SOME INEQUALITIES OF THE HERMITE HADAMARD TYPE FOR PRODUCT OF TWO FUNCTIONS

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Abstaract − In this paper, we shall establish some new inequalities of the Hermite Hadamard type for product of two functions to belong to the class of s-convex functions and the class of h-convex functions. Some results of product of two functions that belong to two classes of different functions are also given.

Keywords − *Hadamard's inequality, Godunova - Levin functions, P-functions, convex functions, s-convex functions, h-convex functions.*

1   Introduction

A real-valued function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *convex* if and only if the following inequality holds

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{1}$$

for all $a, b \in I$ and $t \in [0, 1]$. If (1) is reversed, then f is called *concave.* In particular, if f is a convex function defined on I, then for all $a, b \in I$ with $a < b$ the following well-known double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

is known in the literature as Hermite Hadamard's inequality. Inequality (2) is reversed whenever f is concave. This famous integral inequality is generalized, improved and extended by many mathematicians (see [3, 4, 5, 12] and [14]).

   In 2003, the first time B. G. Pachpatte [9] established two new Hermite Hadamard type inequalities for product of positive convex functions as follows.

Theorem 1.1 ([9])**.** Let f and g be real-valued, non-negative, and convex functions on $[a, b]$ with $a < b$. Then the following inequalities hold

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le M(a,b)/3 + N(a,b)/6, \tag{3}$$

$$2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx + M(a,b)/6 + N(a,b)/3, \tag{4}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

These results were refined by Feixiang Chen [2] in 2013. In the same year, A. Witkowskim [15] proved the following two theorems for convex functions.

Theorem 1.2 ([15])**.** If $f, g : I \subset \mathbb{R} \to \mathbb{R}$ are of the same convexity (i.e. both convex or both concave), then for all $a, b \in I$ with $a < b$ the following inequality holds

$$\frac{1}{(b-a)^2}\int_a^b (b-x)[f(a)g(x)+f(x)g(a)]dx + \frac{1}{(b-a)^2}\int_a^b (x-a)[f(b)g(x)+f(x)g(b)]dx$$

$$\le \frac{1}{b-a}\int_a^b f(x)g(x)dx + M(a,b)/3 + N(a,b)/6, \tag{5}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1. If f and g are of the opposite convexity, then (5) is reversed.

Theorem 1.3 ([15])**.** Let $f, g : I \subset \mathbb{R} \to [0,\infty)$ be convex functions. Then the following inequality holds for all $a, b \in I$ with $a < b$,

$$\frac{1}{b-a}\int_a^b \Big[f\Big(\frac{a+b}{2}\Big)g(x) + g\Big(\frac{a+b}{2}\Big)f(x)\Big]dx$$

$$\le f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) + \frac{1}{2(b-a)}\int_a^b f(x)g(x) + M(a,b)/12 + N(a,b)/6, \tag{6}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1.

By the early year 2014, M. Tunç [13] advanced two new results for product of a s-convex function with a positive h-convex function as follows.

Theorem 1.4 ([13])**.** Let $h : [0,1] \to \mathbb{R}$ be a positive function, $a, b \in [0,\infty)$ with $a < b, f, g : [a, b] \to \mathbb{R}$ functions and $fg \in L_1([a,b]), h \in L_1([0,1])$. If f is h-convex and g is s-convex in the second sense for some fixed $s \in (0,1]$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le M(a,b)\int_0^1 h(t)t^s ds + N(a,b)\int_0^1 h(1-t)t^s dt, \tag{7}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1.

Theorem 1.5 ([13])**.** Let $h : [0,1] \to \mathbb{R}$ be a positive function, $a, b \in [0,\infty)$ with $a < b, f, g : [a, b] \to \mathbb{R}$ functions and $fg \in L_1([a,b]), h \in L_1([0,1])$. If f is h-convex on $[a, b]$ and g is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0,1]$, then

$$\frac{2^{s-1}}{h(1/2)}f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)g(x)dx$$

$$\le M(a,b)\int_0^1 h(1-t)t^s ds + N(a,b)\int_0^1 h(t)t^s dt, \tag{8}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1.

