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$\mathcal{I}^*_{*g}$-CLOSED SETS

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Abstaract — In this paper, we introduce the notion of $\mathcal{I}^*_{*g}$-closed sets and prove that this class of sets is stronger than the class of gs$^*_{\mathcal{I}}$-closed sets as well as the class of $\mathcal{I}_g$-closed sets. Characterizations and properties of $\mathcal{I}^*_{*g}$-closed sets and $\mathcal{I}^*_{*g}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}^*_{*g}$-open sets.

Keywords — **g-closed set, $\mathcal{I}^*_{*g}$-closed set, gs$^*_{\mathcal{I}}$-closed set, weakly $\mathcal{I}_{rg}$-closed set*

1 Introduction and Preliminaries

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(i) A $\in \mathcal{I}$ and B $\subseteq$ A $\Rightarrow$ B $\in \mathcal{I}$ and
(ii) A $\in \mathcal{I}$ and B $\in \mathcal{I} \Rightarrow$ A $\cup$ B $\in \mathcal{I}$ [12].

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^*$: $\wp(X) \rightarrow \wp(X)$, called a local function [12] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A $\subseteq$ X, $A^*(\mathcal{I},\tau)=\{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x)=\{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local functions [[11], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator cl*(.) for a topology $\tau^*(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by cl*(A)=A $\cup$ $A^*(\mathcal{I},\tau)$ [22]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal space or an ideal topological space. N is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal space (X, τ, $\mathcal{I}$) is $\star$-closed [11] (resp. $\star$-dense in itself [9]) if $A^* \subseteq A$ (resp. $A \subseteq A^*$). A subset A

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of an ideal space (X, τ, $\mathcal{I}$) is $\mathcal{I}_g$-closed [2, 16] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and int*(A) will denote the interior of A in (X, τ^*).

A subset A of a space (X, τ) is an α-open [19] (resp. regular open [21], semi-open [13], preopen [15]) set if $A \subseteq$ int(cl(int(A))) (resp. A = int(cl(A)), $A \subseteq$ cl(int(A)), A $\subseteq$ int(cl(A))). The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of A in (X, τ^α) is denoted by $cl_\alpha(A)$. A subset A of a space (X, τ) is said to be g-closed [14] if cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is open. A subset A of a space (X, τ) is said to be $\hat{g}$-closed [23] if cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is semi-open. A subset A of a space (X, τ) is said to be $\hat{g}$-open [23] if its complement is $\hat{g}$-closed. A subset A of a topological space (X, τ) is said to be *g-closed [10] if cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is $\hat{g}$-open in X. The complement of *g-closed set is said to be *g-open. The intersection of all *g-closed sets of X containing a subset A of X is denoted by $^*g$cl(A). An ideal $\mathcal{I}$ is said to be codense [3] or τ-boundary [18] if $\tau \cap \mathcal{I}=\{\emptyset\}$. $\mathcal{I}$ is said to be completely codense [3] if PO(X) $\cap\, \mathcal{I}=\{\emptyset\}$, where PO(X) is the family of all preopen sets in (X, τ). Every completely codense ideal is codense but not the converse [3].

The following Lemmas will be useful in the sequel.

Lemma 1.1. *Let (X, τ, $\mathcal{I}$) be an ideal space and $A \subseteq X$. If $A \subseteq A^*$, then A^* =cl(A^*) =cl(A)=cl*(A) [[20], Theorem 5].*

Lemma 1.2. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then $\mathcal{I}$ is codense if and only if $G \subseteq G^*$ for every semi-open set G in X [[20], Theorem 3].*

Lemma 1.3. *Let (X, τ, $\mathcal{I}$) be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^* \subseteq \tau^\alpha$ [[20], Theorem 6].*

Result 1.4. *If (X, τ) is a topological space, then every closed set is *g-closed but not conversely [10].*

Lemma 1.5. *If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$ ideal space and A is an $\mathcal{I}_g$-closed set, then A is a $\star$-closed set [[16], Corollary 2.2].*

Lemma 1.6. *Every g-closed set is $\mathcal{I}_g$-closed but not conversely [[2], Theorem 2.1].*

Definition 1.7. *A subset G of an ideal topological space (X, τ, $\mathcal{I}$) is said to be*

