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IDENTITIES FOR MULTIPLICATIVE COUPLED FIBONACCI SEQUENCES OF R^{TH} ORDER

Ashok Dnyandeo Godase[1,*] <ashokgodse2012@gmail.com>
Macchindra Dhakne[2] <ijntindia@gamil.com>

[1]Department of Mathematics, V. P. College, Vaijapur, Maharashtra, India
[2]Department of Mathematics, Dr. B. A. M. University, Aurangabad, Maharashtra, India

Abstaract − Many author studied coupled Fibonacci sequences and multiplicative coupled Fibonacci sequences of lower order two, three and four etc. In this paper we defined multiplicative coupled Fibonacci Sequences of r^{th} order under 2^r different schemes. Some new identities for these sequences are established under one specific scheme.

Keywords − *Fibonacci Sequence, Coupled Fibonacci Sequence, Recurrence Relation.*

1 Introduction

The Fibonacci sequence is a source of intresting identities. Many identities have been documented in [14], [15], [16], [17], [21]. A similar interpretation exists for k Fibonacci and k Lucas numbers, many of these identities have been documented in the work of Falcon and Plaza [3], [6], [7], [11], [12], [13]. Many authors defined coupled and multiplicative coupled Fibonacci sequences by varying initial conditions and recurrence relation. Properties of these sequences are documented in[1], [2], [9], [4], [5]. Many authors defined coupled and multiplicative coupled Fibonacci sequences by varying initial conditions and recurrence relation. Properties of these sequences are documented in[1], [2], [9], [4], [5]. In this paper we defined multiplicative coupled Fibonacci sequences of r^{th} order by varying recurrence relation and some identities for these mentioned sequences are also obtained under $2^{r-1^{th}}$ scheme.
Coupled Fibonacci sequences involve two sequences of integers in which the elements of one sequence are part of the generalization of the other and vice versa. K. T. Atanassov [1] was first introduced coupled Fibonacci sequences of second order

[*] *Corresponding Author.*

in additive form and also discussed many curious properties and new direction of generalization of Fibonacci sequence in his series of papers on coupled Fibonacci sequences. He defined and studied about four different ways to generate coupled sequences and called them coupled Fibonacci sequences (or 2-F sequences). The multiplicative Fibonacci Sequences studied by singh-shikwal [?]. K. T. Atanassov [2] notifies four different schemes in multiplicative form for coupled Fibonacci sequences. The analogue of the standard Fibonacci sequence in this form is $x_0 = a$, $x_1 = b$, $x_{n+2} = x_{n+1} \cdot x_n \quad (n \geq 0)$.
Attanasov [1] introduced a new view of generalized Fibonacci sequences by taking a pair of sequences $\{X_i\}_{i=0}^{i=\infty}$ and $\{Y_i\}_{i=0}^{i=\infty}$ and which can be generated by famous Fibonacci formula and gave various identities involving Fibonacci sequence called the coupled Fibonacci sequences.
In this paper we defined multiplicative coupled Fibonacci sequences of r^{th} order by varying recurrence relation and some identities for these mentioned sequences are also obtained under $2^{r-1^{th}}$ scheme.

2 Preliminary and Notations

Definition 2.1. Multiplicative Coupled Fibonacci sequences of third order: Let $\{X_i\}_{i=0}^{i=\infty}$ and $\{Y_i\}_{i=0}^{i=\infty}$ be two infinite sequences and six arbitrary real numbers x_0, x_1, x_2, y_0, y_1, y_2 are given. The Multiplicative coupled Fibonacci sequences of 3^{rd} order are generated by the following eight different ways:
First scheme

$$X_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0$$
$$Y_{n+3} = X_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0$$

Second scheme

$$X_{n+3} = X_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0$$
$$Y_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0$$

Third scheme

$$X_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0$$
$$Y_{n+3} = X_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0$$

Fourth scheme

$$X_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0$$
$$Y_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0$$

Fifth scheme

$$X_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0$$
$$Y_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0$$

Sixth scheme

$$X_{n+3} = X_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0$$
$$Y_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0$$

Seventh scheme

$$X_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot Y_n, \quad n \geq 0$$
$$Y_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot X_n, \quad n \geq 0$$

