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DOUBLE CONNECTED SPACES

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Abstaract − In this paper, we introduce new types of double connected topological spaces. The first one depends on the separated double sets and the other one depends on the quasi-coincident separated double sets. The properties and the relation between them have investigated. Also, we defined and study the component of each type and the properties of these types of component have obtained.

Keywords − *Separated double sets, q-separated double sets, double connected, q-double connected, DC_1-connected, strongly double connected, strongly q-double connected, double components, q-double components and q-hyperconnected in double topological spaces.*

1 Introduction

Atanassov [1, 2, 3, 4] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [5] generalized topological structures in intuitionistic fuzzy case. The concept of intuitionistic sets and the topology on intuitionistic sets was first given by Coker [7, 6].

In 2005, the suggestion of J. G. Garcia et al. [8] that double set is a more appropriate name than flou (intuitionistic) set, and double topology for the flou (intuitionistic) topology. In 2007, Kandil et al. [10] proved the $1-1$ correspondence mapping f between the set of all double sets and the set of all intuitionistic sets defined as: $f(A_1, A_2) = (A_1, A_2^c), A_2^c$ is the complement of A_2. Kandil et al. [9, 10] introduced the concept of double sets, (D-set, for short), double topological spaces,

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(DTS, for short), continuous functions between these spaces and the concept of double point, (D-point, for short).

In this paper, we define (q-)separated double sets, (q)double connected, DC_1-connected, strongly double connected, (q)double components and quasi-hyperconnected in double topological spaces. Moreover, we give some related results to these notions.

2 Preliminary

In this section, we collect some definitions and theorems which will be needed in the sequel. For more details see [9, 10].

Definition 2.1. [10] Let X be a non-empty set.

1. A D-set $\underline{A}$ is an ordered pair $(A_1, A_2) \in P(X)\hat{\times}P(X)$ such that $A_1 \subseteq A_2$.
2. $D(X) = \{(A_1, A_2) \in P(X)\hat{\times}P(X), A_1 \subseteq A_2\}$ is the family of all D-sets on X.
3. Let $\eta_1, \eta_2 \subseteq P(X)$. The product of η_1 and η_2, denoted by $\eta_1 \times \eta_2$, defined by: $\eta_1\hat{\times}\eta_2 = \{(A_1, A_2) : A_1 \in \eta_1, A_1 \in \eta_2, A_1 \subseteq A_2\}$.
4. The D-set $\underline{X} = (X, X)$ is called the universal D-set.
5. The D-set $\underline{\emptyset} = (\emptyset, \emptyset)$ is called the empty D-set.

Definition 2.2. [10] Let $\underline{A} = (A_1, A_2), \underline{B} = (B_1, B_2)$ and $\underline{C} = (C_1, C_2) \in D(X)$.

1. $\underline{A} = \underline{B} \Leftrightarrow A_1 = B_1, A_2 = B_2$.
2. $\underline{A} \subseteq \underline{B} \Leftrightarrow A_1 \subseteq B_1, A_2 \subseteq B_2$.
3. $\underline{A} \cup \underline{B} = (A_1 \cup B_1, A_2 \cup B_2)$.
4. $\underline{A} \cap \underline{B} = (A_1 \cap B_1, A_2 \cap B_2)$.
5. $\underline{A}^c = (A_2^c, A_1^c)$, where $\underline{A}^c$ is the complement of $\underline{A}$.
6. $\underline{A} \setminus \underline{B} = (A_1 \setminus B_2, A_2 \setminus B_1)$.
7. $\underline{A} \cap (\underline{B} \cup \underline{C}) = (\underline{A} \cap \underline{B})\cup(\underline{A} \cup \underline{C})$.
8. Let $x \in X$. Then, the D-sets $\underline{x}_1 = (\{x\}, \{x\})$ and $\underline{x}_{\frac{1}{2}} = (\emptyset, \{x\})$ are said to be D-points in X. The family of all D-points, denoted by $DP(X)$, i.e., $DP(X) = \{\underline{x}_t : x \in X, t \in \{\frac{1}{2}, 1\}\}$.
9. $\underline{x}_1 \underline{\in} \underline{A} \Leftrightarrow x \in A_1$ and $\underline{x}_{\frac{1}{2}} \underline{\in} \underline{A} \Leftrightarrow x \in A_2$.

Definition 2.3. [9] Two D-sets $\underline{A}$ and $\underline{B}$ are said to be a quasi-coincident, denoted by $\underline{A}q\underline{B}$, if $A_1 \cap B_2 \neq \emptyset$ or $A_2 \cap B_1 \neq \emptyset$. $\underline{A}$ is not quasi-coincident with $\underline{B}$, denoted by $\underline{A} \not{q}\underline{B}$, if $A_1 \cap B_2 = \emptyset$ and $A_2 \cap B_1 = \emptyset$.

Theorem 2.4. [9] Let $\underline{A}, \underline{B}, \underline{C} \in D(X)$ and $\underline{x}_t \in DP(X)$. Then,

1. $\underline{A} \not q \underline{B} \Leftrightarrow \underline{A} \subseteq \underline{B}^c$.
2. $\underline{A} \not q \underline{B}, \underline{C} \subseteq \underline{B} \Rightarrow \underline{A} \not q \underline{C}$.

Definition 2.5. [10] Let X be a non-empty set. The family η of D-sets in X is called a double topology on X if it satisfies the following axioms:

1. $\underline{\emptyset}, \underline{X} \in \eta$,
2. If $\underline{A}, \underline{B} \in \eta$, then $\underline{A} \sqcap \underline{B} \in \eta$,
3. If $\{\underline{A}_s : s \in S\} \subseteq \eta$, then $\sqcup_{s \in S} \underline{A}_s \in \eta$.

The pair (X, η) is called a DTS. Each element of η is called an open D-set in X. The complement of open D-set is called closed D-set.

Definition 2.6. [10] Let (X, η) be a DTS and $\underline{A} \in D(X)$. The double closure of $\underline{A}$, denoted by $cl_\eta(\underline{A})$ or $\overline{\underline{A}}$, defined by: $cl_\eta(\underline{A}) = \sqcap\{\underline{B} : \underline{B} \in \eta^c \ and \ \underline{A} \subseteq \underline{B}\}$.

Theorem 2.7. [10] Let (X, η) be a DTS and let $\underline{A}$, $\underline{B}$, $\underline{C} \in D(X)$. Then,

1. $cl_\eta(\underline{A})$ is the smallest closed D-set containing $\underline{A}$.
2. $cl_\eta(\underline{A} \sqcap \underline{B}) \subseteq cl_\eta(\underline{A}) \sqcap cl_\eta(\underline{B})$.
3. $\underline{A} \ q \ \underline{C} \Leftrightarrow cl_\eta(\underline{A}) \ q \ cl_\eta(\underline{C})$, $\underline{C} \in \eta$.

Definition 2.8. [10] Let X be a non-empty set. The family τ of D-sets in X is called a stratified double topology on X if it satisfies the following axioms:

1. $\underline{\emptyset} \in \tau, \underline{X} \in \tau \ and \ (\emptyset, X) \in \tau$,
2. If $\underline{A}, \underline{B} \in \tau$, then $\underline{A} \sqcap \underline{B} \in \tau$,
3. If $\{\underline{A}_s : s \in S\} \subseteq \tau$, then $\sqcup_{s \in S} \underline{A}_s \in \tau$.

The pair (X, τ) is called a stratified DTS.

Definition 2.9. [10] Let X be a non-empty set.

1. $I(X) = \{\underline{\emptyset}, \underline{X}\}$ is a DTS, which is called indiscrete DTS.
2. $i(X)\hat{\times}i(X) = \{\underline{\emptyset}, \underline{X}, (\emptyset, X)\}$ is a DTS, which is called indiscrete stratified DTS, $i(X)$ is the indiscrete topology on X.
3. $D(X) = P(X)\hat{\times}P(X)$ is a DTS, which is called discrete DTS.

