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NEW OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE p-PREINVEX

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Abstaract − In this paper, making use of new an identity, we established new inequalities of Ostrowski's type for the class of p-preinvex functions and gave some midpoint type inequalities.

Keywords − *Preinvex functions, p-preinvex functions, Ostrowski type inequalities.*

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a mapping differentiable in I° and $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, for all $x \in [a, b]$, then the following inequality holds

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \tag{1}$$

for $x \in [a, b]$. This inequality is known in the literature as the Ostrowski inequality ([8]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at the point $x \in [a, b]$. For some results which generalize, improve and extend the inequality (1), we refer the reader to recent papers (see [9, 10]) and the references therein.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function, if

$$f(\lambda x + (1-\lambda)y)) \leq \lambda f(x) + m(1-\lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. We say that f is concave if $-f$ is convex.

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex function is that of invex functions introduced by Hanson in [4]. Weir and Mond [13] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. Pini [12] introduced the concept of prequasiinvex functions as a generalization of invex functions. Later, Mohan and Neogy [9] obtained some properties of generalized preinvex functions.

In [1], I. A. Baloch et. al. introduced the concept of the p-preinvex functions which is generalization of preinvex and harmonically preinvex functions. They also defined the notion of p-prequasiinvex function.

The aim of this paper is to establish some Ostrowski type inequalities for the functions whose derivative in absolute value are p-preinvex. Now, we recall some notions in invexity analysis which will be used through out the paper (see [2,8,14] and references therein).

Definition 1.2. A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}^n$, if for every $x, y \in S$ and $t \in [0, 1]$, we have

$$x + t\eta(y, x) \in S.$$

Note that definition of invex set has a clear geometric interpretation.This definition essentially says that there is a path starting from a point x which is contain in S.We do not require that the point y should be the one of the end points of path. This observation plays an important role in our analysis. Note that, if we demand that y should be an end point of the path for every pair of points, $x, y \in S$, then $\eta(y, x) = y - x$ and corresponding invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(y, x) = y - x$, but converse is not necessarily true, see [15],[18] and references therein.

Definition 1.3. Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $eta : S \times S \to \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η if for every $x, y \in S$ and $t \in [0, 1]$, we have

$$f(x + t\eta(y, x)) \leq tf(x) + (1 - t)f(y).$$

Note that every convex function is a preinvex function, but converse is not true (see [8]). For example, $f(x) = -|x|, x \in \mathbb{R}$, is not a convex function, but it is a preinvex function with respect to

$$\eta(x, y) = \begin{cases} x - y, & xy \geq 0 \\ y - x, & xy < 0 \end{cases}$$

We also need the following assumption regarding the function η which is due to Mohan and Neogy [9].

Condition C: Let $S \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : S \times S \to \mathbb{R}$. For any $x, y \in S$ and any $t \in [0, 1]$,

$$\eta(y, y + t\eta(y, x)) = -t\eta(y, x)$$

$$\eta(x, y + t\eta(y, x)) = (1 - t)\eta(y, x).$$

Note that for every $x, y \in S$ and $t_1, t_2 \in [0, 1]$, from Condition C, we have

$$\eta(y + t_2\eta(y, x), y + t_1\eta(y, x)) = (t_2 - t_1)\eta(y, x).$$

There are many vector functions that satisfy condition C (see [8]), besides the trivial case $\eta(x, y) = x - y$. For example, let $S = \mathbb{R}/\{0\}$ and

$$\eta(x,y) = \begin{cases} x-y, & x>0, y>0 \\ y-x, & x<0, y<0 \\ -y, & \text{otherwise.} \end{cases}$$

Then S is an invex set and η satisfies condition C.
In [3], İ.İscan established the Ostrowski type inequalities for the preinvex function as follow:

Theorem 1.4. Let $S \subset \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$ and $a, b \in S$ with $a < a + \eta(b, a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function and $|f'|$ is preinvex function on S. If f' is integrable on $[a, a + \eta(b, a)]$. Then the following inequality holds:

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \leq \frac{\eta(b,a)}{6}$$

$$\times \left\{ \left[3\left(\frac{x-a}{\eta(b,a)}\right)^2 - 2\left(\frac{x-a}{\eta(b,a)}\right)^3 + \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3 \right] |f'(a)| + \left[1 - 3\left(\frac{x-a}{\eta(b,a)}\right)^2 + 4\left(\frac{x-a}{\eta(b,a)}\right)^3 \right] |f'(b)| \right\} \quad (2)$$

for all $x \in [a, a + \eta(b, a)]$. The constant $\frac{1}{6}$ is best possible in the sense that cannot be replaced by a smaller value.