Eighth scheme

$$X_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot X_n, \quad n \geq 0$$
$$Y_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot Y_n, \quad n \geq 0$$

Definition 2.2. Multiplicative Coupled Fibonacci sequences of r^{th} order: Let $\{X_i\}_{i=0}^{i=\infty}$ and $\{Y_i\}_{i=0}^{i=\infty}$ be two infinite sequences and $2r$ arbitrary real numbers x_0, x_1, x_2, x_3, ..., x_{r-1} and y_0, y_1, y_2, y_3, ..., y_{r-1} are given. The Multiplicative coupled Fibonacci sequences of r^{th} order are generated by the following 2^r different ways:
First scheme

$$X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$

Second scheme

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$

$$\vdots$$

$(2^{r-1})^{th}$ **scheme**
(a) **If r is even**,

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0$$

(b) **If r is odd**,

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$

$$\vdots$$

$(2^r)^{th}$ **scheme**

$$X_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$

n	Y_{n+r}	X_{n+r}
0	$y_0y_1y_2y_3\ldots\ldots\ y_{r-1}$	$x_0x_1x_2x_3\ldots\ldots\ x_{r-1}$
1	$x_0x_1x_2x_3\ldots\ldots\ x_{r-1}y_1y_2y_3\ldots\ldots\ y_{r-1}$	$x_1x_2x_3\ldots\ldots\ x_{r-1}y_0y_1y_2y_3\ldots\ldots\ y_{r-1}$
2	$x_0x_1^2x_2^2x_3^2\ldots\ldots\ x_{r-1}^2y_0y_1y_2^2y_3^2\ldots\ldots\ y_{r-1}^2$	$x_0x_1x_2^2x_3^2\ldots\ldots\ x_{r-1}^2y_0y_1^2y_2^2y_3^2\ldots\ldots\ y_{r-1}^2$
3	$x_0^2x_1^3x_2^4x_3^4\ldots\ldots\ x_{r-1}^4y_0^2y_1^3y_2^3y_3^4\ldots\ldots\ y_{r-1}^4$	$x_0^2x_1^3x_2^3x_3^4\ldots\ldots\ x_{r-1}^4y_0^2y_1^3y_2^4y_3^4\ldots\ldots\ y_{r-1}^4$
4	$x_0^4x_1^5x_2^7x_3^8\ldots\ldots\ x_{r-1}^8y_0^4y_1^6y_2^7y_3^7y_4^8\ldots\ldots\ y_{r-1}^8$	$x_0^4x_1^5x_2^7x_3^7x_4^8\ldots\ldots\ x_{r-1}^8y_0^4y_1^6y_2^7y_3^8\ldots\ldots\ y_{r-1}^8$

Table 1. **First few terms of these sequences under $(2^{r-1})^{th}$(a) scheme**

n	X_{n+r}	Y_{n+r}
0	$y_0y_1y_2y_3\ldots\ldots\ y_{r-1}$	$x_0x_1x_2x_3\ldots\ldots\ x_{r-1}$
1	$x_0x_1x_2x_3\ldots\ldots\ x_{r-1}y_1y_2y_3\ldots\ldots\ y_{r-1}$	$x_1x_2x_3\ldots\ldots\ x_{r-1}y_0y_1y_2y_3\ldots\ldots\ y_{r-1}$
2	$x_0x_1^2x_2^2x_3^2\ldots\ldots\ x_{r-1}^2y_0y_1y_2^2y_3^2\ldots\ldots\ y_{r-1}^2$	$x_0x_1x_2^2x_3^2\ldots\ldots\ x_{r-1}^2y_0y_1^2y_2^2y_3^2\ldots\ldots\ y_{r-1}^2$
3	$x_0^2x_1^3x_2^4x_3^4\ldots\ldots\ x_{r-1}^4y_0^2y_1^3y_2^3y_3^4\ldots\ldots\ y_{r-1}^4$	$x_0^2x_1^3x_2^3x_3^4\ldots\ldots\ x_{r-1}^4y_0^2y_1^3y_2^4y_3^4\ldots\ldots\ y_{r-1}^4$
4	$x_0^4x_1^5x_2^7x_3^8\ldots\ldots\ x_{r-1}^8y_0^4y_1^6y_2^7y_3^7y_4^8\ldots\ldots\ y_{r-1}^8$	$x_0^4x_1^5x_2^7x_3^7x_4^8\ldots\ldots\ x_{r-1}^8y_0^4y_1^6y_2^7y_3^8\ldots\ldots\ y_{r-1}^8$