Theorem 2.10. [10] Let η be a double topology on X. Then, the following collections are ordinary topologies on X :

1. $\pi_1 = \{A_1 : \underline{A} \in \eta\}$.

2. $\pi_2 = \{A_2 : \underline{A} \in \eta\}$.

Definition 2.11. [10] Let (X, η) be a DTS and Y be a non-empty subset of X. Then, $\eta_Y = \{\underline{A \bigcap Y} : \underline{A} \in \eta \; and \; \underline{Y} = (Y, Y)\}$ is a double topology on Y. The DTS (Y, η_Y) is called a double topological subspace of (X, η) ($DT-$subspace, for short).

Definition 2.12. [10] Let (X, η) be a DTS, $\underline{F} \in D(X)$ and Y be a non-empty subset of X. Then, the D-subset over Y, denoted by F^Y, defined by: $F^Y = \underline{F \bigcap Y}$.

Definition 2.13. [10] Consider two ordinary sets X and Y. Let f be a mapping from X into Y. The image of a D-set $\underline{A} \; in \; D(X)$ defined by: $f(\underline{A}) = (f(A_1), f(A_2))$. Also the inverse image of a D-set $\underline{B} \in D(Y)$ defined by: $f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(B_2))$.

Definition 2.14. [10] Let $f : X \to Y$ be a mapping and let (X, η) and (Y, η^*) be DTS. Then, f is called a D-continuous if $f^{-1}(\underline{B}) \in \eta$, whenever $\underline{B} \in \eta^*$.

Theorem 2.15. [10] Let (X, η) and (Y, η^*) be two DTS and let $f : X \to Y$ be a mapping, $\underline{A} \in D(X)$ and $\underline{B} \in D(Y)$. Then, the following conditions are equivalent:

1. f is a D-continuous,
2. $f^{-1}(\underline{B}) \in \eta^c$, $\forall \underline{B} \in \eta^{*c}$,
3. $f(cl_\eta(\underline{A})) \subseteq cl_{\eta^*}(f(\underline{A}))$, $\forall \underline{A} \in D(X)$,
4. $cl_\eta(f^{-1}(\underline{B})) \subseteq f^{-1}(cl_{\eta^*}(\underline{B}))$, $\forall \underline{B} \in D(Y)$,
5. $f^{-1}(int_{\eta^*}(\underline{B})) \subseteq int_\eta(f^{-1}(\underline{B}))$, $\forall \underline{B} \in D(Y)$.

Definition 2.16. [10] Let (X, η) and (Y, η^*) be two DTS and let $f : X \to Y$ be a mapping and $\underline{A} \in D(X)$.

1. f is called D-open if $f(\underline{A}) \in \eta^*$, $\forall \underline{A} \in \eta$.
2. f is called D-closed if $f(\underline{A}) \in \eta^{*c}$, $\forall \underline{A} \in \eta^c$

Theorem 2.17. [10] Let (X, η) and (Y, η^*) be two DTS and let $f : X \to Y$ be a mapping and $\underline{A} \in D(X)$. f is D-closed iff $cl_{\eta^*}(f(\underline{A})) \subseteq f(cl_\eta(\underline{A}))$, $\forall \underline{A} \in D(X)$.

Definition 2.18. [9] Let (X, η) be a DTS. and let $\underline{A} \in D(X)$. $\underline{A}$ is said to be:

1. D-dense if $cl_\eta(\underline{A}) = \underline{X}$.
2. D-nowhere dense if $int_\eta(cl_\eta(\underline{A})) = \underline{\emptyset}$.

3 Connectedness in DTS

Definition 3.1. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$.

1. $\underline{A}, \underline{B}$ are said to be separated double sets (separated D-sets, for short) if $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{\emptyset}$ and $\underline{A} \sqcap cl_\eta(\underline{B}) = \underline{\emptyset}$.

2. $\underline{A}, \underline{B}$ are said to be quasi-coincident separated double sets (q-separated D-sets, for short) if $cl_\eta(\underline{A}) \not q \underline{B}$ and $\underline{A} \not q\, cl_\eta(\underline{B})$.

Proposition 3.2. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$. Then, if $\underline{A}, \underline{B}$ are separated D-sets, then $\underline{A} \sqcap \underline{B} = \underline{\emptyset}$.

Proof. Suppose $\underline{A}, \underline{B}$ are separated D-sets, then $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{\emptyset}$ and $\underline{A} \sqcap cl_\eta(\underline{B}) = \underline{\emptyset}$. But $\underline{A} \subseteq cl_\eta(\underline{A})$, then $\underline{A} \sqcap \underline{B} = \underline{\emptyset}$. Hence, the result.

The following example shows that the converse of Proposition 3.2 is not true in general.

Example 3.3. Let $X = \{a, b, c, d\}$, $\eta = \{\underline{\emptyset}, \underline{X}, (\{a\}, \{a, b\}), (\emptyset, \{a, b\}), (\{b, c, d\}, X)\}$ and $\eta^c = \{\underline{\emptyset}, \underline{X}, (\{c, d\}, \{b, c, d\}), (\{c, d\}, X), (\emptyset, \{a\})\}$. Then, (X, η) is a DTS. Now, $(\{a\}, \{a, b\}) \sqcap (\{c\}, \{c, d\}) = \underline{\emptyset}$, but $\underline{X} = cl_\eta(\{a\}, \{a, b\}) \sqcap (\{c\}, \{c, d\}) = (\{c\}, \{c, d\}) \neq \underline{\emptyset}$.

Proposition 3.4. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$. Then, if $\underline{A}, \underline{B}$ are q-separated D-sets, then $\underline{A} \not q \underline{B}$.

Proof. Suppose $\underline{A}, \underline{B}$ are q-separated D-sets, then $cl_\eta(\underline{A}) \not q\, \underline{B}$ and $\underline{A} \not q\, cl_\eta(\underline{B})$. But $\underline{A} \subseteq cl_\eta(\underline{A})$, then $\underline{A} \not q\, \underline{B}$. Hence, the result.

The following example shows that the converse of Proposition 3.4 is not true in general.

Example 3.5. Let $X = \{a, b, c\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{a, b\})\}$. Then, $\exists (\{a\}, \{a\}), (\{b\}, \{b\}) \in D(X)$ such that $(\{a\}, \{a\}) \not q\, (\{b\}, \{b\})$. But, $cl_\eta(\{a\}, \{a\}) = (\{a, c\}, X)\ q\ (\{b\}, \{b\})$ and $cl_\eta(\{b\}, \{b\}) = (\{b, c\}, X)\ q\ (\{a\}, \{a\})$.

Proposition 3.6. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$. Then, if $\underline{A} \sqcap \underline{B} = \underline{\emptyset}$, then $\underline{A} \not q\, \underline{B}$.

Proof. Suppose that $(A_1, A_2) = \underline{A} \sqcap \underline{B} = (B_1, B_2) = \underline{\emptyset}$, then $A_1 \cap B_1 = \emptyset$ and $A_2 \cap B_2 = \emptyset$, but $A_1 \subseteq A_2, B_1 \subseteq B_2$, then $A_1 \cap B_2 = \emptyset$ and $A_2 \cap B_1 = \emptyset$. Therefore, $\underline{A} \not q\, \underline{B}$.

The following example shows that the converse of Proposition 3.6 is not true in general.

Example 3.7. Let $X = \{a, b\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\{a\}, \{a\}), (\emptyset, \{b\}), (\{a\}, X)\}$. Then, $(\emptyset, \{b\}) \not q (\{a\}, X)$, but $(\emptyset, \{b\}) \sqcap (\{a\}, X) = (\emptyset, \{b\}) \neq \underline{\emptyset}$.

