d\mu_{-1}(y) = T_{n,q,w}^{(h)}(x) \quad (n \ge 0). \tag{1.7}$$

If we let $x = 0$ into the Eq. (1.7), we then have $T_{n,q,w}^{(h)}(0) := T_{n,q,w}^{(h)}$ called n-th h-extension of Carlitz's-type twisted q-Tangent-type number. These numbers can be generated by the following recurrence relation:

$$q^h w (q^2 T_{q,w}^{(h)} + [2]_q)^n + T_{n,q,w}^{(h)} = \begin{cases} 2, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

with the usual convention about replacing $\left(T_{q,w}^{(h)}\right)^n$ by $T_{n,q,w}^{(h)}$.

When $q \to 1^-$ and $w=1$ in the Eq. (1.7), it gives

$$T_{n,q,w}^{(h)}(x) \to T_n(x) := \int_{\mathbb{Z}_p} (2y+x)^n \, d\mu_{-1}(y).$$

Recently, symmetric identities on some special polynomials, e.g. Bernoulli polynomials, Euler polynomails, Genocchi polynomials etc., have been studied by many mathematicians. For instance, Agyuz *et al.* [1] obtained a further investigation for the q-Genocchi numbers and polynomials of higher order under third Dihedral group D_3 and established some closed formulae of the symmetric identities. They also established some known identities for the classical Genocchi numbers and polynomials by using fermionic p-adic q-integral on $\mathbb{Z}_p$. Duran *et al.* [2] investigated some new symmetric identities for q-Genocchi polynomials which are derived from the fermionic p-adic q-integral on $\mathbb{Z}_p$. Duran *et al.* [3] derived symmetric identities involving weighted q-Genocchi polynomials using the fermionic p-adic q-integral on $\mathbb{Z}_p$. Araci *et al.* [5] performed to get some new symmetric identities for q-Frobenius-Euler polynomials under symmetric group of degree five, which are derived from the fermionic p-adic q-integral over the p-adic numbers field. Kim *et al.* [9] introduced new symmetry identities for Carlitz's q-Bernoulli polynomials under symmetric group of degree five. Kim *et al.* [7] investigated some new properties of symmetry for the Carlitz's-type q-Euler polynomials invariant under the symmetric group of degree five. Kim [8] considered new properties of symmetry for the higher-order Carlitz's q-Bernoulli polynomials which derived from p-adic q-integral on $\mathbb{Z}_p$ under the symmetric group of degree five.

In the present paper, we investigate some not only new but also interesting identities for h-extension of Carlitz's-type twisted q-Tangent-type polynomials arising from the fermionic p-adic invariant integral on $\mathbb{Z}_p$ symmetric group of degree five.

2 Symmetric Identities Involving $T_{n,q,w}^{(h)}(x)$ under S_5

For $w_i \in \mathbb{N}$ with $w_i \equiv 1(mod\, 2)$ with $i \in \{1,2,3,4,5\}$, by the Eqs. (1.6) and (1.7), we get

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} \times e^{[w_1w_2w_3w_4 2y + w_1w_2w_3w_4w_5x + w_5w_4w_2w_3i + w_5w_4w_1w_3j + w_5w_4w_1w_2k + w_5w_3w_1w_2s]_q t} d\mu_{-1}(y) \quad (2.1)$$

$$= \lim_{N\to\infty} \sum_{y=0}^{p^N-1} (-1)^y w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} \times e^{[w_1w_2w_3w_4 2y + w_1w_2w_3w_4w_5x + w_5w_4w_2w_3i + w_5w_4w_1w_3j + w_5w_4w_1w_2k + w_5w_3w_1w_2s]_q t}$$

$$= \lim_{N\to\infty} \sum_{l=0}^{w_5-1} \sum_{y=0}^{p^N-1} (-1)^{l+y} w^{w_1w_2w_3w_4(l+w_5y)} q^{hw_1w_2w_3w_4(l+w_5y)}$$

$$\times e^{\left[w_1w_2w_3w_4 2\left(l+w_5y\right)+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q t}.$$