Table 2. **First few terms of these sequences under $(2^{r-1})^{th}$(b) scheme**

Godase-Dhakne [8] obtained many interesting properties of multiplicative coupled Fibonacci sequences of r^{th} order under $(2^r)^{th}$ scheme, some of these are listed below, for every integer $n \geq 0$ and $r \geq 0$

$$X_{n(r+1)} \cdot Y_0 = Y_{n(r+1)} \cdot X_0 \tag{1}$$
$$X_{n(r+1)+1} \cdot Y_1 = Y_{n(r+1)+1} \cdot X_1 \tag{2}$$
$$X_{n(r+1)+2} \cdot Y_2 = Y_{n(r+1)+2} \cdot X_2 \tag{3}$$
$$X_{n(r+1)+3} \cdot Y_3 = Y_{n(r+1)+3} \cdot X_3 \tag{4}$$
$$X_{n(r+1)+m} \cdot Y_m = Y_{n(r+1)+m} \cdot X_m \tag{5}$$
$$\prod_{i=1}^{i=n} X_{ri+1} = \prod_{i=1}^{i=rn} Y_i \tag{6}$$
$$\prod_{i=1}^{i=n} Y_{ri+1} = \prod_{i=1}^{i=rn} X_i \tag{7}$$

3 Methodology

For this research it was decided to consider multiplicative coupled Fibonacci sequences of r^{th} order. When investigating the properties of the mentioned sequences, it is necessary to take into consideration that reader will need special skills and abilities of recurrence relations, master the method of mathematical induction, knowledge on a Fibonacci sequence. All results in this research are proved only using methods of mathematical induction.

4 Main Results

In this section identities for multiplicative coupled Fibonacci sequences of r^{th} order under $(2^{r-1})^{th}$ scheme are established.

Theorem 4.1. For every integer $n \geq 0, r \geq 0$

$$X_{2n(r+1)} \cdot Y_0 = Y_{2n(r+1)} \cdot X_0 \tag{8}$$

Proof. : **Case:(a)**
If r is an even, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0$$

Using Induction Method, For $n = 0$, the result is true because

$$X_o \cdot Y_0 = Y_0 \cdot X_0$$

Now assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0 \tag{9}$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0 \tag{10}$$

Now we prove for $n+1$

$$\begin{aligned}
X_{2n(r+1)+2r+2}\cdot Y_0 &= \left[X_{2n(r+1)+2r+1}\cdot Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdot\cdots Y_{2n(r+1)+r+2}\right]\cdot Y_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdot\cdots Y_{2n(r+1)+r+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdot\cdots Y_{2n(r+1)+r+2}\right]\cdot Y_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdot\cdots Y_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+r}\cdot X_{2n(r+1)+r-1}\cdot Y_{2n(r+1)+r-2}\cdot\cdots Y_{2n(r+1)+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdot\cdots Y_{2n(r+1)+r+2}\right]\cdot Y_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdot\cdots X_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdot\cdots Y_{2n(r+1)}\right]\\
&\cdot\left[X_{2n(r+1)+r-1}Y_{2n(r+1)+r-1}X_{2n(r+1)+r-2}\cdot\cdots X_{2n(r+1)+r+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}X_{2n(r+1)+2r-1}\cdot\cdots Y_{2n(r+1)+r+2}\right]Y_0\\
&= [X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdot\cdots X_{2n(r+1)+r+2}]\\
&\cdot\left[Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdot\cdots Y_{2n(r+1)+1}\right]\\
&\cdot\left[X_{2n(r+1)+r-1}Y_{2n(r+1)+r-1}X_{2n(r+1)+r-2}\cdot\cdots X_{2n(r+1)+r+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}X_{2n(r+1)+2r-1}\cdot\cdots Y_{2n(r+1)+r+2}\right]\left[Y_0X_{2n(r+1)}\right]
\end{aligned}$$

