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REFINED SOFT SETS AND ITS APPLICATIONS

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Abstract - Many disciplines, including engineering, economics, medical science and social science are highly dependent on the task of modeling and computing uncertain data. When the uncertainty is highly complicated and difficult to characterize, classical mathematical approaches are often insufficient to derive effective or useful models. Testifying to the importance of uncertainties that cannot be defined by classical mathematics, researchers are introducing alternative theories every day. In addition to classical probability theory, some of the most important results on this topic are fuzzy sets, intuitionistic fuzzy sets, vague sets, interval-valued fuzzy set and rough sets. But each of these theories has its inherent limitations as pointed out by Molodtsov. For example, in probability theory, we require a large number of experiments in order to check the stability of the system. To define a membership function in case of fuzzy set theory is not always an easy task. Theory of rough sets requires an equivalence relation defined on the universal set under consideration. But in many real life situations such an equivalence relation is very difficult to find due to imprecise human knowledge. Perhaps the above mentioned difficulties associated with these theories are due to their incompatibility with the parameterization tools. Molodtsov introduced soft set theory as a completely new approach for modeling vagueness and uncertainty. This so-called soft set theory is free from the above mentioned difficulties as it has enough parameters. In soft set theory, the problem of setting membership function simply doesn't arise. This makes the theory convenient and easy to apply in practice. Soft set theory has potential applications in various fields including smoothness of functions, game theory, operations research, Riemann integration, probability theory and measurement theory. Most of these applications have already been demonstrated by Molodtsov.

In this paper a new approach called refined soft sets is presented. Mathematically, this so called notion of refined soft sets may seem different from the classical soft set theory but the underlying concepts are very similar. In this paper the concept of refined soft set is introduced and the several operations between refined soft sets and soft sets are discussed. We also present the concept of soft images and soft inverse image of refined soft sets. The concept of image of a refined soft set has been used in a customer query problem.

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1. Introduction

The traditional soft set is a mapping from a parameter to the crisp subset of universe. Molodtsov [15] introduced the theory of soft sets as a generalized tool for modeling complex systems involving uncertain or not clearly defined objects. Soft set is a parameterized general mathematical tool which deals with a collection of approximate descriptions of objects. Each approximate description has two parts, a predicate and an approximate value set. In the soft set theory, the initial description of the object has an approximate nature, and we do not need to introduce the notion of exact solution. The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. Any parameterization we prefer can be used with the help of words and sentences, real numbers, functions, mappings and so on. In recent years, soft set theory have been developed rapidly and focused by many researchers in theory and practice. Maji et al. [14] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Aktas and Cagman [1] compared soft sets to the related concepts of fuzzy sets and rough sets. They also defined the notion of soft groups. Jun [10] applied soft set to the theory of BCK/BCI algebra and introduced the concept of soft BCK/BCI algebra. Jun and park [11] discussed the applications of soft sets in ideal theory of BCK/BCI algebra. Feng et al. [7] defined soft semi rings and several related notions in order to establish a connection between soft sets and semi rings. Furthermore based on [14], Ali et al. [2] introduced some new operations on soft sets and by improving the notion of complement of soft set, proved that certain De Morgan's laws hold in soft set theory. Qin and Hong [17] introduced the notion of soft equality and established lattice structures and soft quotient algebras of soft sets. Chen et al. [6] presented a new definition of soft set parameterization reduction and compared this definition to the related concept of attribute reduction in rough set theory. Kong et al. [13] introduced the notion of normal parameter reduction of soft sets and constructed a reduction algorithm based on the importance degree of parameters. Babitha and Sunil [5] made an attempt to explain the equivalent version of some theories on relations and functions in the background of soft sets. In 2011, Kharal and Ahmad [12] introduced the notion of soft images and soft inverse images and they applied these notions to the problem of medical diagnosis.

In this paper a new approach called refined soft sets is presented. Mathematically, this so called notion of ultra soft sets may seem different from the classical soft set theory but the underlying concepts are very similar. These new type of soft sets satisfy all the basic properties of soft sets. The organization of the paper is as follows: Section 2 briefly reviews some background on soft set. Section 3 focuses on the concepts and operations of refined soft sets. Moreover the basic properties of refined soft sets are presented. In section 4, we propose two different types operations between refined soft sets and soft sets. Section 5 is devoted to the discussion of soft images and soft inverse images of refined soft sets. The last section summarizes all the contributions made and points out future research work.

2. Preliminaries

In this section, some definitions and notions about soft sets are given. These will be useful in later sections.

Let U be an universe set and E be a set of possible parameters with respect to U. Usually parameters are attributes, characteristics or properties of the objects in U. Let P(U) denotes the power set of U and A,B$\subseteq$E.

Definition 2.1: [16] A pair (f, A) is called a soft set over U, where A$\subseteq$E and f is a mapping given by f: A$\rightarrow$ P(U).

In other words, a soft set over U can be regarded as a parameterized family of subsets of U, which gives an approximation(soft) description of the objects in U. For e$\in$A, f(e) may be considered as the set of e-approximate elements of the soft set (f, A).

Definition 2.2: [15] For two soft sets (f, A) and (g, B) over a common universe U, we say that (f, A) is a soft subset of (g, B) if

(i) $A \subseteq B$

(ii) $f(e) \subseteq g(e)$ for $e \in A$.

