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SOME GENERALIZATIONS OF THE BANACH'S CONTRACTION PRINCIPLE ON A COMPLETE COMPLEX VALUED S-METRIC SPACE

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Abstract − In this paper we give some generalizations of the Banach's contraction principle on a complete complex valued S-metric space. We verify our results with an example.

Keywords − *Complex valued S-metric space, Fixed point, Banach's contraction principle.*

1 Introduction

Metric spaces and fixed-point theory have an important role in various areas of mathematics such as analysis, topology, differential equation etc. Fixed-point theory begin with the Banach's contraction principle. Then the principle has been studied and generalized on some metric spaces (see [1], [2], [6], [7] and [8]). Recently, it has been introduced the notion of an S-metric space as a generalization of a metric space [8]. Some mathematicans proved new fixed-point theorems on an S-metric space (see [4], [5], [6], [8], [9] and [10]). Mlaiki presented the concept of a complex valued S-metric space and gave a common fixed-point theorem of two self-mappings on a complex valued S-metric space [3]. The present authors investigated new common fixed-point theorems using the notion of CS-compatibility on a complex valued S-metric space [7].

Let $X = \mathbb{C}$ and the function $S : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be defined by

$$S(x, y, z) = i\left(|x - z| + |y - z|\right),$$

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for all $x, y, z \in \mathbb{C}$. Then the function S is a complex valued S-metric space on $\mathbb{C}$. Let us define the self-mapping $T : \mathbb{C} \to \mathbb{C}$ as follows:

$$Tx = 1 - x,$$

for all $x \in \mathbb{C}$. Then T is a self-mapping on the complete complex valued S-metric space $(\mathbb{C}, S)$. T has a fixed point $x = \frac{1}{2}$, but it does not satisfy the condition of Banach's contraction principle. Therefore it is important to study new generalized fixed-point theorems.

Motivated by the above studies, in this paper, we investigate new fixed-point theorems as generalizations of the Banach's contraction principle on a complete complex valued S-metric spaces. We expect that new generalized fixed-point theorems will be obtained using our main theorems.

In Section 2 we recall some known definitions, lemmas and a theorem. In Section 3 we generalize the Banach's contraction principle on a complete complex valued S-metric space. Also we give an example which satisfies the conditions of our results, but does not satisfy the condition of Banach's contraction principle.

2 Preliminary

In this section we recall some definitions, lemmas and a theorem which is called the Banach's contraction principle.

Let $\mathbb{C}$ be the set of complex numbers and $z_1,\ z_2 \in \mathbb{C}$. The partial order $\precsim$ is defined on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2)$$

and

$$z_1 \prec z_2 \text{ if and only if } Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2).$$

Also we write $z_1 \precsim z_2$ if one of the following conditions hold:

1. $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
2. $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
3. $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$$

and

$$z_1 \precsim z_2,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

Definition 2.1. [3] Let X be a nonempty set. A complex valued S-metric on X is a function $S : X \times X \times X \to \mathbb{C}$ that satisfies the following conditions for all $x, y, z, t \in X$:

(**CS1**) $0 \precsim S(x, y, z)$,

(**CS2**) $S(x, y, z) = 0$ if and only if $x = y = z$,

(**CS3**) $S(x, y, z) \precsim S(x, x, t) + S(y, y, t) + S(z, z, t)$.

The pair (X, S) is called a complex valued S-metric space.

Definition 2.2. [3] Let (X, S) be a complex valued S-metric space. Then

1. A sequence $\{a_n\}$ in X converges to x if and only if for all ε such that $0 \prec \varepsilon \in \mathbb{C}$ there exists a natural number n_0 such that for all $n \geq n_0$, we have $S(a_n, a_n, x) \prec \varepsilon$ and it is denoted by
$$\lim_{n\to\infty} a_n = x.$$
2. A sequence $\{a_n\}$ in X is called a Cauchy sequence if for all ε such that $0 \prec \varepsilon \in \mathbb{C}$ there exists a natural number n_0 such that for all $n, m \geq n_0$, we have $S(a_n, a_n, a_m) \prec \varepsilon$.
3. A complex valued S-metric space (X, S) is called complete if every Cauchy sequence is convergent.

