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A NEW TYPE OF CONVERGENCE IN INTUITIONISTIC FUZZY NORMED LINEAR SPACES

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Abstaract − In the present paper, we introduce a new type of convergence, called standard convergence (or std-convergence), in an intuitionistic fuzzy normed linear space (IFNLS). We have also introduced the concept of std-Cauchyness and proved that these notions are stronger than usual convergence and usual Cauchyness in an IFNLS. Further, we have shown that these two notions are not directly compatible with each other and hence, defined the notion of strong std-convergence which is compatible with std-Cauchy sequences.

Keywords − *Intuitionistic fuzzy normed linear space; std-convergence; strong std-convergence; std-Cauchy sequence.*

1 Introduction

The concepts of convergence and Cauchyness of sequences lay the foundation of structure of any metric space and as such, the study of these concepts are of greatest importance in analysis. Therefore, the study of both weaker and stronger concepts than the usual convergence has always been a well motivated area of research. Some very important work in this direction in connection with fuzzy metric spaces may be found in [9, 11, 14, 17].

Recently, Ricarte and Romaguera [23] have established relationships between the theory of complete fuzzy metric spaces and domain theory by introducing a stronger notion than Cauchy sequence called standard Cauchy sequence. They proved that the famous result due to Edalat and Heckmann [8] which gives a characterization of complete metric spaces with the help of continuous domains could be obtained from their results in fuzzy metrics and in fact, could not be obtained from classical metric. More recently, answer to two well posed questions by Morillas and Sapena [18] was

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given by Gregori and Minana [12] by establishing what conditions must be included in the definition of standard convergence in fuzzy metric spaces so that it remains compatible with the concept of Cauchyness.

The theory of intuitionistic fuzzy sets was introduced by Atanassov [2] which has been extensively used in decision making problems [1] and in E-infinity theory of high energy physics [21]. The concept of intuitionistic fuzzy metric space was introduced by Park [22]. Furthermore, Saadati and Park [24] gave the notion of intuitionistic fuzzy normed space. Some works related to the convergence of sequences in several normed linear spaces in fuzzy setting can be found in [3, 4, 5, 6, 7, 10, 13, 15, 16, 19, 20, 25, 26, 27].

Due to its successful application in connection with fuzzy metric spaces and domain theory [23], in the current paper, we introduce and generalize the notions of standard (std-) convergence and standard (std-) Cauchy sequences in an IFNLS.

2 Preliminary

Throughout the paper $\mathbb{N}$ will denote the set of all natural numbers and $\mathbb{R}$ will denote the set of real numbers. First we collect some preliminary existing definitions in literature.

Definition 2.1. [24] The 5-tuple $(X, \mu, \nu, *, \circ)$ is said to be an IFNLS if X is a linear space, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, and μ, ν fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x,t) + \nu(x,t) \leq 1$,

(b) $\mu(x,t) > 0$,

(c) $\mu(x,t) = 1$ if and only if $x = 0$,

(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,

(f) $\mu(x,t) : (0,\infty) \to [0,1]$ is continuous in t,

(g) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,

(h) $\nu(x,t) < 1$,

(i) $\nu(x,t) = 0$ if and only if $x = 0$,

(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,

(k) $\nu(x,t) \circ \nu(y,s) \geq \nu(x+y, t+s)$,

(l) $\nu(x,t) : (0,\infty) \to [0,1]$ is continuous in t,

(m) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm (IFN). When no confusion arises, an IFNLS will be denoted simply by X.

Example 2.2. Let $(V, ||\cdot||)$ be a normed linear space. Let $a * b = ab$ and $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$ and μ_0, ν_0 be fuzzy sets on $X \times (0, \infty)$ defined as $\mu_0(x, t) = \frac{t}{t+||x||}$, $\nu_0(x, t) = \frac{||x||}{t+||x||}$ for all $t \in (0, \infty)$. Then $(X, \mu_0, \nu_0, *, \circ)$ is an IFNLS.

Remark 2.3. Let $(X, \mu_0, \nu_0, *, \circ)$ be an IFNLS. For $t > 0$, the open ball $B_r^t(x)$ with center x and radius $r \in (0, 1)$ is defined as

$$B_r^t(x) = \{y \in V : \mu(x - y, t) > 1 - r, \nu(x - y, t) < r\}.$$

Considering these open balls as base, the IFN (μ, ν) induces a topology $\tau_{(\mu,\nu)}$ on X.

Definition 2.4. [24] Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > 1 - \epsilon$ and $\nu(x_k - \xi, t) < \epsilon$ for all $k \geq k_0$. It is denoted by $(\mu, \nu) - \lim x = \xi$.

Definition 2.5. [24] Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > 1 - \epsilon$ and $\nu(x_k - x_m, t) < \epsilon$ for all $k, m \geq k_0$.