Using induction hypothesis 9,10.

$$\begin{aligned}
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdot\cdots X_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdot\cdots Y_{2n(r+1)+1}\right]\\
&\cdot\left[X_{2n(r+1)+r-1}Y_{2n(r+1)+r-1}X_{2n(r+1)+r-2}\cdot\cdots X_{2n(r+1)+r+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}X_{2n(r+1)+2r-1}\cdot\cdots Y_{2n(r+1)+r+2}\right]\left[X_0Y_{2n(r+1)}\right]\\
&= \left[X_{2n(r+1)+2r}Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdot\cdots X_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdot\cdots Y_{2n(r+1)+1}\right]\\
&\cdot\left[X_{2n(r+1)+r-1}\cdot Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot\cdots X_{2n(r+1)+1}\cdot Y_{2n(r+1)}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdot\cdots Y_{2n(r+1)+r+2}\right]\cdot X_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdot\cdots X_{2n(r+1)+r+2}\right]\\
&\cdot\left[X_{2n(r+1)+r}\cdot Y_{2n(r+1)+r-1}\cdot X_{2n(r+1)+r-2}\cdot Y_{2n(r+1)+r-3}\cdot\cdots Y_{2n(r+1)+1}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdot\cdots Y_{2n(r+1)+r+2}\right]\cdot X_0\\
&= \left[X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdot\cdots X_{2n(r+1)+r+2}\right]\\
&\cdot\left[Y_{2n(r+1)+2r}\cdot X_{2n(r+1)+2r-1}\cdot\cdots Y_{2n(r+1)+r+2}\cdot X_{2n(r+1)+r+1}\right]\cdot X_0\\
&= \left[Y_{2n(r+1)+2r+1}\cdot X_{2n(r+1)+2r}\cdot Y_{2n(r+1)+2r-1}\cdot X_{2n(r+1)+2r-2}\cdot\cdots X_{2n(r+1)+r+2}\right]\cdot X_0\\
&= Y_{2n(r+1)+2r+2}\cdot X_0
\end{aligned}$$

Case:(b)

If r is an odd, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0$$

Using Induction Method,for $n = 0$, the result is true because

$$X_o \cdot Y_0 = Y_0 \cdot X_0$$

Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0 \tag{11}$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0 \tag{12}$$

Now we prove for, $n + 1$

$$\begin{aligned}
X_{2n(r+1)+2r+2} \cdot Y_0 &= \left[X_{2n(r+1)+2r+1} \cdot Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+r} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots X_{2n(r+1)+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots X_{2n(r+1)}\right] \\
&\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot Y_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots X_{2n(r+1)+1}\right] \\
&\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots X_{2n(r+1)+r+2}\right] \cdot \left[Y_0 \cdot X_{2n(r+1)}\right]
\end{aligned}$$

Using induction hypothesis 11,12

$$
\begin{aligned}
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right] \\
&\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot \left[X_0 \cdot Y_{2n(r+1)}\right] \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right] \\
&\cdot \left[Y_{2n(r+1)+r-1} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdots X_{2n(r+1)+1} \cdot Y_{2n(r+1)}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \\
&\cdot \left[Y_{2n(r+1)+r} \cdot X_{2n(r+1)+r-1} \cdot Y_{2n(r+1)+r-2} \cdot X_{2n(r+1)+r-3} \cdots Y_{2n(r+1)+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0 \\
&= \left[X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+2r} \cdot X_{2n(r+1)+2r-1} \cdots Y_{2n(r+1)+r+2} \cdot X_{2n(r+1)+r+1}\right] \cdot X_0 \\
&= \left[Y_{2n(r+1)+2r+1} \cdot X_{2n(r+1)+2r} \cdot Y_{2n(r+1)+2r-1} \cdot X_{2n(r+1)+2r-2} \cdots Y_{2n(r+1)+r+2}\right] \cdot X_0 \\
&= Y_{2n(r+1)+2r+2} \cdot X_0
\end{aligned}
$$

