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SOME NEW CONCEPTS IN TOPOLOGICAL GROUPS

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Abstaract − In this study, we define a new boundedness concept different from existing definitions. Also we give some theorems and results in topological groups. The new definition more general than boundedness definition in topological vector spaces.

Keywords − *Topological Groups, Boundedness.*

1 Introduction

There exists some works with regards to boundedness of topological groups. Bruguera, Tkachenko and Hejcman have presented another boundedness definitions in topological groups [1], [2]. In 1991, Atkin gave the boundedness concept in uniform spaces which are more general structures than topological groups [3]. Then Hernandez presented Pontryagin duality for topological abelian groups in [4]. If a set is absorbed by every neighbourhood of 0 the set is called as a bounded set in a topological vector space. That is, there exists a number $\varepsilon > 0$ for each neighbourhood U of 0 such that $tA \subseteq U$ for every $|t| < \varepsilon$. The operation of scalar multiplication tA is very important in this definition. There isn't exist this operation in groups so it cannot be applied directly to the topological groups. We know that every topological vector space has an additive topological group structure so the boundedness definition is also generalization of current available boundedness definition in topological vector spaces. Therefore we present a kind of boundedness definition in topological groups so similar to those in topological vector spaces. The new definition is not a generalization of existing boundedness definitions for topological groups.

2 Preliminaries

Let G be an abstract group, A and B be two subsets of G. Then AB is the set of all elements of xy such that $x \in A$ and $y \in B$. The definition of A^2 and $A^m = A^{m-1}A$

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is clear by taking $B = A$ for some $m \in \mathbb{N}$. Further, $A^{-1} = \{a^{-1} : a \in A\}$, $A^{-m} = (A^{-1})^m$ and $A^0 = \{e\}$ for the unit element e of G. Given $x \in A^m$ there exist some $a_1, a_2, ..., a_m \in A$ such that $x = a_1 a_2 ... a_m$. If $x^m \in A^m$ and $e \notin A$ then x^n may not be an element of A^m, for $n < m$. Hence we define the set $A^{\leq m}$ by $x = a_1 a_2 ... a_n$ for $m, n \in \mathbb{N}$, $a_1, a_2, ..., a_n \in A$ and some $n \leq m$. It is clear that $A^m \subseteq A^{\leq m}$ and $A^m = A^{\leq m}$ whenever e is contained by A.

It is known as every topological vector space is an additive group, it is written mU instead of U^m. Then a set B is bounded if and only if there exists a positive integer m depend on U for every symmetrical neighbourhood U such that $B \subseteq U$ or $\frac{1}{m} B \subseteq U$. This is known as boundedness definition in topological vector spaces.

Now we mention that some definitions and propositions in topological groups. Since a topological group has a local basis of symmetrical neighbourhood of the unit element e, a connected topological group G is generated by a neighbourhood U of e i.e. all elements of G is denoted by finite multiplication of elements belong to U [5]. A set S is called as precompact set in a topological group if there exists a finite set F for each neighbourhood U of e such that $S \subseteq FU$. We have known that if a set is bounded then it is metrically bounded i.e. boundedness with respect to the semimetric in a topological vector space. But opposite of this proposition is not correct [6].

Let G be a group and $p : G \to \mathbb{R}$ be a function. p is called an absolute value function on G if satisfies the following properties for each $x, y, a \in G$

(i) $p(x) \geq 0$,

(ii) $p(e) = 0$ and $p(x^{-1}) = p(x)$,

(iii) $p(xy) \leq p(x) + p(y)$,

(iv) If $p(x_n) \to 0$ then $p(a x_n a^{-1}) \to 0$ for every sequence (x_n).

Last condition is unnecessary for abelian groups. The equality $d(x, y) = p(x^{-1} y)$ defines a semimetric generating group topology on G. d is called a left invariant semimetric if $d(ax, ay) = d(x, y)$ for every $x, y, a \in G$. The topology of a topological group first countable comes from a left invariant semimetric [7].