Proposition 3.8. Let (X, η) be a DTS and let $\underline{A}, \underline{B} \in D(X)$. Then, if $\underline{A}, \underline{B}$ are separated D-sets, then $\underline{A}, \underline{B}$ are q-separated D-sets.

Proof. Straightforward.

The following example shows that the converse of Proposition 3.8 is not true in general.

Example 3.9. In Example 3.7, we see that: $(\emptyset, \{b\}) \not q\, cl_\eta(\{a\}, X) = (\{a\}, X)$ and $cl_\eta(\emptyset, \{b\}) = (\emptyset, \{b\}) \not q\, (\{a\}, X)$, but $(\emptyset, \{b\}) = cl_\eta(\emptyset, \{b\}) \sqcap (\{a\}, X) = (\emptyset, \{b\}) \neq \underline{\emptyset}$.

Remark 3.10. $\underline{A} \sqcap \underline{B} = \underline{\emptyset} \not\Leftrightarrow cl_\eta(\underline{A}) \not q\, \underline{B}$ and $\underline{A} \not q\, cl_\eta(\underline{B})$.

Example 3.11.

1. In Example 3.3, we see that: $(\{a\}, \{a, b\}) \sqcap (\{c\}, \{c, d\}) = \underline{\emptyset}$, but $\underline{X} = cl_\eta(\{a\}, \{a, b\})\ q\ (\{c\}, \{c, d\})$.
2. In Example 3.7, we see that: $(\emptyset, \{b\}) \not q\, cl_\eta(\{a\}, X) = (\{a\}, X)$ and $cl_\eta(\emptyset, \{b\}) = (\emptyset, \{b\}) \not q\, (\{a\}, X)$, but $(\emptyset, \{b\}) \sqcap (\{a\}, X) = (\emptyset, \{b\}) \neq \underline{\emptyset}$.

Theorem 3.12. Let (X, η) be a DTS and let $\underline{A}, \underline{B}, \underline{C}, \underline{D} \in D(X)$ such that $\underline{C} \sqsubseteq \underline{A}$ and $\underline{D} \sqsubseteq \underline{B}$. Then, if $\underline{A}, \underline{B}$ are separated D-sets, then $\underline{C}, \underline{D}$ are separated D-sets.

Proof. Suppose $\underline{A}, \underline{B}$ are separated D-sets, then $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{\emptyset}$ and $\underline{A} \sqcap cl_\eta(\underline{B}) = \underline{\emptyset}$. Since $\underline{C} \sqsubseteq \underline{A}$ and $\underline{D} \sqsubseteq \underline{B}$, then $cl_\eta(\underline{C}) \sqsubseteq cl_\eta(\underline{A})$ and $cl_\eta(\underline{D}) \sqsubseteq cl_\eta(\underline{B})$. Implies, $\underline{D} \sqcap cl_\eta(\underline{C}) = \underline{\emptyset}$ and $cl_\eta(\underline{D}) \sqcap \underline{C} = \underline{\emptyset}$. Hence, $\underline{C}, \underline{D}$ are separated D-sets.

Theorem 3.13. Let (X, η) be a DTS and let $\underline{A}, \underline{B}, \underline{C}, \underline{D} \in D(X)$ such that $\underline{C} \sqsubseteq \underline{A}$ and $\underline{D} \sqsubseteq \underline{B}$. Then, if $\underline{A}, \underline{B}$ are q-separated D-sets, then $\underline{C}, \underline{D}$ are q-separated D-sets.

Proof. Suppose $\underline{A}, \underline{B}$ are q-separated D-sets, then $cl_\eta(\underline{A}) \not q\, \underline{B}$ and $\underline{A} \not q\, cl_\eta(\underline{B})$. Since $\underline{C} \sqsubseteq \underline{A}$ and $\underline{D} \sqsubseteq \underline{B}$, then $cl_\eta(\underline{C}) \sqsubseteq cl_\eta(\underline{A})$ and $cl_\eta(\underline{D}) \sqsubseteq cl_\eta(\underline{B})$. Implies, $\underline{D} \not q\, cl_\eta(\underline{C})$ and $\underline{C} \not q\, cl_\eta(\underline{D})$ [by Theorem 2.4]. Hence, $\underline{C}, \underline{D}$ are q-separated D-sets.

Definition 3.14. Let (X, η) be a DTS, and let E be a nonempty subset of X.

1. If there exist two non-empty separated D-sets $\underline{A}, \underline{B} \in D(X)$ such that $\underline{A} \sqcup \underline{B} = \underline{E}$, then the D-sets $\underline{A}$ and $\underline{B}$ form a D-separation of E and it is said to be double disconnected set (D-disconnected set, for short). Otherwise, E is said to be double connected set (D-connected set, for short).
2. If there exist two non-empty q-separated D-sets $\underline{A}, \underline{B} \in D(X)$ such that $\underline{A} \sqcup \underline{B} = \underline{E}$, then the D-sets $\underline{A}$ and $\underline{B}$ form a $qD-$separation of E and it is said to be quasi-coincident double disconnected set ($qD-$disconnected set, for short). Otherwise, E is said to be quasi-coincident double connected set ($qD-$connected set, for short).

Remark 3.15. The D-point $\underline{x}_t$ in any DTS (X, η) is a $qD-$connected set, provided that $\underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$.

Example 3.16. Let $X = \{a, b\}$, $\eta = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{a\}), (\{b\}, X)\}$ and $\eta^c = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{a\}), (\{b\}, X)\}$. Then,

1. $(\emptyset, \{a\}) = cl(\emptyset, \{a\}) \not q\, (\emptyset, \{a\})$ and $(\emptyset, \{a\}) \sqcup (\emptyset, \{a\}) = (\emptyset, \{a\})$. Therefore, $(\emptyset, \{a\})$ is not $qD-$connected set.

2. $(\{b\}, X) = cl(\emptyset, \{b\})\ q\ (\emptyset, \{b\})$ and $(\emptyset, \{b\})\underline{\cup}(\emptyset, \{b\}) = (\emptyset, \{b\})$. Therefore, $(\emptyset, \{b\})$ is $qD-$connected set.

Definition 3.17. Let (X, η) be a DTS.

1. If there exist two non-empty separated D-sets $\underline{A}, \underline{B} \in D(X)$ such that $\underline{A} \cup \underline{B} = \underline{X}$, then $\underline{A}$ and $\underline{B}$ are said to be double division (D-division, for short) for DTS (X, η). (X, η) is said to be double disconnected space (D-disconnected space, for short), if (X, η) has a D-division. Otherwise, (X, η) is said to be double connected space (D-connected space, for short).
2. If there exist two non-empty q-separated D-sets $\underline{A}, \underline{B} \in D(X)$ such that $\underline{A} \cup \underline{B} = \underline{X}$, then $\underline{A}$ and $\underline{B}$ are said to be a double quasi division ($qD-$division, for short) for DTS (X, η). (X, η) is said to be quasi-coincident double disconnected space ($qD-$disconnected space, for short), if (X, η) has a $qD-$division. Otherwise, (X, η) is said to be quasi-coincident double connected space ($qD-$connected space, for short).

Corollary 3.18. 1. Each indiscrete (stratified) DTS is D-connected ($qD-$connected).

2. Each discrete DTS is D-disconnected ($qD-$disconnected).

Proof. 1. It is obvious.

2. Suppose that (X, η) is a discrete DTS, then $\underline{x}_1 \in DP(X)$. Implies, $\underline{X} = \underline{x}_1 \cup \underline{x}_1^c$, $\underline{x}_1 = cl_\eta(\underline{x}_1) \cap \underline{x}_1^c = \emptyset$ and $\underline{x}_1^c = cl_\eta(\underline{x}_1^c) \cap \underline{x}_1 = \underline{\emptyset}$. Therefore, (X, η) is a D-disconnected. Similarly, (X, η) is a $qD-$disconnected.