Lemma 2.3. [3] Let (X, S) be a complex valued S-metric space and $\{a_n\}$ be a sequence in X. Then $\{a_n\}$ converges to x if and only if

$$|S(a_n, a_n, x)| \to 0,$$

as $n \to \infty$.

Lemma 2.4. [3] Let (X, S) be a complex valued S-metric space and $\{a_n\}$ be a sequence in X. Then $\{a_n\}$ is a Cauchy sequence if and only if

$$|S(a_n, a_n, a_m)| \to 0,$$

as $n \to \infty$.

Lemma 2.5. [3] If (X, S) be a complex valued S-metric space then

$$S(x, x, y) = S(y, y, x),$$

for all $x, y \in X$.

Lemma 2.6. [9] Let $(X, S), (Y, S')$ be two S-metric spaces and $f : X \to Y$ be a function. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.

In the next section, we consider two complex valued S-metric spaces in Lemma 2.6.

Now we recall the following theorem which is called the Banach's contraction principle.

Theorem 2.7. [7] Let (X, S) be a complete complex valued S-metric space and T be a self-mapping of X satisfying

$$S(Tx, Tx, Ty) \precsim hS(x, x, y) \tag{1}$$

for all $x, y \in X$ and some $0 \leq h < 1$. Then f has a fixed point in X.

3 Main Results

In this section we prove new generalizations of the Banach's contraction principle.

Theorem 3.1. Let (X, S) be a complete complex valued S-metric space and T be a self-mapping of X. If there exist nonnegative real numbers c_1, c_2, c_3, c_4 satisfying $\max\{c_1 + 3c_3 + 2c_4, c_1 + c_2 + c_3, c_2 + 2c_4\} < 1$ such that

$$\begin{aligned} S(Tx, Tx, Ty) \preceq\ & c_1 S(x, x, y) + c_2 S(Tx, Tx, y) + c_3 S(Ty, Ty, x) \\ & + c_4 \max\{S(Tx, Tx, x), S(Ty, Ty, y)\}, \end{aligned} \tag{2}$$

for all $x, y \in X$, then T has a unique fixed point x in X and T is continuous at x.

Proof. Let $a_0 \in X$ and the sequence $\{a_n\}$ be defined by

$$T^n a_0 = a_n.$$

Assume that $a_n \neq a_{n+1}$ for all n. Using the inequality 2 we obtain

$$\begin{aligned} S(a_n, a_n, a_{n+1}) &= S(Ta_{n-1}, Ta_{n-1}, Ta_n) \preceq c_1 S(a_{n-1}, a_{n-1}, a_n) \\ & \quad + c_2 S(a_n, a_n, a_n) + c_3 S(a_{n+1}, a_{n+1}, a_{n-1}) \\ & \quad + c_4 \max\{S(a_n, a_n, a_{n-1}), S(a_{n+1}, a_{n+1}, a_n)\} \\ &= c_1 S(a_{n-1}, a_{n-1}, a_n) + c_3 S(a_{n+1}, a_{n+1}, a_{n-1}) \\ & \quad + c_4 \max\{S(a_n, a_n, a_{n-1}), S(a_{n+1}, a_{n+1}, a_n)\}. \end{aligned} \tag{3}$$

Using the condition $(CS3)$, we get

$$S(a_{n+1}, a_{n+1}, a_{n-1}) \preceq 2S(a_{n+1}, a_{n+1}, a_n) + S(a_{n-1}, a_{n-1}, a_n). \tag{4}$$

Hence using the inequalities (3), (4) and Lemma 2.5, we have

$$S(a_n, a_n, a_{n+1}) \preceq c_1 S(a_{n-1}, a_{n-1}, a_n) + 2c_3 S(a_{n+1}, a_{n+1}, a_n) + c_3 S(a_{n-1}, a_{n-1}, a_n)$$

$$+c_4S(a_n,a_n,a_{n-1})+c_4S(a_{n+1},a_{n+1},a_n),$$
$$(1-2c_3-c_4)S(a_n,a_n,a_{n+1})\preceq(c_1+c_3+c_4)S(a_{n-1},a_{n-1},a_n)$$

and

$$S(a_n,a_n,a_{n+1})\preceq\frac{c_1+c_3+c_4}{1-2c_3-c_4}S(a_{n-1},a_{n-1},a_n). \tag{5}$$