The main aim of this paper is to give some new inequalities which are similar to the above results for the classes of s-convex (concave) functions and h-convex (concave) functions. As consequences, we also obtain some results for product of two functions belonging to two different classes of functions.

2 Inequalities for the class of s-convex functions

Before stating our main results, we shall recall some notions and definitions. The first notion was introduced by E. K. Godunova and V. I. Levin in 1985 (see [3]).

Definition 2.1 (see [3]). We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a Godunova - Levin function, or that f belongs to the class $Q(I)$, if f is non-negative and for all $a, b \in I$ and $t \in (0, 1)$, the following inequality holds

$$f(ta + (1 - t)b) \leq \frac{f(a)}{t} + \frac{f(b)}{1 - t}. \tag{9}$$

Restricting of the class of functions $Q(I)$ is the class $P(I)$ as follows.

Definition 2.2 (see [3]). We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a P-function, or that f belongs to the class $P(I)$, if f is non-negative and for all $a, b \in I$ and $t \in [0, 1]$, we have

$$f(ta + (1 - t)b) \leq f(a) + f(b). \tag{10}$$

The next concept is s-convex. It was introduced and investigated by Breckner in 1978 as a generalization of convex function.

Definition 2.3 (see [11]). Let $s \in (0, 1]$ be a real number and I be an interval on $[0, \infty)$. A function $f : I \to [0, \infty)$ is said to be s-convex (in the second sense), if

$$f(at + (1 - t)b) \leq t^s f(a) + (1 - t)^s f(b) \tag{11}$$

for all $a, b \in I$ and $t \in [0, 1]$. If (11) is reversed, then f is called to be s-concave.

A more general notion than the above notions is h-convex given in the following definition.

Definition 2.4 (see [11]). Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ is h-convex, or f belongs to the class $SX(h, I)$, if f is non-negative and for all $a, b \in I$ and $t \in (0, 1)$, we have

$$f(ta + (1 - t)b) \leq h(t)f(a) + h(1 - t)f(b). \tag{12}$$

If inequality (12) is reversed, then f is said to be h-concave, or shortly $f \in SV(h, I)$.

In Definition 2.4, if we choose $h(t) = t$, then f is an ordinary convex function; if $h(t) = 1/t$, then f belongs to the class $Q(I)$; if $h(t) = 1$, then f belongs to the class $P(I)$; and if $h(t) = t^s$ for some fixed $s \in (0, 1]$, then f belongs to the class of s-convex functions.

A number of properties and inequalities concerning these classes of functions can be referred to [3, 4, 5] for the classes $Q(I), P(I)$, and [1, 7, 8, 13, 10, 11] for the classes s-convex and h-convex.

We can now state the first main result as follows.

Theorem 2.1. Let $f, g : I \to \mathbb{R}$ be of the same s-convexity (i.e. both s-convex or both s-concave) and $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$M(a,b)\frac{1}{2s+1} + N(a,b)\frac{\Gamma(s+1)^2}{\Gamma(2s+2)} + \frac{1}{b-a}\int_a^b f(x)g(x)dx$$

$$\geq \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s f(a) + (x-a)^s f(b)]g(x)dx$$

$$+ \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s g(a) + (x-a)^s g(b)]f(x)dx. \quad (13)$$

where $\Gamma(\cdot)$ is the Gamma function and $M(a,b), N(a,b)$ are as in Theorem 1.1. If f and g are of the opposite s-convexity, then (13) is reversed.