Theorem 3.19. Let (X, η) be a DTS. Then, the following are equivalent:

1. (X, η) has a D-division,
2. There exist two disjoint closed D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A} \cup \underline{B} = \underline{X}$,
3. There exist two disjoint open D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A} \cup \underline{B} = \underline{X}$.

Proof. $(1 \rightarrow 2)$ Suppose that (X, η) has a D-division $\underline{A}$ and $\underline{B}$, then $\underline{A} \cup \underline{B} = \underline{X}$ and $cl_\eta(\underline{A}) \cap \underline{B} = \underline{\emptyset}$. Implies, $cl_\eta(\underline{A}) \subseteq \underline{B}^c = \underline{X} \setminus \underline{B} \subseteq \underline{A}$, but $\underline{A} \subseteq cl_\eta(\underline{A})$, so that $\underline{A} = cl_\eta(\underline{A})$. Therefore, $\underline{A}$ is a closed D-set. Similarly, we can see that $\underline{B}$ is also a closed D-set. Since $\underline{A} = cl_\eta(\underline{A}), cl_\eta(\underline{A}) \cap \underline{B} = \underline{\emptyset}$, then $\underline{A} \cap \underline{B} = \underline{\emptyset}$. Hence, the result.
$(2 \rightarrow 3)$ Suppose that (X, η) has a D-division $\underline{A}$ and $\underline{B}$ such that $\underline{A}$ and $\underline{B}$ are closed D-sets, then $\underline{A}^c$ and $\underline{B}^c$ are open D-sets, $\underline{A} = \underline{B}^c$ and $\underline{B} = \underline{A}^c$. Therefore, $\underline{A}^c \cup \underline{B}^c = \underline{X}$ and $\underline{A}^c \cap \underline{B}^c = \underline{\emptyset}$. Hence, the result.
$(3 \rightarrow 1)$ Since $\underline{X} = \underline{A} \cup \underline{B}$ such that $\underline{A} \cap \underline{B} = \underline{\emptyset}$ and $\underline{A}, \underline{B}$ are open D-sets, then $\underline{A}^c$ and $\underline{B}^c$ are closed D-sets, $\underline{A} = \underline{B}^c$ and $\underline{B} = \underline{A}^c$. This implies that, $\underline{A} = cl_\eta(\underline{A})$. Therefore, $cl_\eta(\underline{A}) \cap \underline{B} = \underline{\emptyset}$. Similarly, we have $\underline{A} \cap cl_\eta(\underline{B}) = \underline{\emptyset}$. Hence, (X, η) has a D-division.

Theorem 3.20. Let (X, η) be a DTS. Then, the following are equivalent:

1. (X, η) has a $qD-$division,

2. There exist two non quasi-coincident closed D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A} \cup \underline{B} = \underline{X}$,

3. There exist two non quasi-coincident open D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A} \cup \underline{B} = \underline{X}$.

Proof. $(1 \to 2)$ Suppose that (X, η) has a $qD-$division $\underline{A}$ and $\underline{B}$, then $\underline{A} \cup \underline{B} = \underline{X}$ and $cl_\eta(\underline{A}) \not{q} \underline{B}$, Implies, $cl_\eta(\underline{A}) \subseteq \underline{B}^c = \underline{X} \setminus \underline{B} \subseteq \underline{A}$. but $\underline{A} \subseteq cl_\eta(\underline{A})$, so that $\underline{A} = cl_\eta(\underline{A})$. Therefore, $\underline{A}$ is a closed D-set. Similarly, we can see that $\underline{B}$ is also a closed D-set and $\underline{A} \not{q} \underline{B}$ [by theorem 3.4]. Hence, the result.

$(2 \to 3)$ Suppose that (X, η) has a $qD-$division $\underline{A}$ and $\underline{B}$ such that $\underline{A}$ and $\underline{B}$ are closed D-sets, then $\underline{A}^c$ and $\underline{B}^c$ are open D-sets, $\underline{A} = \underline{B}^c$ and $\underline{B} = \underline{A}^c$. Therefore, $\underline{A}^c \cup \underline{B}^c = \underline{X}$ and $\underline{A}^c \not{q} \underline{B}^c$. Hence, the result.

$(3 \to 1)$ Since $\underline{X} = \underline{A} \cup \underline{B}$ such that $\underline{A} \not{q} \underline{B}$ and $\underline{A}, \underline{B}$ are open D-sets, then $\underline{A} \subseteq \underline{X} \setminus \underline{B}$ This implies that, $cl_\eta(\underline{A}) \subseteq \underline{X} \setminus \underline{B}$. Therefore, $cl_\eta(\underline{A}) \not{q} \underline{B}$. Similarly, we have $\underline{A} \not{q}\, cl_\eta(\underline{B})$. Hence, (X, η) has a $qD-$division.

Theorem 3.21. Let (X, η) be a DTS. Then, the following are equivalent:

1. (X, η) is D-connected,

2. $\underline{X}$ cannot be written as the union of two disjoint non-empty closed D-subsets,

3. $\underline{X}$ cannot be written as the union of two disjoint non-empty open D-subsets.

Proof. It follows from Theorem 3.19.

Theorem 3.22. Let (X, η) be a DTS. Then, the following are equivalent:

1. (X, η) is $qD-$connected,

2. $\underline{X}$ cannot be written as the union of two non quasi-coincident closed D-subsets,

3. $\underline{X}$ cannot be written as the union of two non quasi-coincident open D-subsets.

Proof. It follows from Theorem 3.20.

Theorem 3.23. Let (X, η) be a DTS and let Y be a non-empty subset of X. Then, if $\underline{A}$ and $\underline{B}$ are D-sets in Y, then $\underline{A}$ and $\underline{B}$ are separated D-sets in Y if and only if $\underline{A}$ and $\underline{B}$ are separated D-sets in X.

Proof. $cl_\eta(\underline{A}) \sqcap \underline{B} = \underline{Y} \sqcap cl_\eta(\underline{A}) \sqcap \underline{B},\ \underline{B} \subseteq \underline{Y}$
$= \underline{Y} \sqcap \underline{B} \sqcap cl_\eta(\underline{A})$
$= \underline{B} \sqcap \underline{Y} \sqcap cl_\eta(\underline{A})$
$= \underline{B} \sqcap cl_{\eta_Y}(\underline{A})$
$= \underline{\emptyset}$.

Similarly, we have: $cl_\eta(\underline{B}) \sqcap \underline{A} = \underline{\emptyset}$.

Conversely, $cl_{\eta_Y}(\underline{A}) \cap \underline{B} = \underline{Y} \cap cl_{\eta}(\underline{A}) \cap \underline{B} = \underline{Y} \cap (cl_{\eta}(\underline{A}) \cap \underline{B}) = \underline{Y} \cap \underline{\emptyset} = \underline{\emptyset}$.

Similarly, we have: $cl_{\eta_Y}(\underline{B}) \cap \underline{A} = \underline{\emptyset}$. Hence, the result.

Theorem 3.24. Let (X, η) be a DTS and let Y be a non-empty subset of X. Then, if $\underline{A}$ and $\underline{B}$ are D-sets in Y, then $\underline{A}$ and $\underline{B}$ are q-separated D-sets in Y if $\underline{A}$ and $\underline{B}$ are q-separated D-sets in X.