Let $c=\frac{c_1+c_3+c_4}{1-2c_3-c_4}$. Then we find $c<1$ since $c_1+3c_3+2c_4<1$. Using the inequality (5), we obtain

$$S(a_n,a_n,a_{n+1})\preceq c^nS(a_0,a_0,a_1). \tag{6}$$

For all $n,m\in\mathbb{N}$, $n<m$, using the inequality (6) and the condition $(CS3)$, we have

$$\begin{aligned}S(a_n,a_n,a_m) &\preceq 2S(a_n,a_n,a_{n+1})+2S(a_{n+1},a_{n+1},a_{n+2})+\cdots+2S(a_{m-1},a_{m-1},a_m)\\ &\preceq 2(c^n+c^{n+1}+\cdots+c^{m-1})S(a_0,a_0,a_1)\\ &\preceq 2c^n(1+c+\cdots+c^{m-n-1})S(a_0,a_0,a_1)\\ &\preceq 2c^n\frac{1-c^{m-n}}{1-c}S(a_0,a_0,a_1)\\ &\preceq \frac{2c^n}{1-c}S(a_0,a_0,a_1),\end{aligned}$$

which implies

$$|S(a_n,a_n,a_m)|\leq\frac{2c^n}{1-c}\left|S(a_0,a_0,a_1)\right|.$$

Therefore $|S(a_n,a_n,a_m)|\to 0$ as $n,m\to\infty$. Hence $\{a_n\}$ is a Cauchy sequence. Since (X,S) is complete, there exists $x\in X$ such that $\{a_n\}$ converges to x.

Now we show that x is a fixed point of T. Suppose that $Tx\neq x$. Then we get

$$\begin{aligned}S(a_n,a_n,Tx) &= S(Ta_{n-1},Ta_{n-1},Tx)\preceq c_1S(a_{n-1},a_{n-1},x)\\ &\quad +c_2S(a_n,a_n,x)+c_3S(Tx,Tx,a_{n-1})\\ &\quad +c_4\max\{S(a_n,a_n,a_{n-1}),S(Tx,Tx,x)\}\end{aligned}$$

and

$$\begin{aligned}|S(a_n,a_n,Tx)| &\leq c_1\left|S(a_{n-1},a_{n-1},x)\right|+c_2\left|S(a_n,a_n,x)\right|+c_3\left|S(Tx,Tx,a_{n-1})\right|\\ &\quad +c_4\left|\max\{S(a_n,a_n,a_{n-1}),S(Tx,Tx,x)\}\right|.\end{aligned}$$

If we take limit for $n\to\infty$, then using the continuity of S and Lemma 2.5, we have

$$|S(x,x,Tx)|=|S(Tx,Tx,x)|\leq(c_3+c_4)\left|S(Tx,Tx,x)\right|,$$

which is a contradiction since $0\leq c_3+c_4<1$. Hence we obtain $Tx=x$.

Now we show that x is unique. Let y be another fixed point of T such that $x\neq y$. Using the inequality (2) and Lemma 2.5, we have

$$\begin{aligned}S(Tx,Tx,Ty) &= S(x,x,y)\preceq c_1S(x,x,y)+c_2S(x,x,y)\\ &\quad +c_3S(y,y,x)+c_4\max\{S(x,x,x),S(y,y,y)\}\end{aligned}$$

and

$$|S(x,x,y)| \leq (c_1+c_2+c_3)\,|S(x,x,y)|\,,$$

which implies $x=y$ since $c_1+c_2+c_3<1$.