Definition 2.3: [15] The extended *union* of two soft sets (f, A) and (g, B) over a common universe U is the soft set (h, C) where $C = A \cup B$ and $\forall e \in C$

$$h(e) = \begin{cases} f(e) & if\ e \in A - B \\ g(e) & if\ e \in B - A \\ f(e) \cup g(e) & if\ e \in A \cap B \end{cases}$$

We write $(f, A) \cup (g, B) = (h, C)$.

Definition 2.4: [15] The extended *intersection* of two soft sets (f, A) and (g, B) over a common universe U is the soft set (h, C) where $C = A \cup B$ and $\forall e \in C$

$$h(e) = \begin{cases} f(e) & if\ e \in A - B \\ g(e) & if\ e \in B - A \\ f(e) \cap g(e) & if\ e \in A \cap B \end{cases}$$

we write $(f, A) \cap (g, B) = (h, C)$.

Definition 2.5: [15] The complement of a soft set (f, A) is denoted by $(f, A)^{\tilde{c}}$ and is defined by $(f, A)^{\tilde{c}} = (f^c, A)$, where $f^c : A \rightarrow P(U)$ is a mapping defined by $f^c(e) = U - f(e)$ $for\, e \in A$.

Definition 2.6: [15] A soft set (f, A) is called a null soft set denoted by ϕ_{soft} if for all $e \in A,\ f(e) = \phi\,(\text{null set})$.

Definition 2.7: [15] A soft set (f, A) is called an absolute soft set denoted by U_{soft} if for all $e \in A,\ f(e) = U$.

Definition 2.8: [12] Let U, V be two universe sets and A, B be two sets of parameters Let $u : U \to V,\ p : A \to B$ be mappings. Then a mapping $f_{pu} : SS(U)_A \to SS(U)_B$ is defined as:

(i) let (g, A) be a soft set in $SS(U)_A$ Then the image of (g, A) under f_{pu}, denoted by $f_{pu}(g, A)$, is a soft set in $SS(U)_B$ defined by

$$f_{pu}(g, A), = \left(f_{pu}(g), p(A) \right),\ \textit{where for } y \in p(A),$$

$$f_{pu}(g)(y) = \begin{cases} \bigcup\limits_{x \in p^{-1}(y) \cap A} u(g(x)) & \textit{if} \quad p^{-1}(y) \cap A \neq \phi \\ \phi, \textit{ otherwise} \end{cases}$$

(ii) let (h, B) be a soft set in $SS(U)_B$. Then the inverse image of (h, B) under f_{pu}, denoted by $f_{pu}{}^{-1}(h, B)$, is a soft set in $SS(U)_A$ defined by

$$f_{pu}{}^{-1}(h, B) = \left\langle f_{pu}{}^{-1}(h), p^{-1}(B) \right\rangle,\ \textit{where for } x \in A,$$

$$f_{pu}{}^{-1}(h)(x) = \begin{cases} u^{-1}\left(h\left(p(x) \right) \right) & \textit{if} \ \ p(x) \in B \\ \phi, \textit{ otherwise} \end{cases}$$

3. Refined Soft Sets

According to Molodtsov **[15]** a pair (f, A) is called a soft set over U, where A$\subseteq$E and f is a mapping given by f: A$\to$ P(U). In this case f(a)$\subseteq$U for all a$\in$A. But there are many situations in real life problems in which f(a) is itself a soft set for each $a \in A$. Consider the following example:

Among thousands of paper submitted to a journal of Mathematical Science in a particular month, suppose the Editor initially selected 10 papers and forwarded them to two Reviewers to review those papers. Each of the Reviewers will review each paper depending upon the following parameters:

(i) originality of the paper
(ii) applications on real life problems
(iii) general interest on the topic chosen

The Editor will accept or reject a paper depending upon the review report of the Reviewers. $Let\ U = \{ p_1, p_2, p_3, ..., p_{10} \}$ be the universe set of 10 papers. Let

$$B = \{ b_1(opinion\ of\ the 1st\ \mathrm{Re}viewer), b_2(opinion\ of\ the 2nd\ \mathrm{Re}viewer) \},$$

$$A = \{ a_1(\text{originality of the paper}), a_2(\text{applications on real life problems}),$$
$$a_3(\text{general interest on the topic choosen}) \}.$$

$Let\ \zeta : B \to \hat{P}(A,U)$ (where $\hat{P}(A,U)$ denotes the collection of all soft sets over the universe set U) be defined by

$$\zeta(b_1) = \{ a_1 = \{ p_1, p_2, p_7, p_8 \}, a_2 = \{ p_1, p_3, p_5, p_9 \}, a_3 = \{ p_2, p_8, p_{10} \} \},$$

$$\zeta(b_2) = \{ a_1 = \{ p_2, p_7, p_8 \}, a_2 = \{ p_1, p_5, p_9 \}, a_3 = \{ p_8, p_{10} \} \}.$$

Here (ζ, B) is not a traditional soft set. We call these type of sets as refined soft sets.

Definition 3.1: Let U be an universe set and E, F be two sets of parameters such that $E \cap F = \phi$. Let $A \subseteq E$ and $B \subseteq F$. Let us define a soft set (ζ, B) where $\zeta : B \to \hat{P}(A,U)$ is defined by $\zeta(b) = (f_b, A)$ for each $b \in B$ where (f_b, A) is a soft set over U for each $b \in B$. Then we say that (ζ, B) is a soft-soft set. We denote it by $\langle \zeta, B \rangle$.

Example 3.2: Consider the example that has been given in the beginning of the section-3. Then $\langle \zeta, B \rangle$ is a refined soft set.