Proof. According to (11), for all $t \in [0,1]$, we find that the inequality

$$[t^s f(a)+(1-t)^s f(b)-f(ta+(1-t)b)][t^s g(a)+(1-t)^s g(b)-g(ta+(1-t)b)] \geq 0 \quad (14)$$

holds if f and g are of the same s-convexity, else (14) is reversed. Inequality (14) is equivalent to

$$[t^s f(a) + (1-t)^s f(b)][t^s g(a) + (1-t)^s g(b)] + f(ta+(1-t)b)g(ta+(1-t)b)$$
$$\geq [t^s f(a) + (1-t)^s f(b)]g(ta+(1-t)b)$$
$$+ [t^s g(a) + (1-t)^s g(b)]f(ta+(1-t)b).$$

Integrating the above inequality with respect to t over $[0,1]$, we get

$$\int_0^1 [t^s f(a)+(1-t)^s f(b)][t^s g(a)+(1-t)^s g(b)]dt + \int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt$$

$$\geq \int_0^1 [t^s f(a) + (1-t)^s f(b)]g(ta+(1-t)b)dt$$

$$+ \int_0^1 [t^s g(a) + (1-t)^s g(b)]f(ta+(1-t)b)dt. \quad (15)$$

Directly computing, we obtain

$$\int_0^1 [t^s f(a) + (1-t)^s f(b)][t^s g(a) + (1-t)^s g(b)]dt$$

$$= \int_0^1 t^{2s} f(a)g(a)dt + \int_0^1 (1-t)^{2s} f(b)g(b)dt$$

$$+ \int_0^1 t^s(1-t)^s[f(a)g(b) + f(b)g(a)]dt$$

$$= M(a,b)\frac{1}{2s+1} + N(a,b)\int_0^1 t^s(1-t)^s dt.$$

By formulas (1.5.2) and (1.5.5) in [6], we have

$$\int_0^1 t^s(1-t)^s dt = B(s+1, s+1) = \frac{\Gamma(s+1)^2}{\Gamma(2s+2)},$$

where $B(\cdot, \cdot)$ is the Beta function, and so

$$\int_0^1 [t^s f(a) + (1-t)^s f(b)][t^s g(a) + (1-t)^s g(b)]dt = M(a,b)\frac{1}{2s+1} + N(a,b)\frac{\Gamma(s+1)^2}{\Gamma(2s+2)}. \tag{16}$$

Moreover, by substituting $x = ta + (1-t)b$, it is easy to see that

$$\int_0^1 f(ta + (1-t)b)g(ta + (1-tb))dt = \frac{1}{b-a}\int_a^b f(x)g(x)dx, \tag{17}$$

and

$$\int_0^1 [t^s f(a) + (1-t)^s f(b)]g(ta + (1-t)b)dt$$

$$= \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s f(a) + (x-a)^s f(b)]g(x)dx, \tag{18}$$

$$\int_0^1 [t^s g(a) + (1-t)^s g(b)]f(ta + (1-t)b)dt$$

$$= \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s g(a) + (x-a)^s g(b)]g(x)dx. \tag{19}$$

Substituting (16), (17), (18) and (19) in (15), we get the desired result. □

Theorem 2.2. Let $f, g : I \to \mathbb{R}$ be of the same s-convexity and $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$\frac{1}{b-a}\int_a^b f(x)[g(x) + g(a+b-x)]dx + 2^{2s-1}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$\geq \frac{2^s}{b-a}\int_a^b \left[f\left(\frac{a+b}{2}\right)g(x) + g\left(\frac{a+b}{2}\right)f(x)\right]dx. \tag{20}$$

If f and g are of the opposite s-convexity, then (20) is reversed.

Proof. Putting $x_0 = (a+b)/2$ and according to (11), for all $a < x < b$, we have the inequality

$$[f(x) + f(a+b-x) - 2^s f(x_0)][g(x) + g(a+b-x) - 2^s g(x_0)] \geq 0 \tag{21}$$

holds if f and g are of the same s-convexity, else (21) is reversed. Inequality (21) is equivalent to

$$[f(x) + f(a+b-x)][g(x) + g(a+b-x)] + 2^{2s}f(x_0)g(x_0)$$
$$\geq 2^s f(x_0)[g(x) + g(a+b-x)] + 2^s g(x_0)[f(x) + f(a+b-x)].$$