Now we prove that T is continuous at x. For $n \in \mathbb{N}$, using the inequality (2), we get

$$\begin{aligned} S(Ta_n,Ta_n,Tx) \;\preceq\;& c_1S(a_n,a_n,x)+c_2S(Ta_n,Ta_n,x) \\ &+c_3S(Tx,Tx,a_n)+c_4\max\{S(Ta_n,Ta_n,a_n),S(Tx,Tx,x)\}. \end{aligned} \tag{7}$$

Using the condition $(CS3)$, the inequality (7) and Lemma 2.5, we obtain

$$\begin{aligned} S(Ta_n,Ta_n,Tx) \;\preceq\;& c_1S(a_n,a_n,x)+c_2S(Ta_n,Ta_n,x)+c_3S(Tx,Tx,a_n) \\ &+2c_4S(Ta_n,Ta_n,x)+c_4S(a_n,a_n,x) \end{aligned}$$

and

$$(1-c_2-2c_4)S(Ta_n,Ta_n,Tx) \preceq (c_1+c_3+c_4)S(a_n,a_n,x),$$

which implies

$$|S(Ta_n,Ta_n,Tx)| \leq \frac{c_1+c_3+c_4}{1-c_2-2c_4}\,|S(a_n,a_n,x)|\,.$$

If we take limit for $n\to\infty$, then we have

$$|S(Ta_n,Ta_n,Tx)| \to 0.$$

Therefore $\{Ta_n\}$ is convergent to $Tx=x$. Consequently, T is continuous at x by Lemma 2.6. □

Remark 3.2. (1) Theorem 3.1 is a generalization of the Banach's contraction principle on complete complex valued S-metric spaces. Indeed, if we take $c_1=h$ and $c_2=c_3=c_4=0$ in Theorem 3.1, then we obtain the Banach's contraction condition in Theorem 2.7.

(2) If we take the function $S: X\times X\times X \to [0,\infty)$ in Theorem 3.1, Then we have Theorem 3 in [6].

Corollary 3.3. Let (X,S) be a complete complex valued S-metric space and T be a self-mapping of X. If there exist nonnegative real numbers c_1,c_2,c_3,c_4 satisfying $\max\{c_1+3c_3+2c_4, c_1+c_2+c_3, c_2+2c_4\}<1$ such that

$$\begin{aligned} S(T^px,T^px,T^py) \;\preceq\;& c_1S(x,x,y)+c_2S(T^px,T^px,y)+c_3S(T^py,T^py,x) \\ &+c_4\max\{S(T^px,T^px,x),S(T^py,T^py,y)\}, \end{aligned}$$

for all $x,y\in X$ and some $p\in\mathbb{N}$, then T has a unique fixed point x in X and T^p is continuous at x.

Proof. Using the similar arguments in Theorem 3.1, we can easily see that T^p has a unique fixed point x in X and T^p is continuous at x. Also we obtain

$$Tx = TT^px = T^{p+1}x = T^pTx,$$

which implies that Tx is a fixed point of T^p. Consequently we have $Tx = x$ since x is a unique fixed point. □

Theorem 3.4. Let (X, S) be a complete complex valued S-metric space and T be a self-mapping of X. If there exist nonnegative real numbers $c_1, c_2, c_3, c_4, c_5, c_6$ satisfying $\max\{c_1 + c_2 + 3c_4 + c_5 + 3c_6, c_1 + c_3 + c_4 + c_6, 2c_2 + c_3 + 2c_6\} < 1$ such that

$$\begin{aligned} S(Tx, Tx, Ty) \preceq\ & c_1S(x, x, y) + c_2S(Tx, Tx, x) + c_3S(Tx, Tx, y) \\ & + c_4S(Ty, Ty, x) + c_5S(Ty, Ty, y) + c_6 \max\{S(x, x, y), \\ & S(Tx, Tx, x), S(Tx, Tx, y), S(Ty, Ty, x), S(Ty, Ty, y)\}, \end{aligned} \tag{8}$$

for all $x, y \in X$, then T has a unique fixed point x in X and T is continuous at x.

