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SEMIPRIME AND NILPOTENT FUZZY LIE ALGEBRAS

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Abstract – In this paper, we have introduced the concept of semiprime fuzzy Lie algebra and proved that every fuzzy Lie algebra of semiprime (nilpotent) Lie algebra is a semiprime (nilpotent).

Keywords – *Fuzzy Lie Algebra, Semiprime Fuzzy Lie Algebra, Nilpotent Fuzzy Lie Algebra.*

1 Introduction

Zadeh (1965) introduced the concept of fuzzy sets in [1]. Then Liu (1982) extended this concept to rings in [2]. Nanda (1990) defined the notion of fuzzy algebras over fuzzy fields [3]. Yehia (1996) defined the concept of fuzzy Lie algebras over fields [4]. Finally, fuzzy Lie algebras over fuzzy fields are defined by Lilly and Antony (2009) in [5]. In this paper, we first have introduced some basic definitions of fuzzy sets, fuzzy rings and fuzzy Lie algebras which will be used throughout this work. We then have introduced the concept of semiprime fuzzy Lie algebra and proved that every fuzzy Lie algebra of semiprime (nilpotent) Lie algebra is a semiprime (nilpotent).

2 Preliminaries

The following definitions and results are required.

Definition 2.1. [6] A ***Lie algebra*** is a vector space V over a field F on which a product operation $[x \ \ y]$ is defined and satisfies the following axioms

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1) $[x \;\; x] = 0 \;\; \forall x \in V$,

2) $[\lambda.x \;\; y] = \lambda[x \;\; y] = [x \;\; \lambda.y] \;\; \forall x \in V, \forall \lambda \in F$,

3) $[[x \;\; y] \;\; z] + [[y \;\; z] \;\; x] + [[z \;\; x] \;\; y] = 0 \;\; \forall x, y, z \in V$.

We note that the multiplication in a Lie algebra is not associative. But it is anti commutative.

Definition 2.2. [6] Let $\mathcal{G}$ be a Lie algebra over a field F and B be a vector space over F . Then B is called ***left ideal*** if $[\mathcal{G} \;\; B] \subseteq B$.

Clearly, every left ideal (right ideal) of $\mathcal{G}$ is ideal.

Definition 2.3. [6] Let $\mathcal{G}$ be a Lie algebra over a field F . The set

$$Z(\mathcal{G}) = \{x \in \mathcal{G} \; : \; [x \;\; \mathcal{G}] = \{0\}\}$$

is called ***center*** of $\mathcal{G}$. We note that $Z(\mathcal{G})$ is ideal of $\mathcal{G}$.

Definition 2.4. [7] Let $\mathcal{G}$ be a Lie algebra over a field F . Then $\mathcal{G}$ is called ***semiprime*** if $I^2 \neq \{0\}$ for all non-zero ideal I of $\mathcal{G}$, where:

$$I^2 = \left\{ x \in \mathcal{G} \; : \; x = \sum_{i=1}^{n} [y_i \;\; z_i] \; ; \; y_i, z_i \in I \quad \forall i = 1, \dots, n \right\}.$$

Definition 2.5. [8] Let $\mathcal{G}$ be a Lie algebra over a field F . Then $\mathcal{G}$ is called ***nilpotent*** if there exists a positive integer n such that $\mathcal{G}^n = \{0\}$, where

$$\mathcal{G}^1 = \mathcal{G} \;\&\; \dots \;\&\; \mathcal{G}^n = [\mathcal{G}^{n-1} \;\; \mathcal{G}].$$

We note that:

$$\mathcal{G} = \mathcal{G}^1 \supseteq \mathcal{G}^2 \supseteq \cdots \supseteq \mathcal{G}^n \supseteq \cdots$$

Definition 2.6. [1] Let $X \neq \emptyset$ be a set. A ***fuzzy set*** μ of X is a function from X into $[0,1]$, where $([0,1], \leq, \wedge, \vee)$ is distributional complete lattice that has the minimum element 0 and the maximum element 1.

Definition 2.7. [9] Let μ, λ be two fuzzy sets of a $X \neq \emptyset$. Then,

1) $\mu = \lambda \;\Leftrightarrow\; \mu(x) = \lambda(x) \;\; \forall x \in X$.

2) $\mu \subseteq \lambda \;\Leftrightarrow\; \mu(x) \leq \lambda(x) \;\; \forall x \in X$.