Soft set theory basically deals with the opinion of one person depending on some parameters, whereas refined soft set theory deals with the opinion of several persons based on the common set of parameters which makes this theory more convenient and broadly applicable. When all the persons have same opinion, the corresponding refined soft set reduces to an ordinary soft set. Thus one can say that refined soft set is a generalization of traditional soft set. To illustrate this let us consider the following example:

Suppose Mr. X and his wife wants to jointly purchase a house depending upon the following parameters:

(i) Beautiful and cheap
(ii) Wooden

Let $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the set of houses under consideration.

Let $B=\{b_1(\text{opinion of Mr. X}), b_2(\text{opinion of the wife of Mr. X})\}$, and

$A=\{a_1(\textit{beautiful and cheap}),\ a_2(\textit{wooden})\}$ be defined by

$\zeta(b_1)=\{a_1=\{h_1,h_2,h_4\},a_2=\{h_1,h_5\}\}$, $\zeta(b_2)=\{a_1=\{h_1,h_2,h_4\},a_2=\{h_1,h_5\}\}$.

Here (ζ,B) is an refined soft set. But since the opinion of Mr. X and his wife are same based on the same set of parameters, we conclude that (ζ,B) reduces to a soft set (g,A) where

$g(a_1)=\{h_1,h_2,h_4\}$, $g(a_2)=\{h_1,h_5\}$.

Thus we can say that soft set theory deals with collective decisions. On the other hand refined soft set theory deals with individual decisions.

Let U be a universe set and E, F be two sets of parameters such that $E\cap F=\phi$. Let $A\subseteq E$ and $B,C,D\subseteq F$. Let $\langle\zeta,B\rangle$, $\langle\xi,C\rangle$ and $\langle\varsigma,D\rangle$ be three refined soft sets over U, where $\zeta:B\to\hat{P}(A,U)$ is defined by $\zeta(b)=(f_b,A)$ for each $b\in B$; $\xi:C\to\hat{P}(A,U)$ is defined by $\xi(c)=(g_c,A)$ for each $c\in C$ and $\varsigma:D\to\hat{P}(A,U)$ is defined by $\varsigma(d)=(z_d,A)$ for each $d\in D$.

Definition 3.3: The *union* of $\langle\zeta,B\rangle$ and $\langle\xi,C\rangle$ is denoted by $\langle\zeta,B\rangle\tilde{\cup}\langle\xi,C\rangle$ and is defined by the refined soft set $\langle\omega,K\rangle$ where $K=B\cup C$ and $\omega:K\to\hat{P}(A,U)$ is given by

$$\omega(e)=\begin{cases}\zeta(e) & if\ e\in B-C\\ \xi(e) & if\ e\in C-B\\ \zeta(e)\cup\xi(e) & if\ e\in B\cap C\end{cases}$$

Where $\zeta(e)\cup\xi(e)=(f_e,A)\cup(g_e,A)$

Definition 3.4: The *intersection* of $\langle \zeta, B\rangle$ and $\langle \xi, C\rangle$ is denoted by $\langle \zeta, B\rangle \tilde{\cap} \langle \xi, C\rangle$ and is defined by the refined soft set $\langle \vartheta, K\rangle$ where $K = B \cup C$ and $\vartheta : K \to \hat{P}(A,U)$ is given by

$$\vartheta(e) = \begin{cases} \zeta(e) & if\ e \in B - C \\ \xi(e) & if\ e \in C - B \\ \zeta(e) \cap \xi(e) & if\ e \in B \cap C \end{cases}$$

Where $\zeta(e) \cap \xi(e) = (f_e, A) \cap (g_e, A)$

Definition 3.5: The *complement* of $\langle \zeta, B\rangle$ is a refined soft set defined by $\langle \zeta, B\rangle^c$ and is defined by $\langle \zeta, B\rangle^c = \langle \zeta^c, B\rangle$ where $\zeta^c : B \to \hat{P}(A,U)$ is a mapping given by $\zeta^c(b) = (f_b^c, A)$ for $b \in B$ where $f_b^c : A \to \hat{P}(A,U)$ is a mapping defined by $f_b^c(a) = U - f_b(a)$ for $a \in A$.

Definition 3.7: A refined soft set $\langle \zeta, B\rangle$ is called a *null refined soft set* denoted by ϕ^* if $\zeta(b) = (f_b, A) = \phi_{soft}$ for each $b \in B$.

Definition 3.8: A refined soft set $\langle \zeta, B\rangle$ is called an *absolute refined soft set* denoted by U^* if $\zeta(b) = (f_b, A) = U_{soft}$ for each $b \in B$.

Theorem 3.9:

I. $\langle \zeta, B\rangle \tilde{\cup}\, \phi^* = \phi^* \tilde{\cup} \langle \zeta, B\rangle = \langle \zeta, B\rangle$ and $\langle \zeta, B\rangle \tilde{\cap}\, \phi^* = \phi^* \tilde{\cap} \langle \zeta, B\rangle = \phi^*$

II. $\langle \zeta, B\rangle \tilde{\cup}\, U^* = U^* \tilde{\cup} \langle \zeta, B\rangle = U^*$ and $\langle \zeta, B\rangle \tilde{\cap}\, U^* = U^* \tilde{\cap} \langle \zeta, B\rangle = \langle \zeta, B\rangle$

III. $\langle \zeta, B\rangle \tilde{\cup} \langle \xi, C\rangle = \langle \xi, C\rangle \tilde{\cup} \langle \zeta, B\rangle$

IV. $\langle \zeta, B\rangle \tilde{\cap} \langle \xi, C\rangle = \langle \xi, C\rangle \tilde{\cap} \langle \zeta, B\rangle$