3 std-Convergence and std-Cauchy Sequences in IFNLS

Now we are ready to introduce the notions of std-Convergence and std-Cauchy sequences in IFNLS.

Definition 3.1. Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be std-convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon \in (0, 1)$, there exists $k_\epsilon \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > \frac{t}{t+\epsilon}$ and $\nu(x_k - \xi, t) < \frac{\epsilon}{t+\epsilon}$ for all $k \geq k_\epsilon$ and for all $t > 0$. We denote it by $(\mu, \nu) \overset{std}{-} \lim x = \xi$.

Definition 3.2. Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be std-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon \in (0, 1)$, there exists $k_\epsilon \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > \frac{t}{t+\epsilon}$ and $\nu(x_k - x_m, t) < \frac{\epsilon}{t+\epsilon}$ for all $k, m \geq k_\epsilon$ and for all $t > 0$. We call X std-complete if every std-Cauchy sequence is std-convergent in X.

Our first two results show that the notions of std-convergence and std-Cauchy are both stronger than usual convergence and usual Cauchy respectively in an IFNLS.

Theorem 3.3. Let X be an IFNLS and the sequence $x = \{x_k\}$ in X be std-convergent to $\xi \in X$. Then $\{x_k\}$ converges to ξ with respect to the IFN (μ, ν).

Proof. Let $(\mu,\nu) \overset{std}{-} \lim x = \xi$. Then for given $\epsilon > 0$, there exists $k_\epsilon \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > \frac{t}{t+\epsilon}$ and $\nu(x_k - \xi, t) < \frac{\epsilon}{t+\epsilon}$ for all $k \geq k_\epsilon$ and all $t > 0$.

Now since $\frac{\epsilon}{t+\epsilon} < \epsilon$ for all $t > 0$, we have that $\frac{t}{t+\epsilon} = 1 - \frac{\epsilon}{t+\epsilon} > 1 - \epsilon$. Consequently, $\mu(x_k - \xi, t) > \frac{t}{t+\epsilon} > 1 - \epsilon$. In a similar way it can be proved that $\nu(x_k - \xi, t) < \epsilon$ for all $k \geq k_\epsilon$. This proves that $(\mu,\nu) - \lim x = \xi$. □

The following can be proved using techniques in 3.3.

Theorem 3.4. Let X be an IFNLS and the sequence $x = \{x_k\}$ in X be std-Cauchy. Then $\{x_k\}$ is Cauchy with respect to the IFN (μ,ν).

Next we give an example of a sequence in an IFNLS which is std-convergent but not std-Cauchy.

Example 3.5. Consider the usual norm $|\cdot|$ on $\mathbb{R}$ restricted to $[0,\infty)$ and the IFN (μ_0,ν_0) as defined in Example 2.2. Let $X = [0,\infty)$ and define on $X \times (0,\infty)$ the functions μ,ν as

$$\mu(x-y,t) = \begin{cases} 1, & \text{if } x = y \\ \mu_0(x-0,t)\mu_0(0-y,t), & \text{if } x \neq y, \end{cases}$$

and $\nu(x-y,t) = 1 - \mu(x-y,t)$. Then it is a routine verification to check that $(X,\mu,\nu,*,\circ)$ is an IFNLS.

Consider the sequence $\{x_k\}$ in X where $x_k = \frac{1}{k}$ for all $k \in \mathbb{N}$. Let $\epsilon \in (0,1)$. We can choose $k_\epsilon \in \mathbb{N}$ such that $k_\epsilon > \frac{1}{\epsilon}$ and hence $\mu(x_k - 0, t) = \frac{t}{t+\frac{1}{k}} > \frac{t}{t+\epsilon}$ for all $k \geq k_\epsilon$ and all $t > 0$. In a similar fashion it can be proved that $\mu(x_k - 0, t) < \frac{\epsilon}{t+\epsilon}$. So, $\{x_k\}$ is std-convergent to 0 in X.

Now if we assume $\{x_k\}$ to be std-Cauchy, then for each $\epsilon \in (0,1)$, there exists $k_\epsilon \in \mathbb{N}$ such that $\mu(x_k - x_m, t) = \frac{t}{t+\frac{1}{k}} \cdot \frac{t}{t+\frac{1}{m}} > \frac{t}{t+\frac{1}{\epsilon}}$ for all $k, m \geq k_\epsilon$ and all $t > 0$. Thus we have $\frac{t}{(t+\frac{1}{k_\epsilon})(t+\frac{1}{k_\epsilon})} > \frac{1}{t+\epsilon}$ for all $t > 0$. But then we have $\lim_{t\to 0} \frac{t}{(t+\frac{1}{k_\epsilon})(t+\frac{1}{k_\epsilon})} > \lim_{t\to 0} \frac{1}{t+\epsilon}$. This implies that $0 > \frac{1}{\epsilon}$, which is a contradiction. This proves that $\{x_k\}$ can not be std-Cauchy.