1. *$\mathcal{I}_g$-closed [2] if $G^* \subseteq H$ whenever $G \subseteq H$ and H is open in (X, τ, $\mathcal{I}$).*
2. *$\mathcal{I}_{rg}$-closed [17] if $G^* \subseteq H$ whenever $G \subseteq H$ and H is regular open in (X, τ, $\mathcal{I}$).*
3. *$pre^*_{\mathcal{I}}$-open [4] if $G \subseteq$ int*(cl(G)).*
4. *$pre^*_{\mathcal{I}}$-closed [4] if $X \backslash G$ is $pre^*_{\mathcal{I}}$-open.*
5. *$\mathcal{I}$-R closed [1] if G = cl*(int(G)).*
6. *$*$-closed [11] if G = cl*(G) or $G^* \subseteq G$.*

Remark 1.8. *[5] In any ideal topological space, every $\mathcal{I}$-R closed set is $*$-closed but not conversely.*

Definition 1.9. *[5] Let (X, τ, $\mathcal{I}$) be an ideal topological space. A subset G of X is said to be a weakly $\mathcal{I}_{rg}$-closed set if (int(G))* $\subseteq$ H whenever G $\subseteq$ H and H is a regular open set in X.*

Definition 1.10. *[5] Let (X, τ, $\mathcal{I}$) be an ideal topological space. A subset G of X is said to be a weakly $\mathcal{I}_{rg}$-open set if X$\backslash$G is a weakly $\mathcal{I}_{rg}$-closed set.*

Remark 1.11. *[5] Let (X, τ, $\mathcal{I}$) be an ideal topological space. The following diagram holds for a subset G $\subseteq$ X:*

$$\begin{array}{ccccc}
\mathcal{I}_g\text{-closed} & \longrightarrow & \mathcal{I}_{rg}\text{-closed} & \longrightarrow & \text{weakly } \mathcal{I}_{rg}\text{-closed} \\
\uparrow & & & & \uparrow \\
*\text{-closed} & & & & pre^*_{\mathcal{I}}\text{-closed} \\
\uparrow & & & & \\
\mathcal{I}\text{-}R\text{-closed} & & & &
\end{array}$$

These implications are not reversible.

Definition 1.12. *[7, 8] A subset K of an ideal topological space (X, τ, $\mathcal{I}$) is said to be*

1. *semi*-$\mathcal{I}$-open if K $\subseteq$ cl(int*(K)),*
2. *semi*-$\mathcal{I}$-closed if its complement is semi*-$\mathcal{I}$-open.*

Definition 1.13. *[7] The semi*-$\mathcal{I}$-closure of a subset K of an ideal topological space (X, τ, $\mathcal{I}$), denoted by $s^*_{\mathcal{I}}cl(K)$, is defined by the intersection of all semi*-$\mathcal{I}$-closed sets of X containing K.*

Theorem 1.14. *[7] For a subset K of an ideal topological space (X, τ, $\mathcal{I}$), $s^*_{\mathcal{I}}cl(K)$ = K $\cup$ int(cl*(K)).*

Definition 1.15. *[6] Let (X, τ, $\mathcal{I}$) be an ideal topological space and K $\subseteq$ X. K is called*

1. *generalized semi*-$\mathcal{I}$-closed ($gs^*_{\mathcal{I}}$-closed) in (X, τ, $\mathcal{I}$) if $s^*_{\mathcal{I}}cl(K)$ $\subseteq$ O whenever K $\subseteq$ O and O is an open set in (X, τ, $\mathcal{I}$).*
2. *generalized semi*-$\mathcal{I}$-open ($gs^*_{\mathcal{I}}$-open) in (X, τ, $\mathcal{I}$) if X$\backslash$K is a $gs^*_{\mathcal{I}}$-closed set in (X, τ, $\mathcal{I}$).*

2 $\mathcal{I}^*_{*g}$-closed Sets

Definition 2.1. *A subset A of an ideal space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}^*_{*g}$-closed if A* $\subseteq$ U whenever A $\subseteq$ U and U is *g-open.*

Definition 2.2. *A subset A of an ideal space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}^*_{*g}$-open if X−A is $\mathcal{I}^*_{*g}$-closed.*

Theorem 2.3. *If (X, τ, $\mathcal{I}$) is any ideal space, then every $\mathcal{I}^*_{*g}$-closed set is $\mathcal{I}_g$-closed but not conversely.*

Proof. It follows from the fact that every open set is *g-open. □

Example 2.4. *If $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$, then $\{b\}$ is $\mathcal{I}_g$-closed set but not $\mathcal{I}^*_{*g}$-closed set.*

The following Theorem gives characterizations of $\mathcal{I}^*_{*g}$-closed sets.