Theorem 1.5. Let $S \subseteq \mathbb{R}$ be an open invex set with respect to $eta : S \times S \to \mathbb{R}$ and $a, b \in S$ with $a < a + \eta(b, a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function such that $|f'|^q$ is preinvex on $[a, a + \eta(b, a)]$, for some fixed $q > 1$. If f' is integral on $[a, a + \eta(b, a)]$ and η satisfies condition C, then for each $x \in [a, a + \eta(b, a)]$, the following inequality holds

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \frac{(x-a)^2}{\eta(b,a)} \left(\frac{|f'(a)|^q + |f'(x)|^q}{2}\right)^{\frac{1}{q}} + \frac{\big(a+\eta(b,a)-x\big)^2}{\eta(b,a)} \left(\frac{|f'(a+\eta(b,a))|^q + |f'(x)|^q}{2}\right)^{\frac{1}{q}} \right\}, \quad (3)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.6. Let $S \subseteq \mathbb{R}$ be an open invex set with respect to $eta : S \times S \to \mathbb{R}$ and $a, b \in S$ with $a < a + \eta(b, a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function such that $|f'|^q$ is preinvex on $[a, a + \eta(b, a)]$, for some fixed $q \geq 1$. If f' is integral on $[a, a+\eta(b,a)]$ and η satisfies condition C, then for each $x \in [a, a+\eta(b,a)]$, the following inequality holds

$$\left|f(x) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du\right| \leq \eta(b,a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{x-a}{\eta(b,a)}\right)^{2(1-\frac{1}{q})}\right.$$
$$\times\left[\frac{(x-a)^2(3\eta(b,a)-2x+2a)}{6\eta^3(b,a)}|f'(a)|^q + \frac{1}{3}\left(\frac{x-a}{\eta(b,a)}\right)^3|f'(b)|^q\right]^{\frac{1}{q}} + \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{2(1-\frac{1}{q})}$$
$$\left.\times\left[\frac{1}{3}\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3|f'(a)|^q + \left(\frac{1}{6} + \frac{(x-a)^2(2x-3\eta(b,a)-2a)}{6\eta^3(b,a)}\right)|f'(b)|^q\right]^{\frac{1}{q}}\right\} \tag{4}$$

for each $x \in [a, a+\eta(b,a)]$.

Now, we recall the class of the p-preinvex functions [1] which is a generalization of preinvex functions, harmonically preinvex functions and also recall the class of p-prequasiinvex functions :

Definition 1.7. Let $p \in \mathbb{R}/\{0\}$.The set $A_{\eta,p} \subseteq (0,\infty)$is said to be p-invex with respect to $\eta(.,.)$, if for every $x, y \in A$ and $t \in [0,1]$, we have

$$[(1-t)x^p + t(x+\eta(y,x))^p]^{\frac{1}{p}} \in A.$$

The p-invex set $A_{\eta,p}$ is also call a (p,η)-connected set.

Remark 1.8. Note that for $p = 1$, p-invex set becomes invex set and for $p = -1$, p-invex set become to harmonic invex-set.

Definition 1.9. Let $p \in \mathbb{R}/\{0\}$. The function f on the p-invex set $A_{\eta,p}$ is said to be p-preinvex function with respect to η if, where $p \in \mathbb{R}/\{0\}$, , if

$$f\left([(1-t)x^p + t(x+\eta(y,x))^p]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y), \tag{5}$$

for all $x, y \in A_{\eta,p}$ and $t \in [0,1]$.

Remark 1.10. Note that for $p = 1$ p-preinvex functions becomes preinvex functions and for $p = -1$, p-preinvex functions become harmonically preinvex functions.

Theorem 1.11. [1] Let $f : S = [a, a+\eta(b,a)] \to (0,\infty)$ be a p-preinvex function on the interval S° and $a, b \in S^\circ$ with $a < a+\eta(b,a)$. Then the following inequality holds:

$$f\left(\left[\frac{a^p + (a+\eta(b,a))^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{[(a+\eta(b,a))^p - a^p]}\int_a^{a+\eta(b,a)} \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2}$$

Definition 1.12. Let $p \in \mathbb{R}/\{0\}$. The function f on the p-invex set $A_{\eta,p}$ is said to be p-prequasiinvex function with respect to η if, where $p \in \mathbb{R}/\{0\}$, , if

$$f\left([(1-t)x^p + t(x+\eta(y,x))^p]^{\frac{1}{p}}\right) \leq \max\{f(x), f(y)\}, \tag{6}$$

for all $x, y \in A_{\eta,p}$ and $t \in [0,1]$.