From Example 3.5 we observe that the concept of std-convergence and std-Cauchy are not compatible with each other. To settle this, we define the notion of strong std-convergence so that every strong std-convergent sequence is std-Cauchy as well.

Definition 3.6. A sequence $\{x_k\}$ in an IFNLS $(X,\mu,\nu,*,\circ)$ is said to be strong std-convergent if it is both convergent and std-Cauchy with respect to the IFN (μ,ν).

A question naturally arises, whether every strong std-convergent sequence is std-convergent or not. An affirmative answer to this question is given by our next result.

Theorem 3.7. Let $(X,\mu,\nu,*,\circ)$ be an IFNLS and $\{x_k\}$ be a strong std-convergent sequence in X. Then $\{x_k\}$ is std-convergent with respect to the IFN (μ,ν).

Proof. Let $\epsilon \in (0,1)$ and $t > 0$. We assume that $\{x_k\}$ converges to $\xi \in X$ with respect to (μ, ν). Since $\mu(x, \_)$ is continuous for all $x \in X$, we have that $\lim_{m\to\infty} \mu(x_k - x_m, t) = \mu(x_k - \xi, t)$ for all $k \in \mathbb{N}$.

Again since $\{x_k\}$ is std-Cauchy, we have that for $\delta \in (0, \epsilon)$, there exists $k_\delta \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > \frac{t}{t+\delta} > \frac{t}{t+\epsilon}$ for all $k, m \geq k_\delta$ and all $t > 0$. This again implies that $\mu(x_k - \xi, t) = \lim_{m\to\infty} \mu(x_k - x_m, t) \geq \frac{t}{t+\delta} > \frac{t}{t+\epsilon}$ for all $k \geq k_\delta$ and all $t > 0$.

In a similar fashion, it can be proved that $\nu(x_k - \xi, t) < \frac{\epsilon}{t+\epsilon}$ for all $k \geq k_\delta$ and all $t > 0$. Hence $\{x_k\}$ is std-convergent. □

Next we show that the notion of strong std-convergence is free from any ambiguity by showing that strong std-convergent sequences have unique limit. To show this, it is sufficient to prove that a std-convergent sequence has a unique limit, which is the aim of our next result.

Theorem 3.8. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. If a sequence $\{x_k\}$ in X is std-convergent with respect to the IFN (μ, ν), then $(\mu, \nu) \overset{std}{-} \lim x_k$ is unique.

Proof. Let $(\mu, \nu) \overset{std}{-} \lim x_k = \xi_1$ and $(\mu, \nu) \overset{std}{-} \lim x_k = \xi_2$. Given $\epsilon > 0$ and $t > 0$ choose $\gamma \in (0,1)$ such that $(\frac{t}{t+\gamma}) * (\frac{t}{t+\gamma}) > \frac{t}{t+\epsilon}$.

Since $(\mu, \nu) \overset{std}{-} \lim x_k = \xi_1$, there exists $k_1 \in \mathbb{N}$ such that $\mu(x_k - \xi_1, \frac{t}{2}) > \frac{t}{t+\gamma}$ for all $k \geq k_1$ and all $t > 0$. Also since $(\mu, \nu) \overset{std}{-} \lim x_k = \xi_2$, there exists $k_2 \in \mathbb{N}$ such that $\mu(x_k - \xi_2, \frac{t}{2}) > \frac{t}{t+\gamma}$ for all $k \geq k_2$ and all $t > 0$.

Let $k_0 = \max\{k_1, k_2\}$. Then both of the above two conditions hold together for all $k \geq k_0$ and all $t > 0$.

Now we have, for all $k \geq k_0$ and all $t > 0$,

$$
\begin{aligned}
\mu(\xi_1 - \xi_2, t) &\geq \mu(x_k - \xi_1, \frac{t}{2}) * \mu(x_k - \xi_2, \frac{t}{2}) \\
&> (\frac{t}{t+\gamma}) * (\frac{t}{t+\gamma}) \\
&> \frac{t}{t+\epsilon}.
\end{aligned}
$$

Since ϵ was chosen arbitrarily, we must have $\mu(\xi_1 - \xi_2, t) = 1$ for all $t > 0$. Hence we must have $\xi_1 - \xi_2 = 0$, i.e., $\xi_1 = \xi_2$. □

4 Conclusion

In this paper, the concept of std-convergence and std-Cauchy sequences have been introduced. Another concept, called strong std-convergence has also been introduced which is directly compatible with std-Cauchy sequences. These new concepts are stronger than their usual counterparts and as such, they constitute a well motivated area of research. The study of std-statistical convergence, std-ideal convergence, std-lacunary statistical convergence may be suggested as some important future work in this new setting.

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