Theorem 2.5. *If $(X, \tau, \mathcal{I})$ is any ideal space and $A \subseteq X$, then the following are equivalent.*
(a) A is $\mathcal{I}^*_{*g}$-closed.
*(b) $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is *g-open in X.*
(c) For all $x \in cl^*(A)$, $^*gcl(\{x\}) \cap A \neq \emptyset$.
*(d) $cl^*(A) - A$ contains no nonempty *g-closed set.*
*(e) $A^* - A$ contains no nonempty *g-closed set.*

Proof. (a) ⇒ (b) If A is $\mathcal{I}^*_{*g}$-closed, then $A^* \subseteq U$ whenever $A \subseteq U$ and U is *g-open in X and so $cl^*(A) = A \cup A^* \subseteq U$ whenever $A \subseteq U$ and U is *g-open in X. This proves (b).

(b) ⇒ (c) Suppose $x \in cl^*(A)$. If $^*gcl(\{x\}) \cap A = \emptyset$, then $A \subseteq X - {}^*gcl(\{x\})$. By (b), $cl^*(A) \subseteq X - {}^*gcl(\{x\})$, a contradiction, since $x \in cl^*(A)$.

(c) ⇒ (d) Suppose $F \subseteq cl^*(A) - A$, F is *g-closed and $x \in F$. Since $F \subseteq X - A$, then $A \subseteq X - F$, $^*gcl(\{x\}) \cap A = \emptyset$. Since $x \in cl^*(A)$ by (c), $^*gcl(\{x\}) \cap A \neq \emptyset$. Therefore $cl^*(A) - A$ contains no nonempty *g-closed set.

(d) ⇒ (e) Since $cl^*(A) - A = (A \cup A^*) - A = (A \cup A^*) \cap A^c = (A \cap A^c) \cup (A^* \cap A^c) = A^* \cap A^c = A^* - A$, therefore $A^* - A$ contains no nonempty *g-closed set.

(e) ⇒ (a) Let $A \subseteq U$ where U is *g-open set. Therefore $X - U \subseteq X - A$ and so $A^* \cap (X - U) \subseteq A^* \cap (X - A) = A^* - A$. Therefore $A^* \cap (X - U) \subseteq A^* - A$. Since A^* is always closed set, so $A^* \cap (X - U)$ is a *g-closed set contained in $A^* - A$. Therefore $A^* \cap (X - U) = \emptyset$ and hence $A^* \subseteq U$. Therefore A is $\mathcal{I}^*_{*g}$-closed. □

Theorem 2.6. *Every $\star$-closed set is $\mathcal{I}^*_{*g}$-closed but not conversely.*

Proof. Let A be a $\star$-closed, then $A^* \subseteq A$. Let $A \subseteq U$ where U is *g-open. Hence $A^* \subseteq U$ whenever $A \subseteq U$ and U is *g-open. Therefore A is $\mathcal{I}^*_{*g}$-closed. □

Example 2.7. *If $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$, then $\{a, b\}$ is $\mathcal{I}^*_{*g}$-closed set but not $\star$-closed set.*

Theorem 2.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space. For every $A \in \mathcal{I}$, A is $\mathcal{I}^*_{*g}$-closed.*

Proof. Let $A \subseteq U$ where U is *g-open set. Since $A^* = \emptyset$ for every $A \in \mathcal{I}$, then $cl^*(A) = A \cup A^* = A \subseteq U$. Therefore, by Theorem 2.5, A is $\mathcal{I}^*_{*g}$-closed. □

Theorem 2.9. *If $(X, \tau, \mathcal{I})$ is an ideal space, then A^* is always $\mathcal{I}^*_{*g}$-closed for every subset A of X.*