Definition 2.8. [9] Let G be a group. The fuzzy set μ of G is called ***fuzzy group*** of G if it satisfies the following axiom

$$\mu(x-y) \geq \mu(x) \wedge \mu(y) \quad \forall x, y \in G.$$

Definition 2.9. [9] Let G be a group. The mapping $E : G \to [0,1]$ which defined by

$$E(x) = \begin{cases} 1 & if \quad x = 0, \\ 0 & if \quad x \neq 0. \end{cases}$$

is called ***the null fuzzy group*** of G.

Definition 2.10. [9] Let $\{\mu_i\}_{i \in I}$ be a family of fuzzy sets of a $X \neq \varnothing$. Then,

1) $(\bigcap_{i \in I} \mu_i)(x) = \bigwedge_{i \in I} \mu_i(x) \quad \forall x \in X$.
2) $(\bigcup_{i \in I} \mu_i)(x) = \bigvee_{i \in I} \mu_i(x) \quad \forall x \in X$.

If X is a commutative ring with identity, then

3) $(\sum_{i \in I} \mu_i)(x) = \bigvee_{x = \sum_{i \in I} x_i} \left(\bigwedge_{i \in I} \mu_i(x_i) \right) \quad \forall x \in X \;\; ; x_i \in X$.

If μ, φ are two fuzzy sets of the commutative ring X with identity, then

4) $(\mu.\varphi)(x) = \bigvee_{x = \sum_{i=1}^{n} a_i . b_i} \left(\bigwedge_{i=1}^{n} \left(\mu(a_i) \wedge \varphi(b_i) \right) \right) \quad \forall x \in X \;\; ; a_i, b_i \in X$.

If μ is a fuzzy group of the commutative ring X with identity, then

5) $\mu(x) = \mu(-x) \quad \forall x \in X$.

Definition 2.11. [10] Let $\mathscr{G}$ be a Lie algebra over a field F. The fuzzy set μ of $\mathscr{G}$ is called ***fuzzy Lie algebra*** of $\mathscr{G}$ if the following axioms are satisfied

1) $\mu(x-y) \geq \mu(x) \wedge \mu(y) \quad \forall x, y \in \mathscr{G}$,
2) $\mu(\alpha.x) \geq \mu(x) \quad \forall x \in \mathscr{G}, \forall \alpha \in F$,
3) $\mu([x \;\; y]) \geq \mu(x) \wedge \mu(y) \quad \forall x, y \in \mathscr{G}$.

We let $F(\mathscr{G})$ denote the set of all fuzzy Lie algebras of $\mathscr{G}$.

Definition 2.12. [10] Let $\mathcal{G}$ be a Lie algebra over a field F. The fuzzy set μ of $\mathcal{G}$ is called a ***fuzzy Lie ideal*** of $\mathcal{G}$ if the following axioms are satisfied

1) $\mu(x-y) \geq \mu(x) \wedge \mu(y) \;\; \forall x, y \in \mathcal{G}$,
2) $\mu(\alpha.x) \geq \mu(x) \;\; \forall x \in \mathcal{G}, \forall \alpha \in F$,
3) $\mu([x \;\; y]) \geq \mu(x) \;\; \forall x, y \in \mathcal{G}$.

We will define semiprime fuzzy Lie algebra.

Definition 2.13. Let $\mathcal{G}$ be a Lie algebra over a field F. Then $\mu \in F(\mathcal{G})$ is called a ***semiprime*** if for any fuzzy Lie ideal $\lambda \subseteq \mu$ of $\mathcal{G}$, then

$$\lambda^2 = E \;\Rightarrow\; \lambda = E.$$

Definition 2.14. [11] Let $\mathcal{G}$ be a Lie algebra over a field F. Then $\mu \in F(\mathcal{G})$ is called a ***nilpotent*** if there exists a positive integer n such that $\mu^n = E$, where

$$\mu^1 = \mu \;\&\ldots\&\; \mu^n = \mu^{n-1}.\mu.$$

We note that:

$$\mu = \mu^1 \supseteq \mu^2 \supseteq \ldots \supseteq \mu^n \supseteq \ldots,$$

where $\mu^n.\mu$ is defined as in (definition 1.10).

3 The Results

Theorem 3.1. Let $\mathcal{G}$ be a semiprime Lie algebra over a field F, and let $E \neq \mu \in F(\mathcal{G})$. Then μ is semiprime.