Taking

$$\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s}\, w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s}\times q^{h\left(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right)}$$

on the both sides of Eq. (2.1) gives

$$\begin{aligned}&\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s}\, w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s} \qquad (2.2)\\&\times q^{h\left(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right)}\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y}q^{hw_1w_2w_3w_4y}\\&\times e^{\left[w_1w_2w_3w_4 2y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q t}d\mu_{-1}(y)\\&=\lim_{N\to\infty}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}\sum_{l=0}^{w_5-1}\sum_{y=0}^{p^N-1}(-1)^{i+j+k+s+y+l}\times w^{w_1w_2w_3w_4\left(l+w_5y\right)+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s}\\&\times q^{h\left(w_1w_2w_3w_4\left(l+w_5y\right)+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right)}\\&\times e^{\left[w_1w_2w_3w_4 2\left(l+w_5y\right)+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q t}.\end{aligned}$$

Note that the Eq. (2.2) is invariant for any permutation $\sigma\in S_5$. Therefore, we obtain the following theorem.

Theorem 1 *Let* $w_i\in\mathbb{N}$ *with* $w_i\equiv 1(mod\,2)$ *and* $i\in\{1,2,3,4,5\}$. *Then the following*

$$\begin{aligned}&\sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1}\sum_{s=0}^{w_{\sigma(4)}-1}(-1)^{i+j+k+s}\\&\times w^{w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s}\\&\times q^{h\left(w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s\right)}\\&\times\int_{\mathbb{Z}_p} w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}\left(l+w_{\sigma(5)}y\right)}q^{hw_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}\left(l+w_{\sigma(5)}y\right)}\\&\times\exp\Big([w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}2y+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}w_{\sigma(5)}x+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i\\&+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s]_q t\Big)d\mu_{-1}(y)\end{aligned}$$

holds true for any $\sigma\in S_5$.

By Eq. (1.5), one can easily see that

$$\left[w_1w_2w_3w_4 2y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q \qquad (2.3)$$

$$=\left[w_1w_2w_3w_4\right]_q\left[2y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right]_{q^{w_1w_2w_3w_4}}.$$

From Eqs. (2.1) and (2.3), we obtain

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y}q^{hw_1w_2w_3w_4y}e^{\left[w_1w_2w_3w_4 2y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q t}d\mu_{-1}(y) \quad (2.4)$$
$$=\sum_{n=0}^{\infty}\left[w_1w_2w_3w_4\right]_q^n T^{(h)}_{n,q^{w_1w_2w_3w_4},w^{w_1w_2w_3w_4}}\left(w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right)\frac{t^n}{n!}.$$

By Eq. (2.4), we have

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y}q^{hw_1w_2w_3w_4y} \quad (2.5)$$
$$\times\left[w_1w_2w_3w_4 2y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right]_q^n d\mu_{-1}(y)$$
$$=\left[w_1w_2w_3w_4\right]_q^n T^{(h)}_{n,q^{w_1w_2w_3w_4},w^{w_1w_2w_3w_4}}\left(w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right),(n\geq 0).$$

Thus, from Theorem 1 and (2.5), we have the following theorem.

Theorem 2 *For* $w_i\in\mathbb{N}$ *with* $w_i\equiv 1(mod 2)$ *with* $i\in\{1,2,3,4,5\}$, *the following*

$$\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}\right]_q^n\sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1}\sum_{s=0}^{w_{\sigma(4)}-1}(-1)^{i+j+k+s}$$
$$\times w^{w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s}$$
$$\times q^{h\left(w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s\right)}$$
$$\times T^{(h)}_{n,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}},w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}\left(w_{\sigma(5)}x+\frac{w_{\sigma(5)}}{w_{\sigma(1)}}i+\frac{w_{\sigma(5)}}{w_{\sigma(2)}}j+\frac{w_{\sigma(5)}}{w_{\sigma(3)}}k+\frac{w_{\sigma(5)}}{w_{\sigma(4)}}s\right)$$

holds true for any $\sigma\in S_5$.