Proof. Let $A^* \subseteq U$ where U is *g-open. Since $(A^*)^* \subseteq A^*$ [11], we have $(A^*)^* \subseteq U$ whenever $A^* \subseteq U$ and U is *g-open. Hence A^* is $\mathcal{I}^*_{*g}$-closed. □

Theorem 2.10. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then every $\mathcal{I}^*_{*g}$-closed, *g-open set is $\star$-closed set.*

Proof. Since A is $\mathcal{I}^*_{*g}$-closed and *g-open, then A* $\subseteq$ A whenever A $\subseteq$ A and A is *g-open. Hence A is $\star$-closed. □

Corollary 2.11. *If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$ ideal space and A is an $\mathcal{I}^*_{*g}$-closed set, then A is $\star$-closed set.*

Corollary 2.12. *Let (X, τ, $\mathcal{I}$) be an ideal space and A be an $\mathcal{I}^*_{*g}$-closed set. Then the following are equivalent.*

a) A is a $\star$-closed set.
b) cl*(A)$-$A is a *g-closed set.
c) A*$-$A is a *g-closed set.

Proof. (a) $\Rightarrow$ (b) If A is $\star$-closed, then A* $\subseteq$ A and so cl*(A)$-$A=(A $\cup$ A*)$-$A= $\emptyset$. Hence cl*(A)$-$A is *g-closed set.

(b) $\Rightarrow$ (c) Since cl*(A)$-$A=A*$-$A and so A*$-$A is *g-closed set.

(c) $\Rightarrow$ (a) If A*$-$A is a *g-closed set, since A is $\mathcal{I}^*_{*g}$-closed set, by Theorem 2.5, A*$-$A= $\emptyset$ and so A is $\star$-closed. □

Definition 2.13. *A subset A of a topological space (X, τ) is said to be $^*g^*$-closed if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open in X.*

Theorem 2.14. *Every closed set is $^*g^*$-closed but not conversely.*

Example 2.15. *In Example 2.7, {a, b} is $^*g^*$-closed set but not closed set.*

Theorem 2.16. *Every $^*g^*$-closed set is g-closed but not conversely.*

Proof. It follows from the fact that every open set is *g-open. □

Example 2.17. *In Example 2.4, {a} is g-closed set but not $^*g^*$-closed.*

Theorem 2.18. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then every $^*g^*$-closed set is an $\mathcal{I}^*_{*g}$-closed set but not conversely.*

Proof. Let A be a $^*g^*$-closed set. Then cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open. We have cl*(A) $\subseteq$ cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open. Hence A is $\mathcal{I}^*_{*g}$-closed. □

Example 2.19. *In Example 2.4, {a} is $\mathcal{I}^*_{*g}$-closed set but not $^*g^*$-closed.*

Theorem 2.20. *If (X, τ, $\mathcal{I}$) is an ideal space and A is a $\star$-dense in itself, $\mathcal{I}^*_{*g}$-closed subset of X, then A is $^*g^*$-closed.*

Proof. Suppose A is a $\star$-dense in itself, $\mathcal{I}^*_{*g}$-closed subset of X. Let A $\subseteq$ U where U is *g-open. Then by Theorem 2.5 (b), cl*(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open. Since A is $\star$-dense in itself, by Lemma 1.1, cl(A)=cl*(A). Therefore cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is *g-open. Hence A is $^*g^*$-closed. □

Corollary 2.21. *If (X, τ, $\mathcal{I}$) is any ideal space where $\mathcal{I}$={$\emptyset$}, then A is $\mathcal{I}^*_{*g}$-closed if and only if A is $^*g^*$-closed.*

Proof. From the fact that for $\mathcal{I}=\{\emptyset\}$, A*=cl(A) $\supseteq$ A. Therefore A is $\star$-dense in itself. Since A is $\mathcal{I}^*_{*g}$-closed, by Theorem 2.20, A is $^*g^*$-closed. Conversely, by Theorem 2.18, every $^*g^*$-closed set is $\mathcal{I}^*_{*g}$-closed set. □

Corollary 2.22. *If (X, τ, $\mathcal{I}$) is any ideal space where $\mathcal{I}$ is codense and A is a semi-open, $\mathcal{I}^*_{*g}$-closed subset of X, then A is $^*g^*$-closed.*