Integrating the above inequality with respect to x over $[a, b]$, we have

$$\frac{1}{b-a}\int_a^b [f(x) + f(a+b-x)][g(x) + g(a+b-x)]dx + 2^{2s}f(x_0)g(x_0)$$

$$\geq \frac{2^s}{b-a}\int_a^b g(x_0)[f(x) + f(a+b-x)]dx + \frac{2^s}{b-a}\int_a^b f(x_0)[g(x) + g(a+b-x)]dx$$

$$= \frac{2^{s+1}}{b-a}\int_a^b g(x_0)f(x)dx + \frac{2^{s+1}}{b-a}\int_a^b f(x_0)g(x)dx$$

$$= \frac{2^{s+1}}{b-a}\int_a^b [g(x_0)f(x) + f(x_0)g(x)]dx. \tag{22}$$

Besides, we have

$$\int_a^b [f(x) + f(a+b-x)][g(x) + g(a+b-x)]dx$$

$$= \int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx$$

$$+ \int_a^b f(x)g(a+b-x)dx + \int_a^b f(a+b-x)g(x)dx$$

$$= 2\int_a^b f(x)g(x)dx + 2\int_a^b f(x)g(a+b-x)dx$$

$$= 2\int_a^b f(x)[g(x) + g(a+b-x)]dx. \tag{23}$$

Combining (22) and (23), we obtain inequality (20). $\square$

A direct corollary of Theorem 2.2 when we require g is symmetric about $(a+b)/2$ as follows.

Corollary 2.3. Let $f, g : [a, b] \subset [0, \infty) \to \mathbb{R}$ be functions and g is symmetric about $(a+b)/2$. If f and g are of the same s-convexity, then the following inequality holds

$$\frac{2}{b-a}\int_a^b f(x)g(x)dx + 2^{2s-1}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$\geq \frac{2^s}{b-a}\int_a^b \left[f\left(\frac{a+b}{2}\right)g(x) + g\left(\frac{a+b}{2}\right)f(x)\right]dx. \tag{24}$$

If f and g are of the opposite s-convexity, then (24) is reversed.

Proof. By the symmetric about $(a+b)/2$ of g, we find that

$$g(x) = g(a+b-x),$$

for all $x \in [a, b]$. Hence, inequality (20) reduces to inequality (24). $\square$

Corollary 2.4. Let $f, g : [a, b] \subset [0, \infty) \to [0, \infty)$ be two s-concave functions. Then the following inequalities hold

$$\frac{1}{b-a}\int_a^b f(x)[g(x) + g(a+b-x)]dx + 2^{2s-1}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$\geq \frac{2^s}{b-a}\int_a^b \left[f\left(\frac{a+b}{2}\right)g(x) + g\left(\frac{a+b}{2}\right)f(x)\right]dx$$

$$\geq \frac{2^s}{s+1}f\left(\frac{a+b}{2}\right)[g(a) + g(b)] + \frac{2^s}{s+1}g\left(\frac{a+b}{2}\right)[f(a) + f(b)]. \tag{25}$$

Proof. The first inequality in (25) follows immediately from Theorem 2.2. In order to prove the second inequality in (25), we remark that

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b,$$

for all $a < x < b$. By s-concavity of f, we get

$$f(x) = f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) \geq \left(\frac{b-x}{b-a}\right)^s f(a) + \left(\frac{x-a}{b-a}\right)^s f(b).$$

Therefore,

$$\frac{1}{b-a}\int_a^b f(x)dx \geq \frac{1}{(b-a)^{s+1}}\int_a^b [(b-x)^s f(a) + (x-a)^s f(b)]dx$$

$$= \frac{1}{s+1}[f(a) + f(b)]. \tag{26}$$

Analogously, we can point out that

$$\frac{1}{b-a}\int_a^b g(x)dx \geq \frac{1}{s+1}[g(a) + g(b)]. \tag{27}$$

Since the non-negative of f and g, combining (26) and (27), we obtain

$$\frac{2^s}{b-a}\int_a^b \left[f\left(\frac{a+b}{2}\right)g(x) + g\left(\frac{a+b}{2}\right)f(x)\right]dx$$

$$= \frac{2^s}{b-a}\int_a^b f\left(\frac{a+b}{2}\right)g(x)dx + \frac{2^s}{b-a}\int_a^b g\left(\frac{a+b}{2}\right)f(x)dx$$

$$\geq \frac{2^s}{s+1}g\left(\frac{a+b}{2}\right)[f(a) + f(b)] + \frac{2^s}{s+1}f\left(\frac{a+b}{2}\right)[g(a) + g(b)].$$