Hence proof. □

Theorem 4.2. For every integer $n \geq 0, \quad r \geq 0$

$$X_{2n(r+1)+1} \cdot Y_1 = Y_{2n(r+1)+1} \cdot X_1 \tag{13}$$
$$X_{2n(r+1)+2} \cdot Y_2 = Y_{2n(r+1)+2} \cdot X_2 \tag{14}$$
$$X_{2n(r+1)+3} \cdot Y_3 = Y_{2n(r+1)+3} \cdot X_3 \tag{15}$$

Proof. Proof is similar to theorem 4.1 □

Theorem 4.3. For every integer $n \geq 0, r \geq 0$ and $m \geq 0$

$$X_{2n(r+1)+m} \cdot Y_m = Y_{2n(r+1)+m} \cdot X_m \tag{16}$$

Proof. **Case:(a)**
If r is an even, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0$$

Using induction method, for $n = 0$ the result is true because

$$X_m \cdot Y_m = Y_m \cdot X_m$$

Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots Y_n, n \geq 0 \tag{17}$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots X_n, n \geq 0 \tag{18}$$

Now, we prove for $n+1$

$$\begin{aligned}
X_{2n(r+1)+m+2r+2} \cdot Y_m &= \left[X_{2n(r+1)+m+2r+1} \cdot Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots Y_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[Y_{2n(r+1)+m+r} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots Y_{2n(r+1)+m+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m}\right] \\
&\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right] \\
&\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot \left[Y_m \cdot X_{2n(r+1)+m}\right]
\end{aligned}$$

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$$\begin{aligned}
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right] \\
&\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot \left[X_m \cdot Y_{2n(r+1)} + m\right] \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right] \\
&\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdots X_{2n(r+1)+m+1} \cdot Y_{2n(r+1)+m}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot X_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+m+r} \cdot Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-2} \cdot Y_{2n(r+1)+m+r-3} \cdots Y_{2n(r+1)+m+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2}\right] \cdot X_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots Y_{2n(r+1)+m+r+2} \cdot X_{2n(r+1)+m+r+1}\right] \cdot X_m \\
&= \left[Y_{2n(r+1)+m+2r+1} \cdot X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots X_{2n(r+1)+m+r+2}\right] \cdot X_m \\
&= Y_{2n(r+1)+m+2r+2} \cdot X_m
\end{aligned}$$

Case:(b)
If r is odd, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots \cdot X_n, n \geq 0$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots \cdot Y_n, n \geq 0$$

Using induction method, if $n = 0$, the result is true because

$$X_m \cdot Y_m = Y_m \cdot X_m$$

Assume that the result is true for some integer $n \geq 1$

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdots \cdot X_n, n \geq 0 \tag{19}$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdots \cdot Y_n, n \geq 0 \tag{20}$$

Now, we prove for $n + 1$

$$\begin{aligned}
X_{2n(r+1)+m+2r+2} \cdot Y_m &= \left[X_{2n(r+1)+m+2r+1} \cdot Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots \cdot X_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots \cdot Y_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+m+r} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots \cdot X_{2n(r+1)+m+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots \cdot X_{2n(r+1)+m}\right] \\
&\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots \cdot X_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot Y_m \\
&= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdots \cdot Y_{2n(r+1)+m+r+2}\right] \\
&\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdots \cdot X_{2n(r+1)+m+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdots \cdot X_{2n(r+1)+m+r+1}\right] \\
&\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdots \cdot X_{2n(r+1)+m+r+2}\right] \cdot \left[Y_m \cdot X_{2n(r+1)+m}\right]
\end{aligned}$$