Proof. By Lemma 1.2, A is $\star$-dense in itself. By Theorem 2.20, A is $^*g^*$-closed. □

Example 2.23. *If $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$, then $\{a\}$ is $\mathcal{I}^*_{*g}$-closed set but not g-closed.*

Example 2.24. *In Example 2.4, $\{b\}$ is g-closed set but not $\mathcal{I}^*_{*g}$-closed.*

Remark 2.25. *By Example 2.23 and Example 2.24, g-closed sets and $\mathcal{I}^*_{*g}$-closed sets are independent.*

Example 2.26. *In Example 2.4, $\{a\}$ is $\star$-closed set but not $^*g^*$-closed.*

Example 2.27. *In Example 2.7, $\{a, b\}$ is $^*g^*$-closed set but not $\star$-closed.*

Remark 2.28. *By Example 2.26 and Example 2.27, $^*g^*$-closed sets and $\star$-closed sets are independent.*

Remark 2.29. *We have the following implications for the subsets stated above.*

$$\begin{array}{ccccc} \text{closed} & \rightarrow & {}^*g^*\text{-closed} & \rightarrow & g\text{-closed} \\ \downarrow & & \downarrow & & \downarrow \\ \star\text{-closed} & \rightarrow & \mathcal{I}^*_{*g}\text{-closed} & \rightarrow & \mathcal{I}_g\text{-closed} \end{array}$$

Theorem 2.30. *Let (X, τ, $\mathcal{I}$) be an ideal space and $A \subseteq X$. Then A is $\mathcal{I}^*_{*g}$-closed if and only if A=F−N where F is $\star$-closed and N contains no nonempty *g-closed set.*

Proof. If A is $\mathcal{I}^*_{*g}$-closed, then by Theorem 2.5 (e), N=A*−A contains no nonempty *g-closed set. If F=cl*(A), then F is $\star$-closed such that F−N=(A ∪ A*)−(A*−A)=(A ∪ A*) ∩ (A* ∩ A^c)^c=(A ∪ A*) ∩ ((A*)^c ∪ A)=(A ∪ A*) ∩ (A ∪ (A*)^c)=A ∪ (A* ∩ (A*)^c)=A.

Conversely, suppose A=F−N where F is $\star$-closed and N contains no nonempty *g-closed set. Let U be a *g-open set such that A ⊆ U. Then F−N ⊆ U and F ∩ (X−U) ⊆ N. Now A ⊆ F and F* ⊆ F then A* ⊆ F* and so A* ∩ (X−U) ⊆ F* ∩ (X−U) ⊆ F ∩ (X−U) ⊆ N. By hypothesis, since A* ∩ (X−U) is *g-closed, A* ∩ (X−U)=∅ and so A* ⊆ U. Hence A is $\mathcal{I}^*_{*g}$-closed. □

Theorem 2.31. *Let (X, τ, $\mathcal{I}$) be an ideal space and $A \subseteq X$. If $A \subseteq B \subseteq A^*$, then A^*=B^* and B is $\star$-dense in itself.*

Proof. Since A ⊆ B, then A* ⊆ B* and since B ⊆ A*, then B* ⊆ (A*)* ⊆ A*. Therefore A*=B* and B ⊆ A* ⊆ B*. Hence proved. □

Theorem 2.32. *Let (X, τ, $\mathcal{I}$) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq cl^*(A)$ and A is $\mathcal{I}^*_{*g}$-closed, then B is $\mathcal{I}^*_{*g}$-closed.*

Proof. Since A is $\mathcal{I}^*_{*g}$-closed, then by Theorem 2.5 (d), cl*(A)−A contains no nonempty *g-closed set. Since cl*(B)−B ⊆ cl*(A)−A and so cl*(B)−B contains no nonempty *g-closed set. Hence B is $\mathcal{I}^*_{*g}$-closed. □

Corollary 2.33. *Let (X, τ, $\mathcal{I}$) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq A^*$ and A is $\mathcal{I}^*_{*g}$-closed, then A and B are $^*g^*$-closed sets.*

Proof. Let A and B be subsets of X such that $A \subseteq B \subseteq A^* \Rightarrow A \subseteq B \subseteq A^* \subseteq cl^*(A)$ and A is $\mathcal{I}^*_{*g}$-closed. By the above Theorem, B is $\mathcal{I}^*_{*g}$-closed. Since $A \subseteq B \subseteq A^*$, then $A^*=B^*$ and so A and B are ⋆-dense in itself. By Theorem 2.20, A and B are $^*g^*$-closed. □

The following Theorem gives a characterization of $\mathcal{I}^*_{*g}$-open sets.