This proves the desired results.   □

3   Inequalities for the class of h-convex functions

The main purpose of this section is to establish some inequalities for product of two functions to belong to the class of h-convex functions. To do this, we will first denote by I a nonempty interval of the set of real numbers.

Theorem 3.1. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be positive functions satisfying $h_1, h_2, h_1 h_2 \in L_1([0,1])$. Suppose that $f, g : I \to \mathbb{R}$ are of the same h-convexity (i.e. $f \in SX(h_1, I)$ and $g \in SX(h_2, I)$ or $f \in SV(h_1, I)$ and $g \in SV(h_2, I)$) and $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt + \frac{1}{b-a}\int_a^b f(x)g(x)dx$$

$$\geq \int_0^1 [h_1(t)f(a) + h_1(1-t)f(b)]g(ta + (1-t)b)dt$$

$$+ \int_0^1 [h_2(t)g(a) + h_2(1-t)g(b)]f(ta + (1-t)b)dt, \tag{28}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1. If f and g are of the opposite h-convexity, then (28) is reversed.

Proof. According to (12), for all $t \in (0,1)$, we find that the inequality

$$[h_1(t)f(a)+h_1(1-t)f(b)-f(ta+(1-t)b)][h_2(t)g(a)+h_2(1-t)g(b)-g(ta+(1-t)b)] \geq 0 \quad (29)$$

holds if f and g are of the same h-convexity, else (29) is reversed. Inequality (29) is equivalent to

$$[h_1(t)f(a)+h_1(1-t)f(b)][h_2(t)g(a)+h_2(1-t)g(b)]+f(ta+(1-t)b)g(ta+(1-t)b)$$
$$\geq f(ta+(1-t)b)[h_2(t)g(a)+h_2(1-t)g(b)]$$
$$+g(ta+(1-t)b)[h_1(t)f(a)+h_1(1-t)f(b)]. \quad (30)$$

By integrating (30) with respect to t over $[0,1]$ with noting that

$$\int_0^1 h_1(t)h_2(t)dt = \int_0^1 h_1(1-t)h_2(1-t)dt$$

and

$$\int_0^1 h_1(t)h_2(1-t)dt = \int_0^1 h_1(1-t)h_2(t)dt,$$

we get the desired result. $\qquad\square$

Theorem 3.2. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be positive functions satisfying $h_1, h_2, h_1h_2 \in L_1([0,1])$. Suppose that $f, g : I \to \mathbb{R}$ are of the same h-convexity (i.e. $f \in SX(h_1, I)$ and $g \in SX(h_2, I)$ or $f \in SV(h_1, I)$ and $g \in SV(h_2, I)$) and $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$\frac{1}{b-a}\int_a^b f(x)[g(x)+g(a+b-x)]dx + \frac{1}{2h_1(1/2)h_2(1/2)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\geq \frac{1}{b-a}\int_a^b \left[f\left(\frac{a+b}{2}\right)\frac{g(x)}{h_1(1/2)} + g\left(\frac{a+b}{2}\right)\frac{f(x)}{h_2(1/2)}\right]dx. \quad (31)$$

If f and g are of the opposite h-convexity, then (31) is reversed.