Proof. Suppose that $\underline{A}$ and $\underline{B}$ are $qSD-$sets in Y, then $cl_{\eta}(\underline{A}) \not q \underline{B}$ and $cl_{\eta}(\underline{B}) \not q \underline{A}$. So that, $cl_{\eta}(\underline{A}) \subseteq \underline{B}^c$. Thus, $\underline{Y} \cap cl_{\eta}(\underline{A}) \subseteq \underline{Y} \cap \underline{B}^c$. It follows that, $cl_{\eta_Y}(\underline{A}) \subseteq \underline{B}^c$.

Therefore, $cl_{\eta_Y}(\underline{A}) \not q \underline{B}$.

Similarly, we have: $cl_{\eta_Y}(\underline{B}) \not q \underline{A}$. Hence, the result.

Lemma 3.25. Let (Y, η_Y) be a $DT-$subspace of a DTS (X, η). Then, if (Y, η_Y) is a D-connected, then for every pair $\underline{A}$ and $\underline{B}$ of a separated D-subsets of $\underline{X}$ such that $\underline{Y} = \underline{A} \cup \underline{B}$, we have either $\underline{A} = \underline{\emptyset}$ or $\underline{B} = \underline{\emptyset}$.

Proof. Let $\underline{A} \neq \underline{\emptyset} \neq \underline{B}$ and $\underline{Y} = \underline{A} \cup \underline{B}$. Since $\underline{A}, \underline{B} \subseteq \underline{Y}$ and separated D-sets in X, then they are separated D-sets in Y [by Theorem 3.23]. This implies that, (Y, η_Y) is D-disconnected, which a contradiction. Hence, the result.

Lemma 3.26. Let (Y, η_Y) be a $DT-$subspace of a DTS (X, η). Then, if (Y, η_Y) is a $qD-$connected, then for every pair $\underline{A}$ and $\underline{B}$ of a q-separated D-subsets of $\underline{X}$ such that $\underline{Y} = \underline{A} \cup \underline{B}$, we have either $\underline{A} = \underline{\emptyset}$ or $\underline{B} = \underline{\emptyset}$.

Proof. Let $\underline{A} \neq \underline{\emptyset} \neq \underline{B}$ and $\underline{Y} = \underline{A} \cup \underline{B}$. Since $\underline{A}, \underline{B} \subseteq \underline{Y}$ and q-separated D-sets in X, then they are q-separated D-sets in Y [by Theorem 3.24] This implies that, (Y, η_Y) is $qD-$disconnected, which a contradiction. Hence, the result.

Theorem 3.27. Let (X, η) be a DTS and let Y be a non-empty subset of X such that (Y, η_Y) is D-connected. Then, if $\underline{A}$ and $\underline{B}$ are separated D-subsets of X such that $\underline{Y} \subseteq \underline{A} \cup \underline{B}$, then $\underline{Y} \subseteq \underline{A}$ or $\underline{Y} \subseteq \underline{B}$.

Proof. Since $\underline{Y} \subseteq \underline{A} \cup \underline{B}$, then $\underline{Y} = \underline{Y} \cap (\underline{A} \cup \underline{B}) = (\underline{Y} \cap \underline{A}) \cup (\underline{Y} \cap \underline{B})$. By Theorem 3.23, $\underline{Y} \cap \underline{A}$ and $\underline{Y} \cap \underline{B}$ are separated D-sets of Y. Since (Y, η_Y) is D-connected, then $\underline{Y} \cap \underline{A} = \underline{\emptyset}$ or $\underline{Y} \cap \underline{B} = \underline{\emptyset}$ [by Lemma 3.25]. Therefore, $\underline{Y} \subseteq \underline{A}$ or $\underline{Y} \subseteq \underline{B}$.

Theorem 3.28. Let (X, η) be a DTS and let Y be a non-empty subset of X such that (Y, η_Y) is $qD-$connected. Then, if $\underline{A}$ and $\underline{B}$ are q-separated D-subsets of X such that $\underline{Y} \subseteq \underline{A} \cup \underline{B}$, then $\underline{Y} \subseteq \underline{A}$ or $\underline{Y} \subseteq \underline{B}$.

Proof. Since $\underline{Y} \subseteq \underline{A} \cup \underline{B}$, then $\underline{Y} = \underline{Y} \cap (\underline{A} \cup \underline{B}) = (\underline{Y} \cap \underline{A}) \cup (\underline{Y} \cap \underline{B})$. By Theorem 3.24 and Theorem 2.4, $\underline{Y} \cap \underline{A}$ and $\underline{Y} \cap \underline{B}$ are q-separated D-sets of Y. Since (Y, η_Y) is $qD-$connected, then $\underline{Y} \cap \underline{A} = \underline{\emptyset}$ or $\underline{Y} \cap \underline{B} = \underline{\emptyset}$ [by Lemma 3.26]. Therefore, $\underline{Y} \subseteq \underline{A}$ or $\underline{Y} \subseteq \underline{B}$.

Theorem 3.29. Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty D-connected subspaces of DTS (X, η). Then, if $\bigcap_{\alpha \in J} X_\alpha \neq \underline{\emptyset}$, then $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is D-connected subspace of a (X, η).

Proof. Let $Y = \bigcup_{\alpha \in J} X_\alpha$. Choose a D-point $\underline{x_t \in Y}$. Let $\underline{A}$ and $\underline{B}$ be D-division of $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$. Then, $\underline{x_t \in A}$ or $\underline{x_t \in B}$ without loss of generality, we may assume that $\underline{x_t \in A}$. For each $\alpha \in J$, since $(X_\alpha, \eta_{X_\alpha})$ is D-connected. It follows from Theorem 3.27 that, $\underline{X_\alpha} \subseteq \underline{A}$ or $\underline{X_\alpha} \subseteq \underline{B}$. Therefore, we have $\underline{Y} \subseteq \underline{A}$, since $\underline{x_t \in A}$, and then $\underline{B} = \underline{\emptyset}$, which a contradiction. Hence, $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is a D-connected subspace of the DTS (X, η).

Theorem 3.30. Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty $qD-$connected subspaces of DTS (X, η). Then, if $\bigcap_{\alpha \in J} X_\alpha \neq \underline{\emptyset}$, then $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is qD-connected subspace of a (X, η).

Proof. Straightforward.

Theorem 3.31. Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty D-connected subspaces of DTS (X, η). Then, if $X_\alpha \bigcap X_\beta \neq \emptyset$ for arbitrary $\alpha, \beta \in J$, then $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is D-connected subspace of a (X, η).

Proof. Fix an $\alpha_o \in J$. For arbitrary $\beta \in J$, put $A_\beta = X_{\alpha_o} \bigcup X_\beta$. By Theorem 3.29, each $(A_\beta, \eta_{A_\beta})$ is D-connected. Then, $\{(A_\beta, \eta_{A_\beta}) : \beta \in J\}$ is a family non-empty D-connected subspaces of DTS (X, η) and $\bigcap_{\beta \in J} A_\beta = X_{\alpha_o} \neq \emptyset$. Obvious, we have $\bigcup_{\alpha \in J} X_\alpha = \bigcup_{\beta \in J} A_\beta$. It follows from Theorem 3.29 that, $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is D-connected subspace of the DTS (X, η).

Theorem 3.32. Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty $qD-$connected subspaces of DTS (X, η). Then, if $X_\alpha \bigcap X_\beta \neq \emptyset$ for arbitrary $\alpha, \beta \in J$, then $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is $qD-$connected subspace of a (X, η).

Proof. Straightforward.

Theorem 3.33. Let (X, η) be a DTS and let Y be a non-empty subset of X such that (Y, η_Y) is D-connected. Then, if $\underline{Y} \subseteq \underline{A} \subseteq cl_\eta(\underline{Y})$, then (A, η_A) is a D-connected subspace of (X, η). In particular, $(cl_\eta(\underline{Y}), \eta_{cl_\eta(\underline{Y})})$ is a D-connected subspace of (X, η).