Let G is a topological group and $B \subseteq G$. If the set B absorbs every bounded set then B is called a bornivorous.

Let G is a topological group. If every bornivorous in G is a neighbourhood of e then G is called bornological group [5].

3 Main Results

In this section, we will give some new definitions and results in topological groups.

Definition 3.1. Let G be a topological group and $A \subseteq G$. The set A is called as absorbing set if there exist a finite set $F_x \subseteq G$ and a number $m \in \mathbb{N}$ for every $x \in G$ such that $x \in F_x A^m$.

Definition 3.2. Let G be a topological group and $A \subseteq G$. The set A is called as a bounded set if the set is absorbed by every neighbourhood of the unit element e of G i.e. there exist a finite set F and a number $m \in \mathbb{N}$ for every $U \in N_e$ such that $A \subseteq FU^m = \bigcup_{x \in F} \{xU^m\}$.

Proposition 3.3. According to this (boundedness) definition, boundedness of a set A in a topological group $(X, +)$ is equivalent to boundedness of A in the topological vector space X.

Proof. Now we take a subset A is bounded in the topological vector space X. There exists a number $\lambda > 0$ for every $U \in N_0$ such that $A \subseteq \lambda U$. Therefore we get $A \subseteq ([[\lambda]] + 1)U$. If we select $F = \{0\}$ and $([[\lambda]] + 1) = m$ then

$$A \subseteq ([[\lambda]] + 1)U = F + mU = F + U^m$$

Thus the subset A is bounded in the topological group $(X, +)$.

On the contrary, $A \subseteq F + U^m = F + mU$. If we take $F = \{0\}$ and $m = \lambda$ then $A \subseteq \lambda U$. □

Theorem 3.4. Every singleton is bounded in a topological group.

Proof. If we take $F = \{a\}$ and $m = 1$ then $\{a\} \subset \{a\}U = \{aU\}$. This completes the proof, easily. □

Theorem 3.5. Union of two bounded sets is also bounded in a topological group.

Proof. Let A and B be two bounded subsets in a topological group X. There exists a finite set $F \subseteq X$ and a number $m \in \mathbb{N}$ for every $U \in N_e$ such that

$$A \cup B \subseteq FU^n = \underset{x \in F}{\cup} \{xU^m\}$$

We suppose that the above inclusion isn't true. Thus $A \cup B$ isn't covered by FU^m for every finite set $F \subseteq X$ and every number $m \in \mathbb{N}$. Then A isn't covered by FU^m or B isn't covered by FU^m. This contradict with our hypothesis. □

Corollary 3.6. Every subset of a bounded set is bounded in a topological group.

Corollary 3.7. Intersection of two bounded sets is bounded in a topological group.

Theorem 3.8. Every finite set is bounded in a topological group.

Proof. It is easily seen that union of finite number of bounded sets is bounded by induction method since we know that every set is written by union of singletons. □

Theorem 3.9. Every precompact set is bounded in a topological group.

Proof. Let G be a topological group, S be a precompact set in G, U be any neighbourhood of e and V be an other neighbourhood of e such that $VV \subset U$. There exists a finite set F such that $S \subset FV$ by hypothesis then F is bounded. Thus there exist a number $n \in \mathbb{N}$ and a finite set G such that $F \subset GV^n$. Then

$$S \subset FV \subset GV^nV \subset GV^nV^n = G(VV)^n \subset GU^n$$

i.e. S is a bounded set. □

Corollary 3.10. Every compact set is bounded in a topological group.

Lemma 3.11. Let X be a topological group and $x \in X$. Then $xD_r(e) = D_r(x)$.

Proof. $y \in xD_r(e) \Leftrightarrow$ if and only if there exists a point $a \in D_r(e)$ such that $y = xa$. Thus

$$\begin{aligned} a \in D_r(e) &\Leftrightarrow d(e, a) < r \\ &\Leftrightarrow d(e, \tfrac{y}{x}) < r \\ &\Leftrightarrow d(e, x^{-1}y) < r \end{aligned}$$

and also since $d(e, x^{-1}y) = d(xe, xx^{-1}y) = d(x, y)$ then $y \in D_r(x)$. □

Lemma 3.12. Let X be a topological group and $x \in X$ then $xD_r(e)^m \subseteq (xD_r(e))^m$.