Proof: Let $\lambda \subseteq \mu$ be a fuzzy Lie ideal of $\mathcal{G}$ such that $\lambda^2 = E$. Then $\lambda = E$, because if we suppose that $\lambda \neq E$, then

$$\exists\; x \in \mathcal{G} \;;\; \lambda(x) \neq E(x).$$

- If $x = 0$, then $\lambda(x) < 1$. Since

$$1 = \left(\lambda^2\right)(0) = \bigvee_{0=\sum_{i=1}^{n}[a_i \;\; b_i]} \left(\bigwedge_{i=1}^{n} \left(\lambda(a_i) \wedge \lambda(b_i)\right) \right),$$

so there one of forms of 0, for example $0 = \sum_{i=1}^{n}[a_i' \;\; b_i']$ such as

$$1 = \overset{n}{\underset{i=1}{\wedge}} \left(\lambda(a_i') \wedge \lambda(b_i') \right),$$

this implies that

$$1 = \lambda(a_i') = \lambda(b_i') \ ; \ i = 1, \ldots, n,$$

therefore

$$\lambda(0) = \lambda\left(\sum_{i=1}^{n} [a_i' \ \ b_i'] \right) \geq \overset{n}{\underset{i=1}{\wedge}} \lambda\left([a_i' \ \ b_i']\right) \geq \overset{n}{\underset{i=1}{\wedge}} \lambda\left(a_i'\right) = 1.$$

This is a contradiction to be $\lambda(0) < 1$.

- If $x \neq 0$, then $\lambda(x) > 0$, so $[x \ \ \mathscr{G}]$ is ideal of $\mathscr{G}$, because

$$[[x \ \ \mathscr{G}] \ \ \mathscr{G}] = -[[\mathscr{G} \ \ \mathscr{G}] \ \ x] - [[\mathscr{G} \ \ x] \ \ \mathscr{G}] \Rightarrow$$

$$[[x \ \ \mathscr{G}] \ \ \mathscr{G}] + [[\mathscr{G} \ \ x] \ \ \mathscr{G}] = -[[\mathscr{G} \ \ \mathscr{G}] \ \ x] \Rightarrow$$

$$[[x \ \ \mathscr{G}] \ \ \mathscr{G}] = -[[\mathscr{G} \ \ \mathscr{G}] \ \ x] \subseteq [\mathscr{G} \ \ x].$$

Also, $[x \ \ \mathscr{G}] \neq \{0\}$ because, if

$$[x \ \ \mathscr{G}] = \{0\} \ \Rightarrow \ x \in Z(\mathscr{G}) \ \Rightarrow \ Z(\mathscr{G}) \neq \{0\},$$

this implies that $Z(\mathscr{G})$ is non-zero ideal of $\mathscr{G}$ such that $[Z(\mathscr{G}) \ \ Z(\mathscr{G})] = \{0\}$, which is a contradiction to the hypothesis that $\mathscr{G}$ is semiprime. Since $[x \ \ \mathscr{G}]$ is non-zero ideal of $\mathscr{G}$ and $\mathscr{G}$ is semiprime, it implies that

$$[[x \ \ \mathscr{G}] \ \ [x \ \ \mathscr{G}]] \neq \{0\}.$$

Therefore:

$$\exists\, t \in [[x \ \ \mathscr{G}] \ \ [x \ \ \mathscr{G}]] \ ; \ 0 \neq t = \sum_{i=1}^{n} [x_i \ \ y_i] \ ; \ x_i, y_i \in [x \ \ \mathscr{G}] \quad \forall i \in \{1, \ldots, n\},$$

where:

$$x_i, y_i \in [x \ \ \mathscr{G}] \quad \forall i \in \{1, \ldots, n\}.$$

Thus:

$$x_i = [x \ \ z_i] \ \& \ y_i = [x \ \ z_i'] \ ; \ z_i, z_i' \in \mathscr{G} \quad \forall i \in \{1, \ldots, n\},$$

this implies that

$$\lambda(x_i) = \lambda([x \ \ z_i]) \geq \lambda(x) \ \ \& \ \ \lambda(y_i) = \lambda([y \ \ z_i']) \geq \lambda(y) \quad \forall i \in \{1, \ldots, n\},$$

Therefore

$$\left(\lambda^2\right)(t) = \bigvee_{t=\sum_{i=1}^{n}[x_i \quad y_i]} \left(\bigwedge_{i=1}^{n} \left(\lambda(x_i) \wedge \lambda(y_i)\right) \right) \geq \bigwedge_{i=1}^{n} \left(\lambda(x_i) \wedge \lambda(y_i)\right)$$

$$\geq \lambda(x) \wedge \lambda(y) > 0.$$

Since $t \in \mathcal{G} - \{0\}$, then $\left(\lambda^2\right)(t) = E(t) = 0$, this is a contradiction.