Proof. Suppose that (A, η_A) is a D-disconnected subspace of (X, η), then $\underline{A}$ has a D-separation $\underline{F}$ and $\underline{G}$ Implies, $\underline{Y} \subseteq \underline{F}$ or $\underline{Y} \subseteq \underline{G}$ [by Theorem 3.27]. Without loss of generality, we may assume that $\underline{Y} \subseteq \underline{F}$, so $cl_\eta(\underline{Y}) \subseteq cl_\eta(\underline{F}), cl_\eta(\underline{F}) \sqcap \underline{G} = \underline{\emptyset}$. Thus, $cl_\eta(\underline{Y}) \sqcap \underline{G} = \underline{\emptyset}$. Otherwise, $\underline{G} \subseteq \underline{A} \subseteq cl_\eta(\underline{Y})$. Therefore, $(cl_\eta(\underline{Y}))^c \sqcap \underline{G} = \underline{\emptyset}$, which a contradiction with $cl_\eta(\underline{Y}) \sqcap \underline{G} = \underline{\emptyset}$. Also, $\underline{Y} \subseteq cl_\eta(\underline{Y}) \subseteq cl_\eta(\underline{Y})$. This complete the proof.

Theorem 3.34. Let (X, η) be a DTS and let Y be a non-empty subset of X such that (Y, η_Y) is $qD-$connected. Then, if $\underline{Y} \subseteq \underline{A} \subseteq cl_\eta(\underline{Y})$, then (A, η_A) is a $qD-$connected subspace of (X, η). In particular, $(cl_\eta(\underline{Y}), \eta_{cl_\eta(\underline{Y})})$ is a $qD-$connected subspace of (X, η).

Proof. Suppose that (A, η_A) is a $qD-$disconnected subspace of (X, η), then $\underline{A}$ has a $qD-$separation $\underline{F}$ and $\underline{G}$. Implies, $\underline{Y} \subseteq \underline{F}$ or $\underline{Y} \subseteq \underline{G}$ [by Theorem 3.28]. Without loss of generality, we may assume that $\underline{Y} \subseteq \underline{F}$, so $cl_\eta(\underline{Y}) \subseteq cl_\eta(\underline{F}), cl_\eta(\underline{F}) \not{q} \underline{G}$. Thus $cl_\eta(\underline{Y}) \not{q} \underline{G}$. Otherwise, $\underline{G} \subseteq \underline{A} \subseteq cl_\eta(\underline{Y})$, Therefore, $(cl_\eta(\underline{Y}))^c \not{q} \underline{G}$, which a contradiction with $cl_\eta(\underline{Y}) \not{q} \underline{G}$. Also, $\underline{Y} \subseteq cl_\eta(\underline{Y}) \subseteq cl_\eta(\underline{Y})$. This complete the proof.

Theorem 3.35. The image of D-connected under a D-continuous map are *D*-connected.

Proof. Let (X, η) and (Y, τ) be two DTS, where (X, η) is a D-connected and let f be a D-continuous from X onto Y.
Suppose that (Y, τ) is a D-disconnected space, then $\exists\ \underline{A}, \underline{B} \in \tau$ such that $\underline{A} \cap \underline{B} = \underline{\emptyset}$ and $\underline{A} \cup \underline{B} = \underline{Y}$. Implies $\underline{A} \subseteq \underline{B}^c$, thus $f^{-1}(\underline{A}) \subseteq f^{-1}(\underline{B}^c) = (f^{-1}(\underline{B}))^c$. So that, $f^{-1}(\underline{A}) \cap f^{-1}(\underline{B}) = \underline{\emptyset}$ and $f^{-1}(\underline{A} \cup \underline{B}) = f^{-1}(\underline{Y})$. It follows that $f^{-1}(\underline{A}) \cup f^{-1}(\underline{B}) = \underline{X}$, i.e., (X, η) is a D-disconnected, which a contradiction. Hence, (Y, τ) is a D-connected.

Theorem 3.36. The image of $qD-$connected under a D-continuous map are qD-connected.

Proof. Let (X, η) and (Y, τ) be two DTS, where (X, η) is a $qD-$connected and let f be a D-continuous from X onto Y.

Suppose that (Y, τ) is a $qD-$disconnected space, then $\exists\ \underline{A}, \underline{B} \in \tau$ such that $\underline{A} \not\mathrel{q} \underline{B}$ and $\underline{A} \cup \underline{B} = \underline{Y}$. Implies, $\underline{A} \subseteq \underline{B}^c$, thus $f^{-1}(\underline{A}) \subseteq f^{-1}(\underline{B}^c) = (f^{-1}(\underline{B}))^c$. So that, $f^{-1}(\underline{A}) \not\mathrel{q} f^{-1}(\underline{B})$ and $f^{-1}(\underline{A} \cup \underline{B}) = f^{-1}(\underline{Y})$. It follows that $f^{-1}(\underline{A}) \cup f^{-1}(\underline{B}) = \underline{X}$, i.e., (X, η) is a $qD-$disconnected, which a contradiction. Hence, (Y, τ) is a $qD-$connected.

Theorem 3.37. Let (X, η) be a DTS. Then, if (X, η) is a D-disconnected space, then $(X, \pi_i), (i = 1, 2)$ are disconnected spaces.

Proof. Let (X, η) be a D-disconnected. Then, $\exists\ \underline{A}, \underline{B} \in \eta$ such that $(A_1, A_2) = \underline{A} \cap \underline{B} = (B_1, B_2) = \underline{\emptyset}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Thus, $A_1 \cap B_1 = \emptyset, A_2 \cap B_2 = \emptyset, A_1 \cup B_1 = X$ and $A_2 \cup B_2 = X$. Therefore, $A_i \cap B_i = \emptyset, A_i \cup B_i = X$ and $A_i, B_i \in \pi_i,\ (i = 1, 2)$. Hence, (X, π_1) and (X, π_2) are disconnected spaces.

The following Example shows that the converse of Theorem 3.37 is not true in general.

Example 3.38. Let $X = \{a, b, c\}$ and $\pi_1 = \{\emptyset, X, \{a\}, \{b, c\}\}, \pi_2 = \{\emptyset, X, \{b\}, \{a, c\}\}$. Then, (X, π_1) and (X, π_2) are topological spaces and disconnected spaces. Since $\eta = (\pi_1, \pi_2) = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{b\}), (\emptyset, X), (\emptyset, \{a, c\}), (\{a\}, \{a, c\}), (\{b, c\}, X), (\{a\}, X)\}$, then (X, η) is not D-disconnected.

Theorem 3.39. Let (X, η) be a DTS. Then, if (X, η) is a $qD-$disconnected space, then (X, π_1) is disconnected space.

Proof. Let (X, η) be a $qD-$disconnected space. Then, $\exists\ \underline{A}, \underline{B} \in \eta$ such that $(A_1, A_2) = \underline{A} \not\mathrel{q} \underline{B} = (B_1, B_2)$ and $\underline{A} \cup \underline{B} = \underline{X}$. Thus, $A_1 \cap B_2 = \emptyset$ and $A_1 \cup B_1 = X, B_1 \subseteq B_2$. So that $A_1 \cap B_1 = \emptyset$ and $A_1 \cup B_1 = X,\ A_1, B_1 \in \pi_1$. Hence, (X, π_1) is a disconnected.

The following Example shows that the converse of Theorem 3.39 is not true in general.

Example 3.40. In Example 3.38, we see that: (X, π_1) is topological space and disconnected. But, (X, η) is not $qD-$disconnected.

Theorem 3.41. Let $(X, \eta), (X, \eta^*)$ be two DTS. Then, if (X, η) is D-connected and $\eta^* \leq \eta$, then (X, η^*) is also D-connected.