Lemma 3.13. Let X be a topological group and $x \in X$ then $D_r(x)^m \subseteq D_{rm}(x)$.

Proof. If $y \in D_r(x)^m$ there exist $a_1, a_2, ..., a_m \in D_r(x)$ such that $y = a_1 a_2 ... a_m$. Hence

$$\begin{aligned} d(y,x) &= d(a_1 a_2 ... a_m, x) \\ &< d(x,a_1) + d(x,a_2) + ... + d(x,a_m) \\ &< r + r + ... + r \\ &= mr \end{aligned}$$

Thus $y \in D_{rm}(x)$. □

Theorem 3.14. Let G be a semimetric group and $A \subseteq G$ be a bounded set then A is a metrically bounded.

Proof. Let G be a semimetric group and $A \subseteq G$. A set A is bounded if and only if there exists a number $m \in \mathbb{N}$ and a finite set F such that $A \subseteq FD_r(e)^m$. Thus

$$A \subseteq FD_r(e)^m \Leftrightarrow A \subseteq \bigcup_{x \in F} \{xD_r(e)^m\} \subseteq D_{rm}(x).$$

This completes the proof. □

Proposition 3.15. A set is absorbed by each member of a local basis of neighbourhoods of e if and only if this set is bounded.

Proof. Let $B = \{U_\alpha : \alpha \in I\}$ be a basis of neighbourhoods of e in a topological group G. It is easily seen that a subset $A \subseteq G$ is absorbed for every neighbourhood U_α. On the contrary, if every $U \in N_e$ then $U_\alpha \subseteq U$ for every $\alpha \in I$. The set A is absorbed by U_α for $\alpha \in A$ if and only if there exists a finite set F_{U_α} and a number $m \in \mathbb{N}$ such that $A \subseteq F_{U_\alpha} U_\alpha^m \subseteq F_{U_\alpha} U^m$. Thus the set A is bounded. □

Proposition 3.16. Every bounded subset of a topological group is contained by the set $\overline{\{e\}}$.

Proof. Let G be a topological group and S be a bounded subset of G. Now we show that $S \subseteq \overline{\{e\}}$. We assume that $x \in \overline{\{e\}}$ is wrong. $U \cap \{e\} = \emptyset$ for a neighbourhood $U \in N_x$ if and only if $U \subseteq \{e\}^c$ or $\{e\} \subseteq U^c$. There exists a finite set F and $m \in \mathbb{N}$ such that $x \in S \subseteq FW^m$ because S is bounded and $x \in S$ for every $W \in N_e$. There exists $f \in F$ and $w \in W^m$ such that $x = fw$. Then $FW^m \in N_x$. That is $FW^m \cap \{e\} = \bigcup_{f \in F} \{fW^m\} \cap \{e\} \neq \emptyset$. This is a contradiction. □

Theorem 3.17. A set B is bounded if and only if every countable subset of B is bounded in a topological space.

Proof. It is obvious that every countable subset of this set is bounded since if a set is bounded then every subset of this set is bounded.

On the contrary, we assume that every countable subset of B is bounded, but B isn't bounded. There exists a neighbourhood U of e such that B isn't included by FU^m for every number $m \in \mathbb{N}$ and a finite set F. Now, we construct the sequence $\{x_m\}_{m=1}^{\infty}$ such that

$$x_1 \in B \backslash FU, \ x_2 \in B \backslash FU^2, \ ..., x_i \in B \backslash FU^i, ...$$

Obviously the sequence $\{x_m\}_{m=1}^{\infty}$ isn't absorbed by U i.e. $\{x_m\}_{m=1}^{\infty} \subset B$ isn't bounded. This contradict with our hypothesis. □

Now we give definitions of bounded mapping, bornological group and then we prove some theorems connected with these concepts.