2 Main Results

Lemma 2.1. Let S be an open invex set with respect to η and $a, a+\eta(b,a) \in S$ with $a < a+\eta(b,a)$. Suppose that $f: S \to \mathbb{R}$ is differentiable function. If f' is integrable on $[a, a+\eta(b,a)]$ Then, we have following identity

$$f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du$$

$$\begin{aligned}
=\ & \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \\
& \times f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \\
+\ & \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (t-1)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \\
& \times f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \Bigg]
\end{aligned}$$

for all $x \in [a, a+\eta(b,a)]$ and $p \in \mathbb{R}/\{0\}$.

Proof. Let

$$\begin{aligned}
I_1 =\ & \frac{(a+\eta(b,a))^p - a^p}{p} \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \\
& \times f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \\
=\ & t f\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \Bigg|_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} \\
& - \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} f\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \\
=\ & \frac{x^p - a^p}{(a+\eta(b,a))^p - a^p} f(x) - \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} f\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) dt \\
=\ & \frac{x^p - a^p}{(a+\eta(b,a))^p - a^p} f(x) - \frac{p}{(a+\eta(b,a))^p - a^p} \int_a^x \frac{f(u)}{u^{1-p}} du,
\end{aligned}$$

and let

$$\begin{aligned}
I_2 &= \frac{(a+\eta(b,a))^p - a^p}{p} \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (t-1)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \\
&\quad \times f'\Big([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}}\Big) dt \\
&= (t-1) f\Big([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}}\Big)\Big|_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} \\
&\quad - \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} f\Big([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}}\Big) dt \\
&= \Big(1 - \frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}\Big) f(x) - \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} f\Big([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}}\Big) dt \\
&= \Big(1 - \frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}\Big) f(x) - \frac{p}{(a+\eta(b,a))^p - a^p} \int_x^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du.
\end{aligned}$$

Now, by adding I_1 and I_2, we get required result. □

Theorem 2.2. Let $S \subset \mathbb{R}$ be an invex set with respect to $\eta : S\times S \to \mathbb{R}$ and $a, b \in S$ with $a < a + \eta(b, a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function and $|f'|$ is p-preinvex function on S with $p = 2k + 1$ or $p = \frac{n}{m}$, $n = 2r + 1$, $m = 2t + 1$ where $k, r, t \in N$. If f' is integrable on $[a, a + \eta(b, a)]$. Then the following inequality holds:

$$\begin{aligned}
&\Big| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \Big| \\
&\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Big[(S_1 + S_3)|f'(a)| + (S_2 + S_4)|f'(b)| \Big], \qquad (7)
\end{aligned}$$

where

$$S_1 = \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t^2 [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} dt$$

$$S_2 = \int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t(1-t) [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} dt$$

$$S_3 = \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} t(1-t) [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} dt$$

$$S_4 = \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)(1-t) [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} dt$$

Proof. Using Lemma 2.1 and $|f'|$ is p-preinvex on S, we have

$$\Big| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \Big|$$

$$\leq \frac{(a+\eta(b,a))^p-a^p}{p}\Bigg[\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}$$
$$\times\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt$$
$$+\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^1 (t-1)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}$$
$$\times\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt\Bigg]$$

$$\leq \frac{(a+\eta(b,a))^p-a^p}{p}\Bigg[\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}$$
$$\times\Big(t|f'(a)|+(1-t)|f'(b)|\Big)dt$$

$$+\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^1 (t-1)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\Big(t|f'(a)|+(1-t)|f'(b)|\Big)dt\Bigg]$$
$$= \frac{(a+\eta(b,a))^p-a^p}{p}\Bigg[(S_1+S_3)|f'(a)|+(S_2+S_4)|f'(b)|\Bigg].$$

This completes the proof. □

Theorem 2.3. Let $S\subseteq\mathbb{R}$ be an open invex set with respect to $eta: S\times S\to\mathbb{R}$ and $a,b\in S$ with $a<a+\eta(b,a)$. Suppose that $f:S\to\mathbb{R}$ is a differentiable function such that $|f'|^q$ is p-preinvex on $[a,a+\eta(b,a)]$ with $p=2k+1$ or $p=\frac{n}{m}$, $n=2r+1$, $m=2t+1$ where $k,r,t\in N$, for some fixed $q>1$. If f' is integral on $[a,a+\eta(b,a)]$ and η satisfies condition C, then for each $x\in[a,a+\eta(b,a)]$, the following inequality holds

$$\left|f(x)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(u)}{u^{1-p}}du\right|$$
$$\leq \frac{[(a+\eta(b,a))^p-a^p]}{2^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}}\Bigg[\left(S_5|f(a)|^{\frac{q}{q-1}}+S_6|f(b)|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}$$
$$+\left(S_7|f(a)|^{\frac{q}{q-1}}+S_8|f(b)|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\Bigg],$$

where

$$S_5=\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$

$$S_6=\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} (1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$

$$S_7=\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^1 t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$

$$S_8 = \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} dt$$

Proof. Using Lemma 2.1, Holder's inequality and p-preinvexity of $|f'|^{\frac{q}{q-1}}$ on S, we have

$$\left| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \right|$$

$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}}$$

$$\times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right| dt$$

$$+ \int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \left| f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right| dt \Bigg]$$