Lemma 3.2 Let $\mathcal{G}$ be a Lie algebra over a field F, $\mu \in F(\mathcal{G})$ and $\lambda(0) = 1$ for all $\lambda \in F\left(\mathcal{G}\right)$, such as $\lambda \subseteq \mu$. If μ have a series

$$\mu = \mu_1 \supseteq \mu_2 \supseteq \cdots \supseteq \mu_n = E \quad ; \quad \mu_i \in F(\mathcal{G}) \quad \forall 1 \leq i \leq n,$$

such that:

$$\mu_i . \mu_1 \subseteq \mu_{i+1} \quad \forall 1 \leq i \leq n-1,$$

then μ is nilpotent.

Proof: First, we show by induction that:

$$\mu^i \subseteq \mu_i \quad \forall 1 \leq i \leq n$$

For $n = 1$:

$$\mu^1 = \mu = \mu_1 \Rightarrow \mu^1 \subseteq \mu_1,$$

therefore, the result holds for $n = 1$.

Now, we assume that $\mu^k \subseteq \mu_k$ for some $k = 1, \ldots, n-1$, then

$$\mu^{k+1} = \mu^k . \mu \subseteq \mu_k . \mu \subseteq \mu_{k+1}.$$

Now, since $\mu^i \subseteq \mu_i \quad \forall 1 \leq i \leq n$, then $\mu^n \subseteq \mu_n$, this implies that $E \subseteq \mu^n \subseteq \mu_n = E$, thus $\mu^n = E$. This shows that μ is nilpotent.

Theorem 3.3. Let $\mathcal{G}$ be a nilpotent Lie algebra over a field F, $E \neq \mu \in F(\mathcal{G})$ and $\lambda(0) = 1$ for all $\lambda \in F\left(\mathcal{G}\right)$, such as $\lambda \subseteq \mu$. Then μ is nilpotent.

Proof: Since $\mathcal{G}$ is nilpotent, it follows that there is an integer n that satisfies $\mathcal{G}^n = \{0\}$. For every $1 \leq i \leq n$, we define a fuzzy set λ_i by

$$\lambda_i(x) = \begin{cases} \mu(x) & if \quad x \in \mathcal{G}^i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\lambda_i \in F(\mathcal{G})$ for $1 \leq i \leq n$.

This is because

$$1)\ \lambda_i(x)\wedge\lambda_i(y)=\begin{Bmatrix}\mu(x) & if & x\in\mathscr{G}^i, \\ 0 & & \text{otherwise.}\end{Bmatrix}\wedge\begin{Bmatrix}\mu(y) & if & x\in\mathscr{G}^i, \\ 0 & & \text{otherwise.}\end{Bmatrix}$$

$$=\begin{cases}\mu(x)\wedge\mu(y) & if \quad x,y\in\mathscr{G}^i, \\ 0 & \text{otherwise.}\end{cases}$$

$$=\begin{cases}\mu(x)\wedge\mu(y) & if \quad x+y\in\mathscr{G}^i, \\ 0 & \text{otherwise.}\end{cases}$$

$$\leq\begin{cases}\mu(x+y) & if \quad x+y\in\mathscr{G}^i, \\ 0 & \text{otherwise.}\end{cases}$$

$$=\lambda_i(x+y)\quad \forall x,y\in\mathscr{G}.$$

$$2)\ \lambda_i(x)=\begin{cases}\mu(x) & if \quad x\in\mathscr{G}^i, \\ 0 & \text{otherwise}\end{cases}$$

$$\leq\begin{cases}\mu(\alpha.x) & if \quad \alpha.x\in\mathscr{G}^i, \\ 0 & \text{otherwise}\end{cases}$$