Definition 3.18. If a mappings is conserved bounded sets between topological groups then this mapping is called as bounded mapping.

Lemma 3.19. Let f be any homomorphism and $m \in \mathbb{N}$ then $\{f^{-1}(V)\}^m \subseteq f^{-1}(V^m)$.

Proof. For all $z \in f^{-1}(V)^m$ there exist $a_1, a_2, ..., a_m \in f^{-1}(V)$ such that $z = a_1a_2...a_m$.

$f(z) = f(a_1a_2...a_m) = f(a_1)f(a_2)...f(a_m)$ and $f(a_i) \in V$ for all $1 \leq i \leq m$. $f(z) \in V^m$ then $z \in f^{-1}(V^m)$. Since $f(S) \subseteq \underset{x\in F}{\cup} \{f(x)\, f(\{f^{-1}(V^m)\})\}$ and $\{f^{-1}(V)\}^m \subseteq f^{-1}(V^m)$ then $f(S) \subseteq \underset{x\in F}{\cup} \{f(x)\, f(f^{-1}(V^m))\}$. Thus

$$f(S) \subseteq \underset{x\in F}{\cup} \{f(x)\, V^m\} = f(F)V^m$$

□

Theorem 3.20. Every continuous homomorphism between topological groups must be bounded.

Proof. Let G and G' be two topological groups, $f : G \to G'$ be a homomorphism and $S \subseteq G$ be bounded. Also let e and e' be unit elements of G and G', respectively. Since S is bounded there exists a number $m \in \mathbb{N}$ and a finite set F for every neighbourhood U of e such that $S \subseteq FU^m = \underset{x\in F}{\cup} \{xU^m\}$.

$xU^m \in N_x$ because $U^m \in N_e$. If we take $V \in N_{e'}$ then $f^{-1}(V) \in N_e$ and

$$S \subseteq F\left\{f^{-1}(V)\right\}^m = \underset{x\in F}{\cup} \left\{xf^{-1}(V)^m\right\}$$

then

$$f\left(\underset{x\in F}{\cup} \left\{xf^{-1}(V)^m\right\}\right) = \underset{x\in F}{\cup} \left\{f(x)\, f\left(f^{-1}(V)^m\right)\right\}.$$

Hence

$$f(S) \subseteq \underset{x\in F}{\cup} \left\{f(x)\, f(\{f^{-1}(V)\}^m)\right\}$$

and

$$f(S) \subseteq \underset{x\in F}{\cup} \{f(x)V^m\} = f(F)V^m.$$

□

Definition 3.21. Let G be a topological group and $B \subseteq G$. If the set B absorbs every bounded set then B is called as a bornivorous.

Definition 3.22. Let G be a topological group. If every bornivorous in G is a neighbourhood of e then G is called as a bornological group.

Theorem 3.23. Every bornivorous in a semimetric group G is a neighbourhood of e.

Proof. Let B be a bornivorous in G. We assume that B isn't a neighbourhood of e. In this case, the set B^n isn't also a neighbourhood of e for every number $n \in \mathbb{N}$.

The open sphere $D_{\frac{1}{n}}(e) = \{x : d(x,e) < \frac{1}{n}\}$ isn't contained by B, for every number n. So this sphere isn't contained the sets B^n because they aren't also neighbourhood of e. Then $\{D_{\frac{1}{n}}(e)\} \backslash B^n \neq \phi$ for every number n. The sequence $\{x_m\}_{m=1}^{\infty}$ which is constructed the style that

$$x_1 \in \{D_1(e)\} \backslash B,\ x_2 \in \{D_{\frac{1}{2}}(e)\} \backslash B^{\leq 2}, ...$$

isn't absorbed by the set B. But the sequence is bounded since $\{x_m\}_{m=1}^{\infty}$ is absorbed by neighbourhood $D_1(e)$ of e. This case is contrary to the fact that B is a bornivorous. □

Remark 3.24. Obviously every neighbourhood of e is a bornivorous. Also it is understand that every semimetric group is a bornological group by above theorem.