V. $\langle \zeta, B\rangle \tilde{\cup} (\langle \xi, C\rangle \tilde{\cup} \langle \varsigma, D\rangle) = (\langle \zeta, B\rangle \tilde{\cup} \langle \xi, C\rangle) \tilde{\cup} \langle \varsigma, D\rangle$

VI. $\langle \zeta, B\rangle \tilde{\cap} (\langle \xi, C\rangle \tilde{\cap} \langle \varsigma, D\rangle) = (\langle \zeta, B\rangle \tilde{\cap} \langle \xi, C\rangle) \tilde{\cap} \langle \varsigma, D\rangle$

VII. $\langle \zeta, B\rangle \tilde{\cup} (\langle \xi, C\rangle \tilde{\cap} \langle \varsigma, D\rangle) = (\langle \zeta, B\rangle \tilde{\cup} \langle \xi, C\rangle) \tilde{\cap} (\langle \zeta, B\rangle \tilde{\cup} \langle \varsigma, D\rangle)$

VIII. $\langle \zeta, B\rangle \tilde{\cap} (\langle \xi, C\rangle \tilde{\cup} \langle \varsigma, D\rangle) = (\langle \zeta, B\rangle \tilde{\cap} \langle \xi, C\rangle) \tilde{\cup} (\langle \zeta, B\rangle \tilde{\cap} \langle \varsigma, D\rangle)$

IX. $(\langle \zeta, B\rangle \tilde{\cup} \langle \xi, C\rangle)^c = \langle \zeta, B\rangle^c \tilde{\cap} \langle \xi, C\rangle^c$

X. $(\langle \zeta, B\rangle \tilde{\cap} \langle \xi, C\rangle)^c = \langle \zeta, B\rangle^c \tilde{\cup} \langle \xi, C\rangle^c$

4. Operations between Refined Soft Sets and Soft Sets

Let us consider U as a universe set and E, F be two sets of parameters such that $E \cap F = \phi$. Let $A \subseteq E$ and $B, C \subseteq F$. Let us consider a refined soft set $\langle \zeta, B \rangle$ where $\zeta : B \to \hat{P}(A, U)$ is defined by $\zeta(b) = (f_b, A)$ for each $b \in B$. Let (g, B) be a soft set over U. Then

Definition 4.1:
(i) The operation "$\langle \zeta, B \rangle$ necessary(g, B)" denoted by $\langle \zeta, B \rangle \square (g, B)$ is defined by the soft refined set $\langle \zeta, B \rangle \square (g, B) = (\upsilon, B)$ *where* for $b \in B$, $\upsilon(\widehat{b}) = \bigcap_{a \in A} \left(\left(U - f_b(a) \right) \cup g(b) \right)$

(ii) The operation "$\langle \zeta, B \rangle$ possibility(g, B)" denoted by $\langle \zeta, B \rangle \Diamond (g, B)$ is defined by the refined soft set $\langle \zeta, B \rangle \Diamond (g, B) = \langle \psi, B \rangle$ *where* for $b \in B$, $\psi(b) = \bigcup_{a \in A} \left(f_b(a) \cap g(b) \right)$

Theorem 4.2: Let (g_1, B) *and* (g_2, B) be two soft sets over U. Then

(i) $\langle \zeta, B \rangle \square U_{soft} = U_{soft}$ *and* $\langle \zeta, B \rangle \Diamond \phi_{soft} = \phi_{soft}$

(ii) $\langle \zeta, B \rangle \square (g_1, B) \subseteq (g, B) \subseteq \langle \zeta, B \rangle \Diamond (g_1, B)$

(iii) $(g_1, B) \tilde{\subseteq} (g_2, B) \Rightarrow \langle \zeta, B \rangle \square (g_1, B) \tilde{\subseteq} \langle \zeta, B \rangle \square (g_2, B)$

(iv) $(g_1, B) \tilde{\subseteq} (g_2, B) \Rightarrow \langle \zeta, B \rangle \Diamond (g_1, B) \tilde{\subseteq} \langle \zeta, B \rangle \Diamond (g_2, B)$

(v) $\langle \zeta, B \rangle \square \left((g_1, B) \cap (g_2, B) \right) = \left(\langle \zeta, B \rangle \square (g_1, B) \right) \cap \left(\langle \zeta, B \rangle \square (g_2, B) \right)$

(vi) $\langle \zeta, B \rangle \square \left((g_1, B) \cup (g_2, B) \right) \supseteq \left(\langle \zeta, B \rangle \square (g_1, B) \right) \tilde{\cup} \left(\langle \zeta, B \rangle \square (g_2, B) \right)$

(vii) $\langle \zeta, B \rangle \Diamond \left((g_1, B) \cap (g_2, B) \right) \subseteq \left(\langle \zeta, B \rangle \Diamond (g_1, B) \right) \tilde{\cap} \left(\langle \zeta, B \rangle \Diamond (g_2, B) \right)$

(viii) $\langle \zeta, B \rangle \Diamond \left((g_1, B) \cup (g_2, B) \right) = \left(\langle \zeta, B \rangle \Diamond (g_1, B) \right) \tilde{\cup} \left(\langle \zeta, B \rangle \Diamond (g_2, B) \right)$

(ix) $\left(\langle \zeta, B \rangle \square (g_1, B) \right)^c = \langle \zeta, B \rangle \Diamond (g_1, B)^c$

(x) $\left(\langle \zeta, B \rangle \Diamond (g_1, B) \right)^c = \langle \zeta, B \rangle \square (g_1, B)^c$

Proof: (i)-(iv) are straight forward.