Theorem 2.34. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. Then A is $\mathcal{I}^*_{*g}$-open if and only if F ⊆ int*(A) whenever F is *g-closed and F ⊆ A.*

Proof. Suppose A is $\mathcal{I}^*_{*g}$-open. If F is *g-closed and F ⊆ A, then X−A ⊆ X−F and so cl*(X−A) ⊆ X−F by Theorem 2.5 (b). Therefore F ⊆ X−cl*(X−A)=int*(A). Hence F ⊆ int*(A).

Conversely, suppose the condition holds. Let U be a *g-open set such that X−A ⊆ U. Then X−U ⊆ A and so X−U ⊆ int*(A). Therefore cl*(X−A) ⊆ U. By Theorem 2.5 (b), X−A is $\mathcal{I}^*_{*g}$-closed. Hence A is $\mathcal{I}^*_{*g}$-open. □

Corollary 2.35. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. If A is $\mathcal{I}^*_{*g}$-open, then F ⊆ int*(A) whenever F is closed and F ⊆ A.*

Theorem 2.36. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. If A is $\mathcal{I}^*_{*g}$-open and int*(A) ⊆ B ⊆ A, then B is $\mathcal{I}^*_{*g}$-open.*

Proof. Since A is $\mathcal{I}^*_{*g}$-open, then X−A is $\mathcal{I}^*_{*g}$-closed. By Theorem 2.5 (d), cl*(X−A)−(X−A) contains no nonempty *g-closed set. Since int*(A) ⊆ int*(B) which implies that cl*(X−B) ⊆ cl*(X−A) and so cl*(X−B)−(X−B) ⊆ cl*(X−A)−(X−A). Hence B is $\mathcal{I}^*_{*g}$-open. □

The following Theorem gives a characterization of $\mathcal{I}^*_{*g}$-closed sets in terms of $\mathcal{I}^*_{*g}$-open sets.

Theorem 2.37. *Let (X, τ, $\mathcal{I}$) be an ideal space and A ⊆ X. Then the following are equivalent.*

(a) A is $\mathcal{I}^*_{*g}$-closed.
(b) A ∪ (X−A) is $\mathcal{I}^*_{*g}$-closed.*
(c) A−A is $\mathcal{I}^*_{*g}$-open.*

Proof. (a) ⇒ (b) Suppose A is $\mathcal{I}^*_{*g}$-closed. If U is any *g-open set such that A ∪ (X−A*) ⊆ U, then $X-U \subseteq X-(A \cup (X-A^*))=X \cap (A \cup (A^*)^c)^c=A^* \cap A^c=A^*-A$. Since A is $\mathcal{I}^*_{*g}$-closed, by Theorem 2.5 (e), it follows that X−U=∅ and so X=U. Now A ∪ (X−A*) ⊆ X and so $(A \cup (X-A^*))^* \subseteq X^* \subseteq X=U$. Hence A ∪ (X−A*) is $\mathcal{I}^*_{*g}$-closed.

(b) ⇒ (a) Suppose A ∪ (X−A*) is $\mathcal{I}^*_{*g}$-closed. If F is any *g-closed set such that F ⊆ A*−A, then F ⊆ A* and F ⊈ A. Hence X−A* ⊆ X−F and A ⊆ X−F. Therefore

A $\cup$ (X$-$A*) $\subseteq$ A $\cup$ (X$-$F)= X$-$F and X$-$F is *g-open. Since (A $\cup$ (X$-$A*))* $\subseteq$ X$-$F $\Rightarrow$ A* $\cup$ (X$-$A*)* $\subseteq$ X$-$F and so A* $\subseteq$ X$-$F $\Rightarrow$ F $\subseteq$ X$-$A*. Since F $\subseteq$ A*, it follows that F=$\emptyset$. Hence A is $\mathcal{I}^*_{*g}$-closed.