Proposition 3.25. Let G and H be two topological groups, $f : G \to H$ be a bounded homomorphism. If $A \subseteq G$ is a bornivorous, then $f(A)$ is also a bornivorous in H.

Proof. Let we take $y \in f(S)$. Then

$$\begin{aligned} y \in f(S) &\Rightarrow f(x) \in f(S) \\ &\Rightarrow x \in S \\ &\Rightarrow x \in FA^n \end{aligned}$$

Thus $f(S) \subseteq f(FA^n) = f(\underset{x \in F}{\cup}\{xA^n\}) = f(\underset{x \in F}{\cup}\{x\}))f(A^n)$. $f(A)$ is a bornivorous in H because $f(F)$ is a finite set. □

Proposition 3.26. Let G and H be two topological groups, $f : G \to H$ be a bounded homomorphism. If $B \subseteq f(G)$ is a bornivorous in H, then $f^{-1}(B)$ is also a bornivorous in G.

Theorem 3.27. Let G be a bornological group. In this case, every bounded homomorphism f which is defined from G into any topological group H is continuous.

Proof. Let U be a neighbourhood of e in H then the set U absorbs every bounded set in H. Thus the set U is a bornivorous. $f^{-1}(U)$ is a bornivorous in G by above proposition and G is also a neighbourhood of e by hypothesis i.e. f is continuous on e. So f is continuous in everywhere. □

Proposition 3.28. Let (X, τ) and (Y, τ') be any topological groups and $f : X \to Y$ be a continuous homomorphism. If $A \subseteq X$ is bounded then $(f\bar{(A)}) \subseteq Y$ is bounded.

Proof. Let we take any $V \in N_{e'}$ so $f^{-1}(V) \in N_e$. Since A is bounded set, there exists a finite set F and a number $m \in \mathbb{N}$ such that $A \subseteq Ff^{-1}(V)^m$. Thus $f(A) \subseteq f(F)f(f^{-1}(V)^m) \subseteq f(F)f(f^{-1}(V^m)) \subseteq f(F)V^m$ and then $(f\bar{(A)}) \subseteq f(F)V^{m+1}$ i.e. $(f\bar{(A)})$ is bounded. □

Proposition 3.29. Let $(X_i, \tau_i)_{i \in I}$ is any family of topological groups, $X = \underset{i \in I}{\Pi} X_i$ and $\Pi_i : X \to X_i$ be the projection. $A \subseteq X$ is bounded if and only if $\Pi_i(A) \subseteq X_i$ is bounded for every $i \in I$.

Proof. If A is bounded in (X,τ) there exists a finite set F and $m\in\mathbb{N}$ such that $A\subseteq\left(\Pi_i^{-1}(V_i)\right)^m F$. $\Pi_i(A)\subseteq\Pi_i\left(\Pi_i^{-1}(V_i)^m\right)\Pi_i(F)$. Then

$$\Pi_i(A)\subseteq\Pi_i\left(\{\Pi_i^{-1}(V_i)\}^m\right)\Pi_i(F)\subseteq\Pi_i\left(\Pi_i^{-1}(V_i^m)\right)\Pi_i(F)\subseteq V_i^m\Pi_i(F).$$

On the contrary, let $\Pi_i(A)$ is bounded in (X_i,τ_i) for every $i\in I$.

We take any $V\in N_e$. For every $i\in I$ and $V_i\in N_i(e_i)$, $V=\prod\limits_{i\in I}V_i$. There exists a finite set F_i and $m\in\mathbb{N}$ such that $\Pi_i(A)\subseteq V_i^m F_i$ because $\Pi_i(A)$ is bounded for every $i\in I$. Let we take $\Pi_i(A)=A_i$. Therefore $\prod\limits_{i\in I}A_i\subseteq\prod\limits_{i\in I}(V_i^m F_i)=\left(\prod\limits_{i\in I}V_i\right)^m\prod\limits_{i\in I}F_i$. Thus $A\subseteq V^mF$ i.e. $A\subseteq X$ is bounded. □

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