(v) Let $\langle \zeta, B \rangle \square \left((g_1, B) \cap (g_2, B) \right) = (\upsilon, \widehat{B})$. Then for $b \in B$, we have

$$\upsilon(b) = \bigcap_{a \in A} \left(\left(U - f_b(a) \right) \cup \left(g_1(b) \cap g_2(b) \right) \right)$$
$$= \bigcap_{a \in A} \left(\left(\left(U - f_b(a) \right) \cup g_1(b) \right) \cap \left(\left(U - f_b(a) \right) \cup g_2(b) \right) \right)$$

Again $\left(\langle\zeta,B\rangle\square\left(g_1,B\right)\right)\cap\left(\langle\zeta,B\rangle\square\left(g_2,B\right)\right)=\left(\upsilon_1,B\right)\cap\left(\upsilon_2,B\right)=\left(\upsilon_3,B\right)$ where for $b\in B$, we have

$$\begin{aligned}\upsilon_3(b)&=\upsilon_1(b)\cap\upsilon_2(b)\\&=\bigcap_{a\in A}\left(\left(\left(U-f_b\left(a\right)\right)\cup g_1\left(b\right)\right)\cap\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup g_2\left(b\right)\right)\right)\\&=\bigcap_{a\in A}\left(\left(\left(U-f_b\left(a\right)\right)\cup g_1\left(b\right)\right)\cap\left(\left(U-f_b\left(a\right)\right)\cup g_2\left(b\right)\right)\right)\end{aligned}$$

Hence $\langle\zeta,B\rangle\square\left(\left(g_1,B\right)\cap\left(g_2,B\right)\right)=\left(\langle\zeta,B\rangle\square\left(g_1,B\right)\right)\cap\left(\langle\zeta,B\rangle\square\left(g_2,B\right)\right)$.

(vi) Let $\langle\zeta,B\rangle\square\left(\left(g_1,B\right)\cup\left(g_2,B\right)\right)=\left(\upsilon,B\right)$. Then for $b\in B$, we have

$$\begin{aligned}\upsilon(b)&=\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup\left(g_1\left(b\right)\cup g_2\left(b\right)\right)\right)\\&=\bigcap_{a\in A}\left(\left(\left(U-f_b\left(a\right)\right)\cup g_1\left(b\right)\right)\cup\left(\left(U-f_b\left(a\right)\right)\cup g_2\left(b\right)\right)\right)\end{aligned}$$

Again $\left(\langle\zeta,B\rangle\square\left(g_1,B\right)\right)\cup\left(\langle\zeta,B\rangle\square\left(g_2,B\right)\right)=\left(\upsilon_1,B\right)\cup\left(\upsilon_2,B\right)=\left(\upsilon_3,B\right)$ where for $b\in B$, we have

$$\begin{aligned}\upsilon_3(b)&=\upsilon_1(b)\cup\upsilon_2(b)\\&=\bigcap_{a\in A}\left(\left(\left(U-f_b\left(a\right)\right)\cup g_1\left(b\right)\right)\cup\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup g_2\left(b\right)\right)\right)\end{aligned}$$

Hence $\langle\zeta,B\rangle\square\left(\left(g_1,B\right)\cap\left(g_2,B\right)\right)=\left(\langle\zeta,B\rangle\square\left(g_1,B\right)\right)\cap\left(\langle\zeta,B\rangle\square\left(g_2,B\right)\right)$.

Since $g_1\left(b\right)\subseteq g_1\left(b\right)\cup g_2\left(b\right)$ *and* $g_2\left(b\right)\subseteq g_1\left(b\right)\cup g_2\left(b\right),$ we have

$\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup g_1\left(b\right)\right)\subseteq\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup\left(g_1\left(b\right)\cup g_2\left(b\right)\right)\right)$ *and*

$\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup g_2\left(b\right)\right)\subseteq\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup\left(g_1\left(b\right)\cup g_2\left(b\right)\right)\right)$. Consequently,

$$\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup g_1\left(b\right)\right)\cup\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup g_2\left(b\right)\right)\subseteq\bigcap_{a\in A}\left(\left(U-f_b\left(a\right)\right)\cup\left(g_1\left(b\right)\cup g_2\left(b\right)\right)\right).$$

So $\langle\zeta,B\rangle\square\left(\left(g_1,B\right)\cup\left(g_2,B\right)\right)\supseteq\left(\langle\zeta,B\rangle\square\left(g_1,B\right)\right)\tilde{\cup}\left(\langle\zeta,B\rangle\square\left(g_2,B\right)\right).$

(vii)-(viii) can be proved similarly.

(ix) Let $\langle \zeta, B\rangle \Diamond (g_1, B)^c = \langle \psi, B\rangle$ *where* for $b \in B$, we have $\psi(b) = \bigcup_{a\in A}\left(f_b(a) \cap (U - g(b))\right)$

Again for $\left(\langle \zeta, B\rangle \square (g_1, B)\right)^c = (\upsilon, B)$ and for $b \in B$, we have

$$\upsilon(b) = U - \left(\bigcap_{a\in A} \left((U - f_b(a)) \cup g(b) \right) \right)$$
$$= \bigcup_{a\in A}\left(f_b(a) \cap (U - g(b))\right)$$

Consequently, $\left(\langle \zeta, B\rangle \square (g_1, B)\right)^c = \langle \zeta, B\rangle \Diamond (g_1, B)^c$.