Proof. Let $a_0 \in X$ and the sequence $\{a_n\}$ be defined by

$$T^n a_0 = a_n.$$

Assume that $a_n \neq a_{n+1}$ for all n. Using the inequality 8, the condition $(CS3)$ and Lemma 2.5, we obtain

$$\begin{aligned} S(a_n, a_n, a_{n+1}) =\ & S(Ta_{n-1}, Ta_{n-1}, Ta_n) \preceq c_1S(a_{n-1}, a_{n-1}, a_n) + c_2S(a_n, a_n, a_{n-1}) \\ & + c_3S(a_n, a_n, a_n) + c_4S(a_{n+1}, a_{n+1}, a_{n-1}) + c_5S(a_{n+1}, a_{n+1}, a_n) \\ & + c_6 \max\{S(a_{n-1}, a_{n-1}, a_n), S(a_n, a_n, a_{n-1}), S(a_n, a_n, a_n), \\ & S(a_{n+1}, a_{n+1}, a_{n-1}), S(a_{n+1}, a_{n+1}, a_n)\} \\ =\ & c_1S(a_{n-1}, a_{n-1}, a_n) + c_2S(a_n, a_n, a_{n-1}) + c_4S(a_{n+1}, a_{n+1}, a_{n-1}) \\ & + c_5S(a_{n+1}, a_{n+1}, a_n) + c_6 \max\{S(a_{n-1}, a_{n-1}, a_n), S(a_n, a_n, a_{n-1}), \\ & S(a_{n+1}, a_{n+1}, a_{n-1}), S(a_{n+1}, a_{n+1}, a_n)\} \\ \preceq\ & (c_1 + c_2 + c_4 + c_6)S(a_{n-1}, a_{n-1}, a_n) \\ & + (2c_4 + c_5 + 2c_6)S(a_{n+1}, a_{n+1}, a_n) \end{aligned}$$

and

$$S(a_n, a_n, a_{n+1}) \preceq \frac{c_1 + c_2 + c_4 + c_6}{2c_4 + c_5 + 2c_6} S(a_{n-1}, a_{n-1}, a_n). \tag{9}$$

Let $c = \frac{c_1+c_2+c_4+c_6}{2c_4+c_5+2c_6}$. Then we find $c < 1$ since $c_1 + c_2 + 3c_4 + c_5 + 3c_6 < 1$. Using the inequality (9), we obtain

$$S(a_n, a_n, a_{n+1}) \preceq c^n S(a_0, a_0, a_1). \tag{10}$$

For all $n, m \in \mathbb{N}$, $n < m$, using the inequality (10) and the condition $(CS3)$, we have

$$S(a_n, a_n, a_m) \preceq \frac{2c^n}{1 - c} S(a_0, a_0, a_1),$$

which implies

$$|S(a_n, a_n, a_m)| \preceq \frac{2c^n}{1-c} |S(a_0, a_0, a_1)|.$$

Therefore $|S(a_n, a_n, a_m)| \to 0$ as $n, m \to \infty$. Hence $\{a_n\}$ is a Cauchy sequence. Since (X, S) is complete, there exists $x \in X$ such that $\{a_n\}$ converges to x.

Now we show that x is a fixed point of T. Suppose that $Tx \neq x$. Then we get

$$\begin{aligned} S(a_n, a_n, Tx) &= S(Ta_{n-1}, Ta_{n-1}, Tx) \preceq c_1 S(a_{n-1}, a_{n-1}, x) + c_2 S(a_n, a_n, a_{n-1}) \\ &+c_3 S(a_n, a_n, x) + c_4 S(Tx, Tx, a_{n-1}) + c_5 S(Tx, Tx, x) \\ &+c_6 \max\{S(a_{n-1}, a_{n-1}, x), S(a_n, a_n, a_{n-1}), S(a_n, a_n, x), \\ &S(Tx, Tx, a_{n-1}), S(Tx, Tx, x)\} \end{aligned}$$

and

$$\begin{aligned} |S(a_n, a_n, Tx)| &\leq c_1 |S(a_{n-1}, a_{n-1}, x)| + c_2 |S(a_n, a_n, a_{n-1})| + c_3 |S(a_n, a_n, x)| \\ &+c_4 |S(Tx, Tx, a_{n-1})| + c_5 |S(Tx, Tx, x)| \\ &+c_6 \left| \begin{array}{c} \max\{S(a_{n-1}, a_{n-1}, x), S(a_n, a_n, a_{n-1}), S(a_n, a_n, x), \\ S(Tx, Tx, a_{n-1}), S(Tx, Tx, x)\} \end{array} \right|. \end{aligned}$$

If we take limit for $n \to \infty$, then using the continuity of S and Lemma 2.5, we have

$$|S(Tx, Tx, x)| \leq (c_4 + c_5 + c_6) |S(Tx, Tx, x)|,$$

which is a contradiction since $0 \leq c_4 + c_5 + c_6 < 1$. Hence we obtain $Tx = x$.