Proof. The proof runs as in the proof of Theorem 2.2. Here, in order to obtain the desired result, we start with observing that the following inequality

$$[h_1(1/2)f(x)+h_1(1/2)f(a+b-x)-f(x_0)][h_2(1/2)g(x)+h_2(1/2)g(a+b-x)-g(x_0)] \geq 0,$$

where $x_0 = (a+b)/2$, holds for all $a < x < b$ if f and g are of the same h-convexity, else the above inequality is reversed. $\qquad\square$

Corollary 3.3. For the same hypotheses as in Theorem 3.2. If we require that g is symmetric about $(a+b)/2$, then inequality (31) reduces to

$$\frac{2}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{2h_1(1/2)h_2(1/2)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\geq \frac{1}{b-a}\int_a^b \left[f\left(\frac{a+b}{2}\right)\frac{g(x)}{h_1(1/2)} + g\left(\frac{a+b}{2}\right)\frac{f(x)}{h_2(1/2)}\right]dx. \quad (32)$$

Proof. The above corollary is obtained from Theorem 3.2 with noting that g is symmetric about $(a+b)/2$. $\qquad\square$

Corollary 3.4. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be positive functions with $h_1, h_2, h_1 h_2 \in L_1([0,1])$. Suppose that $f : I \to \mathbb{R}$ is non-negative h_1-concave function and $g : I \to \mathbb{R}$ is non-negative h_2-concave function satisfying $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequalities hold

$$\frac{1}{b-a} \int_a^b f(x)[g(x) + g(a+b-x)]dx + \frac{1}{2h_1(1/2)h_2(1/2)} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)$$

$$\geq \frac{1}{b-a} \int_a^b \left[f\left(\frac{a+b}{2}\right) \frac{g(x)}{h_1(1/2)} + g\left(\frac{a+b}{2}\right) \frac{f(x)}{h_2(1/2)} \right] dx$$

$$\geq f\left(\frac{a+b}{2}\right) \frac{g(a) + g(b)}{h_1(1/2)} \int_0^1 h_2(t)dt + g\left(\frac{a+b}{2}\right) \frac{f(a) + f(b)}{h_2(1/2)} \int_0^1 h_1(t)dt. \quad (33)$$

Proof. The first inequality in (33) is similar to Theorem 3.2. The second inequality in (33) is proved as follows. For all $a < x < b$, we have

$$g(x) = g\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) \geq h_2\left(\frac{b-x}{b-a}\right)g(a) + h_2\left(\frac{x-b}{b-a}\right)g(b).$$

Integrating the above inequality over $[a, b]$ and substituting $t = (x-a)/(b-a)$, we get

$$\frac{1}{b-a} \int_a^b g(x)dx \geq \frac{1}{b-a} \int_a^b \left(h_2\left(\frac{b-x}{b-a}\right)g(a) + h_2\left(\frac{x-b}{b-a}\right)g(b) \right) dx$$

$$= \int_0^1 [h_2(1-t)g(a) + h_2(t)g(b)]dt$$

$$= g(a) \int_0^1 h_2(1-t)dt + g(b) \int_0^1 h_2(t)dt$$

$$= [g(a) + g(b)] \int_0^1 h_2(t)dt. \quad (34)$$

Multiplying (34) by non-negative quantity $\frac{1}{h_1(1/2)} f(\frac{a+b}{2})$, we obtain

$$\frac{1}{b-a} \int_a^b f\left(\frac{a+b}{2}\right) \frac{g(x)}{h_1(1/2)} dx \geq f\left(\frac{a+b}{2}\right) \frac{g(a) + g(b)}{h_1(1/2)} \int_0^1 h_2(t)dt. \quad (35)$$

Analogously, we can point out that

$$\frac{1}{b-a} \int_a^b g\left(\frac{a+b}{2}\right) \frac{f(x)}{h_2(1/2)} dx \geq g\left(\frac{a+b}{2}\right) \frac{f(a) + f(b)}{h_2(1/2)} \int_0^1 h_1(t)dt. \quad (36)$$

Combining (35) and (36) reduces to the desired result. $\square$

4 Inequalities for product of different kinds of convex functions

In this section, we shall give some inequalities for product of different kinds of convex functions as corollaries of Theorem 3.1 and 3.2 in the previous section.