$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t^q dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} [(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} \right.$$

$$\left. \times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right|^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}}$$

$$+ \left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)^q dt \right)^{\frac{1}{q}} \left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} \right.$$

$$\left. \times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right|^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \Bigg]$$

$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \Bigg[\left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t^q dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} [(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} \right.$$

$$\left. \times (t|f(a)|^{\frac{q}{q-1}} + (1-t)|f(b)|^{\frac{q}{q-1}}) dt \right)^{\frac{q-1}{q}}$$

$$+ \left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} (1-t)^q dt \right)^{\frac{1}{q}}$$

$$\times \left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1} [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}} (t|f(a)|^{\frac{q}{q-1}} + (1-t)|f(b)|^{\frac{q}{q-1}}) dt \right)^{\frac{q-1}{q}} \Bigg]$$

$$= \frac{[(a+\eta(b,a))^p - a^p]}{2^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}} \left[\left(S_5|f(a)|^{\frac{q}{q-1}} + S_6|f(b)|^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} + \left(S_7|f(a)|^{\frac{q}{q-1}} + S_8|f(b)|^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \right]$$

The proof is completed. □

Theorem 2.4. Let $S \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : S \times S \to \mathbb{R}$ and $a, b \in S$ with $a < a + \eta(b,a)$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function such that $|f'|^q$ is preinvex on $[a, a+\eta(b,a)]$ with $p = 2k+1$ or $p = \frac{n}{m}$, $n = 2r+1$, $m = 2t+1$ where $k, r, t \in N$, for some fixed $q \geq 1$. If f' is integral on $[a, a+\eta(b,a)]$ and η satisfies condition C, then for each $x \in [a, a+\eta(b,a)]$, the following inequality holds

$$\left| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \right|$$
$$\leq \frac{[(a+\eta(b,a))^p - a^p]}{2^{1+\frac{1}{r}}(r+1)^{\frac{1}{r}}} \left[\left(S_9|f(a)|^q + S_{10}|f(b)|^q \right)^{\frac{1}{q}} + \left(S_{11}|f(a)|^q + S_{12}|f(b)|^q \right)^{\frac{1}{q}} \right],$$

where

$$S_9 = \int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}} dt$$

$$S_{10} = \int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} (1-t)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}} dt$$

$$S_{11} = \int_{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}}^1 t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}} dt$$

$$S_{12} = \int_{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}}^1 (1-t)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}} dt$$

Proof. Using Lemma 2.1, Holder's inequality and p-preinvexity of $|f'|^q$ on S, we have

$$\left| f(x) - \frac{p}{[(a+\eta(b,a))^p - a^p]} \int_a^{a+\eta(b,a)} \frac{f(u)}{u^{1-p}} du \right|$$
$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \left[\int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} t[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \right.$$
$$\times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right| dt$$

$$\left. + \int_{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}}^1 (1-t)[(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1-p}{p}} \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right| dt \right]$$

$$\leq \frac{(a+\eta(b,a))^p - a^p}{p} \left[\left(\int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} t^r dt \right)^{\frac{1}{r}} \left(\int_0^{\frac{x^p - a^p}{(a+\eta(b,a))^p - a^p}} [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}} \right. \right.$$
$$\left. \times \left| f'\left([(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{1}{p}} \right) \right|^q dt \right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1}(1-t)^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}\right.$$

$$\left.\times\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(a+\eta(b,a))^p-a^p}{p}\left[\left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}} t^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}\right.\right.$$

$$\left.\times(t|f(a)|^q+(1-t)|f(b)|^q)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1}(1-t)^r dt\right)^{\frac{1}{r}}$$

$$\left.\times\left(\int_{\frac{x^p-a^p}{(a+\eta(b,a))^p-a^p}}^{1}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}(t|f(a)|^q+(1-t)|f(b)|^q)dt\right)^{\frac{1}{q}}\right]$$

$$=\frac{[(a+\eta(b,a))^p-a^p]}{2^{1+\frac{1}{r}}(r+1)^{\frac{1}{r}}}\left[\left(S_9|f(a)|^q+S_{10}|f(b)|^q\right)^{\frac{1}{q}}+\left(S_{11}|f(a)|^q+S_{12}|f(b)|^q\right)^{\frac{1}{q}}\right]$$

The proof is completed. □

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