Now we show that x is unique. Let y be another fixed point of T such that $x \neq y$. Using the inequality (8) and Lemma 2.5, we have

$$\begin{aligned} S(Tx, Tx, Ty) &= S(x, x, y) \preceq c_1 S(x, x, y) + c_2 S(x, x, x) + c_3 S(x, x, y) \\ &+c_4 S(y, y, x) + c_5 S(y, y, y) + c_6 \max\{S(x, x, y), \\ &S(x, x, x), S(x, x, y), S(y, y, x), S(y, y, y)\} \end{aligned}$$

and

$$|S(x, x, y)| \leq (c_1 + c_3 + c_4 + c_6) |S(x, x, y)|,$$

which implies $x = y$ since $c_1 + c_3 + c_4 + c_6 < 1$.

Now we prove that T is continuous at x. For $n \in \mathbb{N}$, using the inequality (8), the condition $(CS3)$ and Lemma 2.5, we obtain

$$\begin{aligned} S(Ta_n, Ta_n, Tx) &\preceq c_1 S(a_n, a_n, x) + c_2 S(Ta_n, Ta_n, a_n) + c_3 S(Ta_n, Ta_n, x) \\ &\quad + c_4 S(Tx, Tx, a_n) + c_5 S(Tx, Tx, x) \\ &\quad + c_6 \max\{S(a_n, a_n, x), S(Ta_n, Ta_n, , a_n), S(Ta_n, Ta_n, x), \\ &\quad S(Tx, Tx, a_n), S(Tx, Tx, x)\} \\ &\preceq c_1 S(a_n, a_n, x) + 2c_2 S(Ta_n, Ta_n, x) + c_2 S(a_n, a_n, x) \\ &\quad + c_3 S(Ta_n, Ta_n, x) + c_4 S(Tx, Tx, a_n) \\ &\quad + c_6 \max\{S(a_n, a_n, x), 2S(Ta_n, Ta_n, x) + S(a_n, a_n, x), \\ &\quad S(Ta_n, Ta_n, x)\} \\ &= (c_1 + c_2 + c_4 + c_6) S(a_n, a_n, x) + (2c_2 + c_3 + 2c_6) S(Tx, Tx, Ta_n) \end{aligned}$$

and

$$(1 - 2c_2 - c_3 - 2c_6) S(Ta_n, Ta_n, Tx) \preceq (c_1 + c_2 + c_4 + c_6) S(a_n, a_n, x),$$

which implies

$$|S(Ta_n, Ta_n, Tx)| \leq \frac{c_1 + c_2 + c_4 + c_6}{1 - 2c_2 - c_3 - 2c_6} |S(a_n, a_n, x)|.$$

If we take limit for $n \to \infty$, then we have

$$|S(Ta_n, Ta_n, Tx)| \to 0.$$

Therefore $\{Ta_n\}$ is convergent to $Tx = x$. Consequently, T is continuous at x by Lemma 2.6. □

Remark 3.5. (1) Theorem 3.4 is a generalization of Banach's contraction principle on complete complex valued S-metric spaces. Indeed, if we take $c_1 = h$ and $c_2 = c_3 = c_4 = c_5 = c_6 = 0$ in Theorem 3.4, then we obtain the Banach's contraction condition in Theorem 2.7.

(2) If we take the function $S : X \times X \times X \to [0, \infty)$ in Theorem 3.4, Then we have Theorem 4 in [6].