(x) Proof is similar to (ix).

5. Soft Images and Soft Inverse Images of Refined Soft Sets

Definition 5.1: Let U, V be two universe sets and E_1, E_2, F_1, F_2 be four universe sets of parameters such that $E_i \cap F_j = \phi$ for $i, j = 1,2$. . Let $A_1 \subseteq E_1$ and $A_2 \subseteq E_2$ and $B \subseteq F_1$ and $C \subseteq F_2$. Let $u : U \to V, p : B \to C$ *and* $q : A_1 \to A_2$ be mappings. Let $SS(U)_B^{A_1}$ and $SS(V)_C^{A_2}$ be two families of refined soft sets. Then a mapping $f_{qpu} : SS(U)_B^{A_1} \to SS(U)_C^{A_2}$ is defined as:

(i) Let $\langle \zeta, B\rangle$ be a refined soft set in $SS(U)_B^{A_1}$ and $\zeta(b) = (r_b, A_1)$ for each $b \in B$. Then the image of $\langle \zeta, B\rangle$ under f_{qpu}, denoted by $f_{q\hat{p}u}\langle \zeta, B\rangle$, is a refined soft set in $SS(U)_C^{A_2}$ defined by $f_{qpu}\langle \zeta, B\rangle = \left\langle f_{qpu}(\zeta), p(B)\right\rangle$, where for $c \in p(B)$, $f_{qpu}(\zeta)(c) = (z_c, A_2)$ where for $a^{//} \in A_2$

$$z_c\left(a^{//}\right) = \begin{cases} \bigcup_{c^/ \in p^{-1}(c)\cap B} \left[\bigcup_{a^/ \in q^{-1}\left(a^{//}\right)\cap A_1} u\left(r_{c^/}\left(a^/\right)\right)\right] & if \;\; p^{-1}(c) \cap B \neq \phi \;\; and \;\; q^{-1}\left(a^{//}\right) \cap A_1 \neq \phi \\ \phi, \; otherwise & \end{cases}$$

(ii) Let $\langle \varsigma, C\rangle$ be a refined soft set in $SS(U)_C^{A_2}$ and $\varsigma(c) = (k_c, A_2)$ for each $c \in C$. Then the inverse image of $\langle \varsigma, C\rangle$ under f_{qpu}, denoted by $f_{qpu}^{\ -1}\langle \varsigma, C\rangle$, is a refined soft set in $SS(U)_B^{A_1}$ defined by

$f_{qpu}^{\ -1}\langle \varsigma, C\rangle = \left\langle f_{qpu}^{\ -1}(\varsigma), p^{-1}(C)\right\rangle$, where for $b \in p^{-1}(C)$, $f_{qpu}^{\ -1}(\varsigma)(b) = (t_b, A_1)$ where for

$$a^{/} \in A_1,\ t_b\left(a^{/}\right)=\begin{cases} u^{-1}\left(k_{p(b)}\left(q\left(a^{/}\right)\right)\right) \ if \ \ p(b)\neq\phi \\ \phi,\ otherwise \end{cases}$$

The refined soft function f_{qpu} is called surjective if p, q, u are all surjective. The refined soft function f_{qpu} is called injective if p, q, u are all injective.

Example 5.2: Let $U=\{h_1,h_2,h_3,h_4,h_5,h_6\}$, $V=\{v_1,v_2,v_3\}$ and $u:U\to V$ be defined by

$u(h_1)=v_1, u(h_2)=v_3, u(h_3)=v_3, u(h_4)=v_1, u(h_5)=v_2, u(h_1)=v_1.$

Let $B=\{b_1,b_2\}$, $B=\{c_1,c_2\}$ and $p:B\to C$ be defined by $p(b_1)=c_1, p(b_2)=c_1.$

Let $A_1=\{\alpha_1,\alpha_2,\alpha_3\}$, $A_2=\{\beta_1,\beta_2,\beta_3\}$ *and* $q:A_1\to A_2$ be defined by

$q(\alpha_1)=\beta_1, q(\alpha_2)=\beta_1, q(\alpha_3)=\beta_2.$

$p(B)=\{p(b):b\in B\}=\{c_1\}$ and so $c\in p(B)\Rightarrow c=c_1$ *and* $p^{-1}(c)=p^{-1}(c_1)=\{b_1,b_2\}.$

Let $r_{b_1}(\alpha_1)=\{h_1,h_2,h_4\}$, $r_{b_1}(\alpha_2)=\{h_1,h_3,h_5\}$, $r_{b_1}(\alpha_3)=\{h_2,h_3,h_6\}$,
$r_{b_2}(\alpha_1)=\{h_1,h_3,h_4,h_6\}$, $r_{b_2}(\alpha_2)=\{h_3,h_4,h_5\}$ and $r_{b_2}(\alpha_3)=\{h_2,h_5\}$.

Here $a^{//}\in A_2=\{\beta_1,\beta_2,\beta_3\}$.
$z_c(\beta_1)=u\left(r_{b_1}(\alpha_1)\right)\cup u\left(r_{b_1}(\alpha_2)\right)=u\left(\{h_1,h_2,h_4\}\right)\cup u\left(\{h_1,h_3,h_5\}\right)=\{v_1,v_3\}\cup\{v_1,v_2,v_3\}=\{v_1,v_2,v_3\}$,
$z_c(\beta_2)=u\left(r_{b_1}(\alpha_3)\right)\cup u\left(r_{b_2}(\alpha_3)\right)=u\left(\{h_2,h_3,h_6\}\right)\cup u\left(\{h_2,h_5\}\right)=\{v_1,v_3\}\cup\{v_1,v_2,v_3\}=\{v_1,v_2,v_3\}$,
$z_c(\beta_3)=\{\ \}$.