(b) $\Leftrightarrow$ (c) Since X$-$(A*$-$A)=X $\cap$ (A* $\cap$ A^c)^c=X $\cap$ ((A*)^c $\cup$ A)=(X $\cap$ (A*)^c) $\cup$ (X $\cap$ A)=A $\cup$ (X$-$A*), it is obvious. □

Theorem 2.38. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then every subset of X is $\mathcal{I}^*_{*g}$-closed if and only if every *g-open set is $\star$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}^*_{*g}$-closed. If U $\subseteq$ X is *g-open, then U is $\mathcal{I}^*_{*g}$-closed and so U* $\subseteq$ U. Hence U is $\star$-closed. Conversely, suppose that every *g-open set is $\star$-closed. If U is *g-open set such that A $\subseteq$ U $\subseteq$ X, then A* $\subseteq$ U* $\subseteq$ U and so A is $\mathcal{I}^*_{*g}$-closed. □

The following Theorem gives a characterization of normal spaces in terms of $\mathcal{I}^*_{*g}$-open sets.

Theorem 2.39. *Let (X, τ, $\mathcal{I}$) be an ideal space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(a) X is normal.

(b) For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}^*_{*g}$-open sets U and V such that A $\subseteq$ U and B $\subseteq$ V.

(c) For any closed set A and open set V containing A, there exists an $\mathcal{I}^*_{*g}$-open set U such that A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ V.

Proof. (a) $\Rightarrow$ (b) The proof follows from the fact that every open set is $\mathcal{I}^*_{*g}$-open.

(b) $\Rightarrow$ (c) Suppose A is closed and V is an open set containing A. Since A and X$-$V are disjoint closed sets, there exist disjoint $\mathcal{I}^*_{*g}$-open sets U and W such that A $\subseteq$ U and X$-$V $\subseteq$ W. Since X$-$V is *g-closed and W is $\mathcal{I}^*_{*g}$-open, X$-$V $\subseteq$ int*(W) and so X$-$int*(W) $\subseteq$ V. Again U $\cap$ W=$\emptyset$ $\Rightarrow$ U $\cap$ int*(W)=$\emptyset$ and so U $\subseteq$ X$-$int*(W) $\Rightarrow$ cl*(U) $\subseteq$ X$-$int*(W) $\subseteq$ V. U is the required $\mathcal{I}^*_{*g}$-open sets with A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ V.

(c) $\Rightarrow$ (a) Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}^*_{*g}$-open set U such that A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ X$-$B. Since U is $\mathcal{I}^*_{*g}$-open, A $\subseteq$ int*(U). Since $\mathcal{I}$ is completely codense, by Lemma 1.3, $\tau^* \subseteq \tau^\alpha$ and so int*(U) and X$-$cl*(U) are in τ^α. Hence A $\subseteq$ int*(U) $\subseteq$ int(cl(int(int*(U))))=G and B $\subseteq$ X$-$cl*(U) $\subseteq$ int(cl(int(X$-$cl*(U))))=H. G and H are the required disjoint open sets containing A and B respectively, which proves (a). □

Remark 2.40. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. By Remark 1.11, Definition 1.15, Definition 2.1 and Theorem 2.3, the following diagram holds for a subset G $\subseteq$ X:*

$$\begin{array}{ccccccc} \text{gs}^*_{\mathcal{I}}\text{-closed} & & & & & & \\ \uparrow & & & & & & \\ \mathcal{I}^*_{*g}\text{-closed} & \longrightarrow & \mathcal{I}_g\text{-closed} & \longrightarrow & \mathcal{I}_{rg}\text{-closed} & \longrightarrow & \text{weakly } \mathcal{I}_{rg}\text{-closed} \end{array}$$

These implications are not reversible.

Example 2.41. *If X = {a, b, c, d}, τ = {$\emptyset$, X, {a}, {b, c}, {a, b, c}} and $\mathcal{I}$ = {$\emptyset$, {d}}, then {a} is gs$^*_{\mathcal{I}}$-closed set but not $\mathcal{I}^*_{*g}$-closed.*

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