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$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdot \dots \cdot Y_{2n(r+1)+m+r+2}\right]$$
$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdot \dots \cdot Y_{2n(r+1)+m+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot \dots \cdot X_{2n(r+1)+m+r+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdot \dots \cdot Y_{2n(r+1)+m+r+2}\right] \cdot \left[X_m \cdot Y_{2n(r+1)+m}\right]$$
$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdot \dots \cdot Y_{2n(r+1)+m+r+2}\right]$$
$$\cdot \left[X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdot \dots \cdot Y_{2n(r+1)+m+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+r-1} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot \dots \cdot X_{2n(r+1)+m+1} \cdot Y_{2n(r+1)+m}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdot \dots \cdot Y_{2n(r+1)+m+r+2}\right] \cdot X_m$$
$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdot \dots \cdot Y_{2n(r+1)+m+r+2}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+r} \cdot X_{2n(r+1)+m+r-1} \cdot Y_{2n(r+1)+m+r-2} \cdot X_{2n(r+1)+m+r-3} \cdot \dots \cdot Y_{2n(r+1)+m+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdot \dots \cdot Y_{2n(r+1)+m+r+2}\right] \cdot X_m$$
$$= \left[X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdot \dots \cdot Y_{2n(r+1)+m+r+1}\right]$$
$$\cdot \left[Y_{2n(r+1)+m+2r} \cdot X_{2n(r+1)+m+2r-1} \cdot \dots \cdot Y_{2n(r+1)+m+r+2} \cdot X_{2n(r+1)+m+r+1}\right] \cdot X_m$$
$$= \left[Y_{2n(r+1)+m+2r+1} \cdot X_{2n(r+1)+m+2r} \cdot Y_{2n(r+1)+m+2r-1} \cdot X_{2n(r+1)+m+2r-2} \cdot \dots \cdot Y_{2n(r+1)+m+r+2}\right] \cdot X_m$$
$$= Y_{2n(r+1)+m+2r+2} \cdot X_m$$

Hence proof. □

Theorem 4.4. For every integer $n \geq 0, \quad r \geq 0$

$$\prod_{i=1}^{i=n} X_{ri+1} \cdot Y_{ri+1} = \prod_{i=1}^{i=rn} Y_i \cdot X_i$$

Proof. Using induction method, for $n = 1$, the result is true because

$$\begin{aligned} X_{r+1} \cdot Y_{r+1} &= [Y_r \cdot Y_{r-1} \cdot Y_{r-2} \cdot \dots \cdot Y_1] \cdot [X_r \cdot X_{r-1} \cdot X_{r-2} \cdot \dots \cdot X_1] \\ &= [Y_r \cdot X_r] \cdot [Y_{r-1} \cdot X_{r-1}] \cdot [Y_{r-2} \cdot X_{r-2}] \cdot \dots \cdot [Y_1 \cdot X_1] \\ &= \prod_{i=1}^{i=r} Y_i \cdot X_i \end{aligned}$$

Assume that the result is true for some integer $n \geq 1$ (a) If r is an even, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdot \dots \cdot Y_n, n \geq 0 \quad (21)$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdot \dots \cdot X_n, n \geq 0 \quad (22)$$

(b) If r is an odd, then

$$X_{n+r} = X_{n+r-1} \cdot Y_{n+r-2} \cdot X_{n+r-3} \cdot \dots \cdot X_n, n \geq 0 \quad (23)$$
$$Y_{n+r} = Y_{n+r-1} \cdot X_{n+r-2} \cdot Y_{n+r-3} \cdot \dots \cdot Y_n, n \geq 0 \quad (24)$$

Now, we prove for $n+1$

$$\prod_{i=1}^{i=n+1} X_{ri+1} \cdot Y_{ri+1} = \prod_{i=1}^{i=n} [X_{ri+1} \cdot Y_{ri+1}] \cdot \left[X_{r(n+1)+1} \cdot X_{r(n+1)+1}\right]$$

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$$= \prod_{i=1}^{i=rn} [X_i \cdot Y_i] \cdot [X_{rn+r+1} \cdot Y_{rn+r+1}]$$
$$= \prod_{i=1}^{i=rn} [X_i \cdot Y_i] \cdot [Y_{rn+r} \cdot X_{rn+r}] \cdot [Y_{rn+r-1} \cdot X_{rn+r-1}] \cdot [Y_{rn+r-2} \cdot X_{rn+r-2}] \cdots \cdot [Y_{rn+1} \cdot X_{rn+1}]$$
$$= \prod_{i=1}^{i=rn+r} [Y_i \cdot X_i]$$

Hence proof. □

5 Conclusion

Identities of multiplicative coupled Fibonacci sequences of r^{th} order under $2^{r-1^{th}}$ scheme are described in this paper, this idea can be extended for other schemes and multiplicative coupled Fibonacci sequences of r^{th} order with negative integers.

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