Hence $f_{qpu}\langle\zeta,B\rangle=\left\{c_1=\left\{\beta_1=\{v_1,v_2,v_3\},\ \beta_2=\{v_1,v_2,v_3\},\ \beta_3=\{\ \}\right\}\right\}$.
$p^{-1}(C)=\{b_1,b_2\}$.

Let $k_{c_1}(\beta_1)=\{v_1,v_2\}$, $k_{c_1}(\beta_2)=\{v_3\}$, $k_{c_1}(\beta_3)=\{v_1,v_3\}$, $k_{c_2}(\beta_1)=\{v_2\}$, $k_{c_2}(\beta_2)=\{v_2,v_3\}$
and $k_{c_2}(\beta_3)=\{v_1,v_2,v_3\}$.
Then $t_{b_1}(\alpha_1)=u^{-1}\left(k_{c_1}\left(q(\alpha_1)\right)\right)=u^{-1}\left(k_{c_1}(\beta_1)\right)=u^{-1}\left(\{v_1,v_2\}\right)=\{h_1,h_4,h_5,h_6\}$,
$t_{b_1}(\alpha_2)=u^{-1}\left(k_{c_1}\left(q(\alpha_2)\right)\right)=u^{-1}\left(k_{c_1}(\beta_1)\right)=u^{-1}\left(\{v_1,v_2\}\right)=\{h_1,h_4,h_5,h_6\}$,

$$t_{b_1}(\alpha_3)=u^{-1}\left(k_{c_1}\left(q(\alpha_3)\right)\right)=u^{-1}\left(k_{c_1}(\beta_2)\right)=u^{-1}\left(\{v_3\}\right)=\{h_2,h_3\},$$
$$t_{b_2}(\alpha_1)=u^{-1}\left(k_{c_1}\left(q(\alpha_1)\right)\right)=u^{-1}\left(k_{c_1}(\beta_1)\right)=u^{-1}\left(\{v_1,v_2\}\right)=\{h_1,h_4,h_5,h_6\},$$
$$t_{b_2}(\alpha_2)=u^{-1}\left(k_{c_1}\left(q(\alpha_2)\right)\right)=u^{-1}\left(k_{c_1}(\beta_1)\right)=u^{-1}\left(\{v_1,v_2\}\right)=\{h_1,h_4,h_5,h_6\},$$
$$t_{b_2}(\alpha_3)=u^{-1}\left(k_{c_1}\left(q(\alpha_3)\right)\right)=u^{-1}\left(k_{c_1}(\beta_2)\right)=u^{-1}\left(\{v_3\}\right)=\{h_2,h_3\}.$$

Hence $f_{qpu}{}^{-1}\langle\varsigma,C\rangle=\Big\{b_1=\big\{\alpha_1=\alpha_2=\{h_1,h_4,h_5,h_6\},\alpha_3=\{h_2,h_3\}\big\},$

$$b_2=\big\{\alpha_1=\alpha_2=\{h_1,h_4,h_5,h_6\},\alpha_3=\{h_2,h_3\}\big\}\Big\}.$$

6. Application

The concept of image of a refined soft set can be used in a customer query problem. Suppose the following is a narration by a customer to a shopkeeper:

"I mainly need an android smart phone with long battery life and minimum 1GB of RAM. There should be 3G type network connectivity in the mobile. The rear and front camera should be a minimum of 5MP and 2MP respectively. Can you please give me some idea about the cost and the OS version of a smart phone which has 4.8 inch or 5 inch display size?"

According to the demand of the customer and based on the availability of the smart phones in the shop, let us consider the following refined soft set on the universe of smart phones $U=\{m_1, m_2, m_3, \ldots, m_{10}\}$:

$$\langle g,B\rangle=\begin{cases} battery(b_1)=\begin{cases} high\ importance(a_1)=\{m_1,m_2,m_7\} \\ medium\ importance(a_2)=\{m_4,m_5,m_6\} \\ low\ importance(a_3)=\{m_3,m_8,m_9,m_{10}\}\end{cases} \\ camera(b_2)=\begin{cases} high\ importance(a_1)=\{m_1,m_4,m_6\} \\ medium\ importance(a_2)=\{m_2,m_3,m_8\} \\ low\ importance(a_3)=\{m_5,m_7,m_9,m_{10}\}\end{cases}\end{cases}$$

where $B=\{b_1,b_2\}$ and let $A=\{a_1,a_2,a_3\}$. Let $V=\{o_1$ (*jellybean*), o_2 (*kitkat*), o_3 (*lollipop*)$\}$ be the set of android versions; $A_2=\{a_1''$ (*high cost* (*more than 10,000*)), a_1'' (*medium cost* (*between 7000-10,000*)), a_1'' (*low cost* (*less than 7000*))$\}$; $C=\{c_1$ (*display size is 4.8 inch*), c_2 (*display size is 5 inch*)$\}$.

Let $u:U\to V$ be defined by $u(m_1)=u(m_3)=u(m_9)=o_1$, $u(m_2)=u(m_5)=u(m_7)=o_2$, $u(m_4)=u(m_6)=u(m_8)=u(m_{10})=o_3$. Let $p:B\to C$ be defined by $p(b_1)=c_1$, $p(b_2)=c_2$. Let

$q:A_1\to A_2$ be defined by $q(a_1)=a_3''$, $q(a_2)=a_2''$ $q(a_3)=a_1''$.