Corollary 3.6. Let (X, S) be a complete complex valued S-metric space and T be a self-mapping of X. If there exist nonnegative real numbers $c_1, c_2, c_3, c_4, c_5, c_6$ satisfying $\max\{c_1 + c_2 + 3c_4 + c_5 + 3c_6, c_1 + c_3 + c_4 + c_6, 2c_2 + c_3 + 2c_6\} < 1$ such that

$$\begin{aligned} S(T^p x, T^p x, T^p y) &\preceq c_1 S(x, x, y) + c_2 S(T^p x, T^p x, x) + c_3 S(T^p x, T^p x, y) \\ &\quad + c_4 S(T^p y, T^p y, x) + c_5 S(T^p y, T^p y, y) + c_6 \max\{S(x, x, y), \\ &\quad S(T^p x, T^p x, x), S(T^p x, T^p x, y), S(T^p y, T^p y, x), S(T^p y, T^p y, y)\}, \end{aligned}$$

for all $x, y \in X$ and some $p \in \mathbb{N}$, then T has a unique fixed point x in X and T^p is continuous at x.

Proof. It follows from Theorem 3.4 by the same argument used in the proof of Corollary 3.3. □

In the following example we give a self-mapping satisfying the conditions of our results, but does not satisfy the condition of the Banach's contraction principle.

Example 3.7. Let $X = \mathbb{R}$ and the function $S : X \times X \times X \to \mathbb{C}$ be defined as

$$S(x, y, z) = e^{it}(|x - z| + |x + z - 2y|),$$

for all $x, y, z, t \in \mathbb{R}$. Then $(\mathbb{R}, S)$ is a complete complex valued S-metric space. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} x + 70 & \text{if} \quad x \in \{0, 6\} \\ 65 & \text{if} \quad \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Therefore T satisfies the inequality (2) in Theorem 3.1 for $c_1 = c_2 = c_3 = 0$, $c_4 = \frac{1}{4}$ and the inequality (8) in Theorem 3.4 for $c_1 = c_3 = c_4 = c_5 = 0$, $c_2 = c_6 = \frac{1}{5}$. So T has a unique fixed point $x = 65$. But T does not satisfy the Banach's contraction condition in Theorem 2.7. Indeed, for $x = 6$, $y = 2$, we obtain

$$S(Tx, Tx, Ty) = S(76, 76, 65) = 22e^{it} \precsim hS(x, x, y) = hS(6, 6, 2) = 8he^{it}$$

and

$$\left|22e^{it}\right| = 22 \leq \left|8he^{it}\right| = 8h,$$

which is a contradiction $h < 1$.

References

[1] A. Azam, M. Arshad and I. Beg, *Banach contraction principle on cone rectangular metric spaces*, Applicable Analysis and Discrete Mathematics 3 (2009) 236-241.

[2] S. U. Khan and A. Bano, *Common fixed point theorems for f-contraction mappings in TVS-valued cone metric space*, Journal of New Theory 2 (13) (2016) 96-103.

[3] N. M. Mlaiki, *Common Fixed Points in Complex S-Metric Space*, Advances in Fixed Point Theory 4 (4) (2014) 509-524.

[4] N. Y. Özgür and N. Taş, *Some fixed point theorems on S-metric spaces*, submitted for publication.

[5] N. Y. Özgür and N. Taş, *Some new contractive mappings on S-metric spaces and their relationships with the mapping* (**S25**), submitted for publication.

[6] N. Y. Özgür and N. Taş, *Some generalizations of fixed point theorems on S-metric spaces*, Essays in Mathematics and Its Applications in Honor of Vladimir Arnold, New York, Springer, 2016.

[7] N. Y. Özgür and N. Taş, *Common fixed points of continuous mappings on complex valued S-metric spaces*, submitted for publication.

[8] S. Sedghi, N. Shobe and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Matematicki Vesnik 64 (3) (2012) 258-266.

[9] S. Sedghi and N. V. Dung, *Fixed point theorems on S-metric spaces*, Matematicki Vesnik 66 (1) (2014) 113-124.

[10] S. Sedghi, İ. Altun, N. Shobe and M. A. Salahshour, *Some properties of S-metric spaces and fixed point results*, Kyungpook Mathematical Journal 54 (2014) 113-122.