Proof. Suppose that (X, η^*) is a D-disconnected and $\eta^* \leq \eta$, then $\exists\ \underline{A}, \underline{B} \in \eta^*$ such that $\underline{A} \cap \underline{B} = \underline{\emptyset}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Implies, $\underline{A}, \underline{B} \in \eta, \underline{A} \cap \underline{B} = \underline{\emptyset}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Therefore, (X, η) is a D-disconnected, which a contradiction. Hence, (X, η^*) is D-connected.

Theorem 3.42. Let $(X, \eta), (X, \eta^*)$ be two DTS. Then, if (X, η) is $qD-$connected and $\eta^* \leq \eta$, then (X, η^*) is also $qD-$connected.

Proof. Suppose that (X, η^*) is a $qD-$disconnected and $\eta^* \leq \eta$, then $\exists\ \underline{A}, \underline{B} \in \eta^*$ such that $\underline{A} \not{q}\ \underline{B}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Implies, $\underline{A}, \underline{B} \in \eta, \underline{A} \not{q}\ \underline{B}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Therefore, (X, η) is a $qD-$disconnected, which a contradiction. Hence, (X, η^*) is $qD-$connected.

Theorem 3.43. Every $qD-$connected space is D-connected.

Proof. Suppose that (X, η) is a D-disconnected space, then $\exists\ \underline{A}, \underline{B} \in D(X)$ such that $cl_\eta(\underline{A}) \cap \underline{B} = \underline{\emptyset},\ \underline{A} \cap cl_\eta(\underline{B}) = \underline{\emptyset}$ and $\underline{A} \cup \underline{B} = \underline{X}$. It follows from Proposition 3.6 that $cl_\eta(\underline{A}) \not{q}\ \underline{B},\ \underline{A} \not{q}\ cl_\eta(\underline{B})$ and $\underline{A} \cup \underline{B} = \underline{X}$. Therefore, (X, η) is a $qD-$disconnected space, which a contradiction. Hence, (X, η) is D-connected space.

Definition 3.44. The DTS (X, η) is said to be:

1. DC_1-disconnected, if (X, η) has a proper open and closed D-set in X.

2. DC_1-connected, if (X, η) is not DC_1-disconnected.

Corollary 3.45. Let (X, η) be a DTS or stratified DTS. Then, if the only open and closed D-sets are $\underline{\emptyset}, \underline{X}$ in (X, η) and $\underline{\emptyset}, \underline{X}$ and $(\emptyset, X)$ in stratified DTS, then (X, η) is $qD-$connected.

Proposition 3.46. Every DC_1-connected space is $qD-$connected.

Proof. Suppose that (X, η) is a $qD-$disconnected space, then $\exists\ \underline{A} \neq \underline{\emptyset}, \underline{B} \neq \underline{\emptyset} \in \eta$ such that $\underline{A} \not{q}\ \underline{B}$ and $\underline{A} \cup \underline{B} = \underline{X}$. Implies $\underline{A} \subseteq \underline{B}^c$ and $\underline{B}^c \subseteq \underline{A}$, so that $\underline{A} = \underline{B}^c$. Therefore, $\underline{A}$ is a proper open and closed D-set in X. Thus, (X, η) be a DC_1-disconnected, which a contradiction. Hence, (X, η) is DC_1-connected.

The converse of Proposition3.46 is not true in general.

Example 3.47. Let $X = \{a, b, c\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{b\}), (\{a\}, \{a\}), (\{a\}, \{a, b\}), (\{c\}, \{c\}), (\{c\}, \{b, c\}), (\{a, c\}, \{a, c\}), (\{a, c\}, X)\}$, $\eta^c = \{\underline{\emptyset}, \underline{X}, (\{a, c\}, X), (\{b, c\}, \{b, c\}), (\{c\}, \{b, c\}), (\{a, b\}, \{a, b\}), (\{a\}, \{a, b\}), (\{b\}, \{b\}), (\emptyset, \{b\})\}$. Then, (X, η) is a DTS and $qD-$connected space. But, (X, η) is not DC_1-connected because, $\exists\ (\emptyset, \{b\}), (\{a\}, \{a, b\}), (\{c\}, \{b, c\})$ and $(\{a, c\}, X)$ are open and closed D-sets.

Example 3.48. Let $X = \{a, b, c\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{c\}), (\emptyset, \{a, b\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, X), (\{a\}, \{a\}), (\{a\}, \{a, b\}), (\{a\}, \{a, c\}), (\{a\}, X), (\{b\}, \{b\}), (\{b\}, \{a, b\}), (\{b\}, \{b, c\}), (\{b\}, X), (\{a, b\}, \{a, b\}), (\{a, b\}, X)\}$,

$\eta^c = \{\underline{\emptyset}, \underline{X}, (\{b, c\}, X), (\{a, c\}, X), (\{a, b\}, X), (\{c\}, X), (\{b\}, X), (\{a\}, X), (\emptyset, X), (\{b, c\}, \{b, c\}), (\{c\}, \{b, c\}), (\{b\}, \{b, c\}), (\emptyset, \{b, c\}), (\{a, c\}, \{a, c\}), (\{c\}, \{a, c\}), (\{a\}, \{a, c\}), (\emptyset, \{a, c\}), (\{c\}, \{c\}), (\emptyset, \{c\})\}$.

Then, (X, η) is stratified DTS and $qD-$connected space. But, (X, η) is not DC_1-connected because, $\exists$ $(\emptyset, \{c\}), (\{a\}, X), (\{a\}, \{a, c\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\{b\}, \{b, c\}), (\{b\}, X)$, and $(\{a, b\}, X)$ are open and closed D-sets.

Example 3.49. From Example 3.48 (X, η) is stratified DTS and $qD-$connected, but $\exists$ $(\emptyset, \{c\})$,
$(\{a\}, X), (\{a\}, \{a, c\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\{b\}, \{b, c\}), (\{b\}, X)$, and $(\{a, b\}, X)$, are open and closed D-sets.

Corollary 3.50. For a DTS (X, η) we have the following implication:
DC_1-connected $\rightarrow$ $qD-$connected $\rightarrow$ D-connected.

Definition 3.51. The DTS (X, η) is said to be:

1. Strongly double connected (strongly $SD-$connected, for short), if there exist no non-empty closed D-sets $\underline{A}, \underline{B} \in X$ such that $\underline{A} \cap \underline{B} = \underline{\emptyset}$.

2. Strongly $SD-$disconnected, if (X, η) is not strongly $SD-$connected.

Proposition 3.52. (X, η) is strongly $SD-$connected if and only if there exist no open D-sets $\underline{A}, \underline{B}$ in X such that $\underline{A} \neq \underline{X} \neq \underline{B}$ and $\underline{A} \cup \underline{B} = \underline{X}$.

Proof. Let $\underline{A}, \underline{B}$ be open D-sets in X such that $\underline{A} \neq \underline{X} \neq \underline{B}$. If we take $\underline{C} = \underline{A}^c$ and $\underline{D} = \underline{B}^c$, then $\underline{C}$ and $\underline{D}$ become closed D-sets in X and $\underline{C} \neq \underline{\emptyset} \neq \underline{D}$, $\underline{C} \cap \underline{D} = \underline{\emptyset}$, which a contradiction.
Conversely, it is obvious.

Proposition 3.53. Strongly $SD-$connectedness does not imply DC_1-connectedness, and DC_1- connectedness does not imply $StD-$connectedness.

Example 3.54. In Example3.7, we see that: (X, η) is strongly $SD-$connected, but it is not DC_1- connected, for $\exists$ $(\emptyset, \{b\})$ is both open and closed D-set.