Then $p(B)=\{p(b):b\in B\}=\{c_1,c_2\}$.

Now $f_{qpu}(\langle g,B\rangle)=\langle f_{qpu},p(B)\rangle$.

where for $c\in p(B)$, $f_{qpu}(C)=(Z_c,A_2)$,

where for $a''\in A_2$, $Z_c(a'')=\bigcup_{c'\in p^{-1}(c)\cap B}\ \bigcup_{a'\in q^{-1}(a'')\cap A_1} u(r_{c'}(a'))$

So

$$
\begin{aligned}
Z_{c_1}(a_1'') &= \bigcup_{c'\in\{b_1\}}\ \bigcup_{a'\in\{a_3\}} u(r_{c'}(a')) \\
&= u(r_{b_1}(a_3)) \\
&= u(\{m_3,m_8,m_9,m_{10}\}) \\
&= \{o_1,o_3\}
\end{aligned}
$$

$$
\begin{aligned}
Z_{c_1}(a_2'') &= \bigcup_{c'\in\{b_1\}}\ \bigcup_{a'\in\{a_2\}} u(r_{c'}(a')) \\
&= u(r_{b_1}(a_2)) \\
&= u(\{m_4,m_5,m_6\}) \\
&= \{o_2,o_3\}
\end{aligned}
$$

$$
\begin{aligned}
Z_{c_1}(a_3'') &= \bigcup_{c'\in\{b_1\}}\ \bigcup_{a'\in\{a_1\}} u(r_{c'}(a')) \\
&= u(r_{b_1}(a_1)) \\
&= u(\{m_1,m_2,m_7\}) \\
&= \{o_1,o_2\}
\end{aligned}
$$

$$
\begin{aligned}
Z_{c_2}(a_1'') &= \bigcup_{c'\in\{b_2\}}\ \bigcup_{a'\in\{a_3\}} u(r_{c'}(a')) \\
&= u(r_{b_2}(a_3)) \\
&= u(\{m_5,m_7,m_9,m_{10}\}) \\
&= \{o_1,o_2,o_3\}
\end{aligned}
$$

$$
\begin{aligned}
Z_{c_2}\left(a_2''\right) &= \bigcup_{c'\in\{b_2\}} \bigcup_{a'\in\{a_2\}} u\left(r_{c'}\left(a'\right)\right) \\
&= u\left(r_{b_2}\left(a_2\right)\right) \\
&= u\left(\{m_2, m_3, m_8\}\right) \\
&= \{o_1, o_2, o_3\}
\end{aligned}
$$

$$
\begin{aligned}
Z_{c_2}\left(a_3''\right) &= \bigcup_{c'\in\{b_2\}} \bigcup_{a'\in\{a_1\}} u\left(r_{c'}\left(a'\right)\right) \\
&= u\left(r_{b_2}\left(a_1\right)\right) \\
&= u\left(\{m_1, m_4, m_6\}\right) \\
&= \{o_1, o_3\}
\end{aligned}
$$

Thus

$$
f_{qpu}\left(\langle g, B\rangle\right) = \begin{cases} c_1 = \begin{cases} a_1'' = \{o_1, o_3\} \\ a_2'' = \{o_2, o_3\} \\ a_3'' = \{o_1, o_2\} \end{cases} \\ c_2 = \begin{cases} a_1'' = \{o_1, o_2, o_3\} \\ a_2'' = \{o_1, o_2, o_3\} \\ a_3'' = \{o_1, o_3\} \end{cases} \end{cases}
$$

$$
= \begin{cases} display\ size\ is\ 4.8\ inch = \begin{cases} high\ cost = \{o_1, o_3\} \\ medium\ cost = \{o_2, o_3\} \\ low\ cost = \{o_1, o_2\} \end{cases} \\ display\ size\ is\ 5\ inch = \begin{cases} high\ cost = \{o_1, o_2, o_3\} \\ medium\ cost = \{o_1, o_2, o_3\} \\ low\ cost = \{o_1, o_3\} \end{cases} \end{cases}
$$

Thus

(i) Considering the display size 4.8 inch and high cost, the preferred operating systems are o_1 and o_3.

(ii) Considering the display size 4.8 inch and medium cost, the preferred operating systems are o_2 and o_3.

(iii) Considering the display size 4.8 inch and low cost, the preferred operating systems are o_1 and o_2.

(iv) Considering the display size 5 inch and high cost, the preferred operating systems are o_1, o_2 and o_3.

(v) Considering the display size 5 inch and medium cost, the preferred operating systems are o_1, o_2 and o_3.

(vi) Considering the display size 5 inch and low cost, the preferred operating systems are o_1 and o_3.

7. Conclusion and Future Works

Soft set theory is a general method for solving problem of uncertainty. In the present paper the structure of refined soft set is discussed together with their operations and basic properties. Moreover the concept of soft image and soft inverse image in refined soft set theory context are presented which may be useful in medical expert system.

With the motivation of ideas presented in this paper one can think of similarity measures, Cartesian products and relations on refined soft sets. Further studies on the topology generated by the refined soft sets or refined soft set relations may be done so that we may brood over the topological side of refined soft sets or refined soft set relations. Moreover the refined soft sets and the refined soft set relations can be extended in fuzzy refined soft sets and fuzzy refined soft set relations respectively and thus one can get more affirmative solution in decision making problems in real life situations. It is hoped that the combinations of refined soft sets, fuzzy sets and rough sets will generate potentially interesting some new research direction.

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