$$=\lambda_i(\alpha.x)\quad \forall x\in\mathscr{G}\ ,\ \forall\alpha\in F.$$

$$3)\ \lambda_i(x)\wedge\lambda_i(y)=\begin{Bmatrix}\mu(x) & if & x\in\mathscr{G}^i, \\ 0 & & \text{otherwise.}\end{Bmatrix}\wedge\begin{Bmatrix}\mu(y) & if & x\in\mathscr{G}^i, \\ 0 & & \text{otherwise.}\end{Bmatrix}$$

$$=\begin{cases}\mu(x)\wedge\mu(y) & if \quad x,y\in\mathscr{G}^i, \\ 0 & \text{otherwise.}\end{cases}$$

$$\leq\begin{cases}\mu([x\quad y]) & if \quad [x\quad y]\in[\mathscr{G}^i\quad \mathscr{G}^i]=[\mathscr{G}^i\quad \mathscr{G}]\subseteq\mathscr{G}^{i+1}\subseteq\mathscr{G}^i, \\ 0 & \text{otherwise.}\end{cases}$$

$$=\lambda_i([x\quad y])\quad \forall x,y\in\mathscr{G}.$$

Now, we will prove that $\lambda_n = E$. Thus

$$\lambda_n(x)=\begin{cases}\mu(x) & if \quad x\in\mathscr{G}^n=\{0\}, \\ 0 & \text{otherwise.}\end{cases}$$

$$=\begin{cases}\mu(0) & if \quad x=0, \\ 0 & \text{otherwise.}\end{cases}$$

$$=\begin{cases}1 & if \quad x=0, \\ 0 & \text{otherwise.}\end{cases}$$

$$=E(x)\ \ \forall x\in\mathscr{G}.$$

Thus $\lambda_n = E$.

Let $a,b \in \mathscr{G}$ and $1 \le i \le n-1$. Then:

- If $a \notin \mathscr{G}^i$, then $\lambda_i(a) \wedge \lambda_1(b) = 0 \le \lambda_{i+1}([a\ \ b])$.
- If $a \in \mathscr{G}^i$, then

$$\begin{aligned}\lambda_i(a) \wedge \lambda_1(b) &= \mu(a) \wedge \mu(b)\\ &\le \mu([a\ \ b])\\ &= \lambda_{i+1}([a\ \ b]) \quad ; \quad [a\ \ b] \in [\mathscr{G}^i\ \ \mathscr{G}] = \mathscr{G}^{i+1},\end{aligned}$$

thus:

$$\lambda_i(a) \wedge \lambda_1(b) \le \lambda_{i+1}([a\ \ b]) \quad \forall 1 \le i \le n-1 \quad ...(*).$$

Hence $\lambda_i . \lambda_1 \subseteq \lambda_{i+1}$ for $1 \le i \le n-1$. This is because:

$$\left(\lambda_i . \lambda_1\right)(x) = \bigvee_{x = \sum_{j=1}^{m} [a'_j\ \ b'_j]} \left(\bigwedge_{j=1}^{m} \left(\lambda_i(a'_j) \wedge \lambda_1(b'_j) \right) \right),$$

so there one of forms of x , for example $x = \sum_{j=1}^{m} [a_j\ \ b_j]$ such as

$$\begin{aligned}\left(\lambda_i . \lambda_1\right)(x) &= \bigwedge_{j=1}^{m} \left(\lambda_i(a_j) \wedge \lambda_1(b_j) \right)\\ &\overset{(*)}{\le} \bigwedge_{j=1}^{m} \left(\lambda_{i+1}\left([a_j\ \ b_j]\right) \right)\\ &\le \lambda_{i+1}\left(\sum_{j=1}^{n} [a_j\ \ b_j] \right)\\ &= \lambda_{i+1}(x) \quad \forall x \in \mathscr{G}.\end{aligned}$$

Hence there is a series

$$\mu = \lambda_1 \supseteq \lambda_2 \supseteq \cdots \supseteq \lambda_n = E \quad ; \quad \lambda_i \in F(\mathscr{G}) \quad \forall 1 \le i \le n,$$

such that:

$$\lambda_i . \lambda_1 \subseteq \lambda_{i+1} \quad \forall 1 \le i \le n-1,$$

thus μ is nilpotent (Lemma 3.2).

4 Conclusions

In this paper, we have discussed the concepts of semiprime fuzzy Lie algebra over a field and nilpotent fuzzy Lie algebra over a field. Also, we expected that several results about Lie algebras can be extended to the concept of fuzzy Lie algebras over field.

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