It is easy to show by using the definition of $[x]_q$ that

$$\left[2y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right]^n_{q^{w_1w_2w_3w_4}} \quad (2.6)$$
$$=\sum_{m=0}^{n}\binom{n}{m}\left(\frac{\left[w_5\right]_q}{\left[w_1w_2w_3w_4\right]_q}\right)^{n-m}\left[w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s\right]^{n-m}_{q^{w_5}}$$
$$\times q^{m\left(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s\right)}\left[2y+w_5x\right]^m_{q^{w_1w_2w_3w_4}}.$$

Taking $\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} d\mu_{-1}(y)$ on the both sides of Eq. (2.6) gives

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y} \left[2y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right]_{q^{w_1w_2w_3w_4}}^n d\mu_{-1}(y) \quad (2.7)$$

$$=\sum_{m=0}^{n}\binom{n}{m}\left(\frac{[w_5]_q}{[w_1w_2w_3w_4]_q}\right)^{n-m}[w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s]_{q^{w_5}}^{n-m}$$

$$\times q^{m(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s)}\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y}[2y+w_5x]_{q^{w_1w_2w_3w_4}}^m d\mu_{-1}(y)$$

$$=\sum_{m=0}^{n}\binom{n}{m}\left(\frac{[w_5]_q}{[w_1w_2w_3w_4]_q}\right)^{n-m}[w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s]_{q^{w_5}}^{n-m}$$

$$\times q^{m(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s)}T^{(h)}_{m,q^{w_1w_2w_3w_4},w^{w_1w_2w_3w_4}}(w_5x).$$

By the Eq. (2.7), we have

$$[w_1w_2w_3w_4]_q^n\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s}w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s} \quad (2.8)$$

$$\times q^{h(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s)}$$

$$\times\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} q^{hw_1w_2w_3w_4y}\left[2y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right]_{q^{w_1w_2w_3w_4}}^n d\mu_{-1}(y)$$

$$=\sum_{m=0}^{n}\binom{n}{m}[w_1w_2w_3w_4]_q^m[w_5]_q^{n-m}T^{(h)}_{m,q^{w_1w_2w_3w_4},w^{w_1w_2w_3w_4}}(w_5x)\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}$$

$$\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s}w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s}$$

$$\times q^{(m+h)(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s)}\times[w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s]_{q^{w_5}}^{n-m}$$

$$=\sum_{m=0}^{n}\binom{n}{m}[w_1w_2w_3w_4]_q^m[w_5]_q^{n-m}T^{(h)}_{m,q^{w_1w_2w_3w_4},w^{w_1w_2w_3w_4}}(w_5x)C_{n,q^{w_5},w^{w_5}}(w_1,w_2,w_3,w_4\,|\,m),$$

where

$$C_{n,q,w}(w_1,w_2,w_3,w_4\,|\,m) \quad (2.9)$$

$$=\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s}w^{w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s}$$

$$\times q^{(m+h)(w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s)}[w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s]_q^{n-m}.$$

As a result, by (2.9), we arrive at the following theorem.

Theorem 3 *Let* $w_i \in \mathrm{N}$ *with* $w_i \equiv 1(mod\, 2)$ *with* $i \in \{1,2,3,4,5\}$. *For* $n \geq 0$, *the following expression*

$$\sum_{m=0}^{n} \binom{n}{m} \left[w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)} \right]_q^m \left[w_{\sigma(5)} \right]_q^{n-m}$$

$$\times T^{(h)}_{m,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}},w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}} \left(w_{\sigma(5)} x \right) C_{n,q^{w_{\sigma(5)}},w^{w_{\sigma(5)}}} (w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}, w_{\sigma(4)} \mid m)$$

holds true for some $\sigma \in S_5$.

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