Proposition 4.1. Let $h : [0,1] \to \mathbb{R}$ be an integrable positive function and $s \in (0,1]$ is a real number. Suppose that $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ is a h-convex (concave) function and $g : [a,b] \subset [0,\infty) \to \mathbb{R}$ is a s-convex (concave, respectively) function satisfying $fg \in L_1([a,b])$. Then the following inequality holds

$$M(a,b) \int_0^1 h(t)t^s dt + N(a,b) \int_0^1 h(t)(1-t)^s dt + \frac{1}{b-a} \int_a^b f(x)g(x)dx$$

$$\geq \int_0^1 [h(t)f(a) + h(1-t)f(b)]g(ta + (1-t)b)dt$$

$$+ \int_0^1 [t^s g(a) + (1-t)^s g(b)]f(ta + (1-t)b)dt, \quad (37)$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1. If f is h-convex (concave) and g is s-concave (convex, respectively), then (37) is reversed.

Proof. If choosing $h_1(t) = h(t)$ and $h_2(t) = t^s$ for all $t \in [0,1]$ in Theorem 3.1, then we obtain the desired result. $\square$

Proposition 4.2. Let $h : [0,1] \to \mathbb{R}$ be an integrable positive function satisfying $h(t) = h(1-t)$ for all $t \in [0,1/2]$. If $f : I \subset \mathbb{R} \to \mathbb{R}$ is h-convex and $g : I \to \mathbb{R}$ is P-function satisfying $fg \in L_1(I)$. Then, for all $a,b \in I$ with $a < b$, the following inequality holds

$$[M(a,b) + N(a,b)] \int_0^1 h(t)dt + \frac{1}{b-a} \int_a^b f(x)g(x)dx$$

$$\geq \frac{g(a) + g(b)}{b-a} \int_a^b f(x)dx + [f(a) + f(b)] \int_0^1 h(t)g(ta + (1-t)b)dt, \quad (38)$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1. If f is h-concave and g is P-function, then (38) is reversed.

Proof. If choosing $h_1(t) = h(t)$ and $h_2(t) = 1$ for all $t \in [0,1]$ in Theorem 3.1, then we obtain the desired result. $\square$

Proposition 4.3. Let $h : [0,1] \to \mathbb{R}$ be a integrable positive function and $s \in (0,1]$ is a real number. If $f : I \subset [0,\infty) \to \mathbb{R}$ is h-convex (concave) and $g : I \to \mathbb{R}$ is s-convex (concave, respectively) satisfying $fg \in L_1(I)$. Then, for all $a,b \in I$ with $a < b$, the following inequality holds

$$\frac{1}{b-a} \int_a^b f(x)[g(x) + g(a+b-x)]dx + \frac{2^{s-1}}{h(1/2)} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)$$

$$\geq \frac{1}{b-a} \int_a^b \left[f\left(\frac{a+b}{2}\right) \frac{g(x)}{h(1/2)} + 2^s g\left(\frac{a+b}{2}\right) f(x) \right] dx. \quad (39)$$

If f is h-convex (concave) and g is s-concave (convex, respectively), then (40) is reversed.

Proof. If choosing $h_1(t) = h(t)$ and $h_2(t) = t^s$ for all $t \in [0,1]$ in Theorem 3.2, then we obtain the desired result. $\square$

Proposition 4.4. If $f : I \subset \mathbb{R} \to \mathbb{R}$ is convex and $g : I \to \mathbb{R}$ is P-function satisfying $fg \in L_1(I)$. Then, for all $a, b \in I$ with $a < b$, the following inequality holds

$$\frac{1}{b-a} \int_a^b f(x)[g(x) + g(a + b - x)]dx + f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$\geq \frac{1}{b-a} \int_a^b \left[2f\left(\frac{a+b}{2}\right)g(x) + g\left(\frac{a+b}{2}\right)f(x)\right]dx. \quad (40)$$

If f is concave and g is P-function, then (40) is reversed.

Proof. If choosing $h_1(t) = t$ and $h_2(t) = 1$ for all $t \in [0, 1]$ in Theorem 3.2, then we obtain the desired result. $\qquad \square$

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