Example 3.55. In Example3.3, we see that: (X, η) is DC_1-connected, but it is not strongly $SD-$connected, for

$$\exists\ (\{a\}, \{a, b\}), (\{b, c, d\}, X) \in \eta \text{ and } (\{a\}, \{a, b\}) \,\underline{\cup}\, (\{b, c, d\}, X) = \underline{X}.$$

Definition 3.56. Let (X, η) be a DTS and $\underline{Y} \subseteq \underline{X}$ with $\underline{x}_t \underline{\in} DP(Y)$. The union of all D-connected subsets of $\underline{Y}$ containing the D-point $\underline{x}_t$ is called double component (D-component, for short) of Y with respect to $\underline{x}_t$, denoted by $\underline{C}(\underline{Y}, \underline{x}_t)$, i.e.,
$\underline{C}(\underline{Y}, \underline{x}_t) = \underline{\cup}\{\underline{A} \subseteq \underline{Y} : \underline{x}_t \underline{\in} \underline{A} \textit{ and } \underline{A} \textit{ is a } D-\textit{connected set}\}$.

Remark 3.57. The D-component $\underline{C}(\underline{Y},\underline{x}_t)$ is the largest D-connected subset of Y containing $\underline{x}_t$.

Definition 3.58. Let (X,η) be a DTS and $\underline{Y} \subseteq \underline{X}$ with $\underline{x}_t \underline{\in} DP(Y), \underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$. Then, the union of all $qD-$connected subsets of $\underline{Y}$ containing the D-point $\underline{x}_t$ is called quasi-coincident double component ($qD-$component, for short) of Y with respect to $\underline{x}_t$, denoted by $\underline{Cq}(\underline{Y},\underline{x}_t)$, i.e., $\underline{Cq}(\underline{Y},\underline{x}_t) = \cup\{\underline{A} \subseteq \underline{Y} : \underline{x}_t \underline{\in} \underline{A}\ and\ \underline{A}\ is\ a\ qD-connected\ set\}$.

Remark 3.59. The $qD-$component $\underline{Cq}(\underline{Y},\underline{x}_t)$ is the largest $qD-$connected subset of Y containing $\underline{x}_t$.

Theorem 3.60. Every D-component of a DTS is a closed D-set.

Proof. Let (X,η) be a DTS and let $\underline{C}(\underline{Y},\underline{x}_t)$ be a D-component of the DTS (X,η) with respect to an arbitrary D-point $\underline{x}_t \in DP(X)$. Then, $\underline{C}(\underline{Y},\underline{x}_t)$ is a D-connected subset of X [Theorem 3.29]. Also, by Theorem 3.33 $cl_\eta(\underline{C}(\underline{Y},\underline{x}_t))$ is D-connected subset of $\underline{X}$ containing $\underline{x}_t$, then $cl_\eta(\underline{C}(\underline{Y},\underline{x}_t)) \subseteq \underline{C}(\underline{Y},\underline{x}_t)$. But, $\underline{C}(\underline{Y},\underline{x}_t) \subseteq cl_\eta(\underline{C}(\underline{Y},\underline{x}_t))$. Hence, $\underline{C}(\underline{Y},\underline{x}_t) = cl_\eta(\underline{C}(\underline{Y},\underline{x}_t))$, which shows that the D-component $\underline{C}(\underline{Y},\underline{x}_t)$ is a closed D-set.

Theorem 3.61. Every $qD-$component of a DTS is a closed D-set.

Proof. Let (X,η) be a DTS and let $\underline{Cq}(\underline{Y},\underline{x}_t)$ be a $qD-$component of the DTS (X,η) with respect to an arbitrary D-point $\underline{x}_t \in DP(X)$. Then, $\underline{Cq}(\underline{Y},\underline{x}_t)$ is a $qD-$connected subset of X [Theorem 3.30]. Also, by Theorem 3.34 $cl_\eta(\underline{Cq}(\underline{Y},\underline{x}_t))$ is $qD-$connected subset of $\underline{X}$ containing $\underline{x}_t$, then $cl_\eta(\underline{Cq}(\underline{Y},\underline{x}_t)) \subseteq \underline{Cq}(\underline{Y},\underline{x}_t)$. But, $\underline{Cq}(\underline{Y},\underline{x}_t) \subseteq cl_\eta(\underline{Cq}(\underline{Y},\underline{x}_t))$. Hence, $\underline{Cq}(\underline{Y},\underline{x}_t) = cl_\eta(\underline{Cq}(\underline{Y},\underline{x}_t))$, which shows that the qD-component $\underline{Cq}(\underline{Y},\underline{x}_t)$ is a closed D-set.

Theorem 3.62. Let (X,η) be a DTS. Then, each D-point in X is contained in exactly one D-component of X.

Proof. Let $\underline{x}_t \underline{\in} \underline{X}$ and consider the collection:
$\underline{C} = \{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ D-connected\ set\}$.
Then, we have:

1. $\underline{C} \neq \underline{\emptyset}$, for the D-point $\underline{x}_t$ is a D-connected subset of X. Then, $\underline{x}_t \underline{\in} \underline{C}$.

2. $\underline{\bigcap}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ D-connected\ set\} \neq \underline{\emptyset}$. Since $\underline{x}_t \underline{\in} \underline{Y}$, $\forall\ \underline{Y} \underline{\in} \underline{C}$.

3. $\underline{\bigcup}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ D-connected\ set\}$, having non null double intersection, is D-connected subset of X containing $\underline{x}_t$.

4. $\underline{\bigcup}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ D-connected\ set\}$ is the largest D-connected subset of $\underline{X}$ containing $\underline{x}_t$, which is the D-component $\underline{C}(\underline{X},\underline{x}_t)$ of X with respect to $\underline{x}_t$ and containing $\underline{x}_t$ from Definition 3.56.

Now, suppose that $\underline{C}^*(\underline{X}, \underline{x}_t)$ be another D-component containing $\underline{x}_t$, then $\underline{C}^*(\underline{X}, \underline{x}_t)$ is D-connected subset of X containing $\underline{x}_t$. Since $\underline{C}(\underline{X}, \underline{x}_t)$ is D-component containing $\underline{x}_t$, then $\underline{C}^*(\underline{X}, \underline{x}_t) \subseteq \underline{C}(\underline{X}, \underline{x}_t)$. Again, since $\underline{C}^*(\underline{X}, \underline{x}_t)$ is D-component containing $\underline{x}_t$, then $\underline{C}(\underline{X}, \underline{x}_t) \subseteq \underline{C}^*(\underline{X}, \underline{x}_t)$. Therefore, $\underline{C}(\underline{X}, \underline{x}_t) = \underline{C}^*(\underline{X}, \underline{x}_t)$. Hence, $\underline{x}_t$ is contained in exactly one D-component of X.

Theorem 3.63. Let (X, η) be a DTS. Then, each D-point in X, $\underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$, is contained in exactly one $qD-$component of X.

Proof. Let $\underline{x}_t \underline{\in} \underline{X}$ and consider the collection:
$\underline{Cq} = \{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ qD - connected\ set\}$.
Then, we have:

1. $\underline{Cq} \neq \underline{\emptyset}$, for the D-point $\underline{x}_t$ is a $qD-$connected subset of X, $\underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$. Then, $\underline{x}_t \underline{\in} \underline{Cq}$.
2. $\underline{\bigcap}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ qD{-}connected\ set\} \neq \underline{\emptyset}$. Since $\underline{x}_t \underline{\in} \underline{Y}$, $\forall\ \underline{Y} \underline{\in} \underline{C}$.
3. $\underline{\bigcup}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ qD - connected\ set\}$, having non null double intersection, is $qD-$connected subset of X containing $\underline{x}_t$.
4. $\underline{\bigcup}\{\underline{Y} \subseteq \underline{X} : \underline{x}_t \underline{\in} \underline{Y}\ and\ \underline{Y}\ is\ a\ qD{-}connected\ set\}$ is the largest $qD-$connected subset of $\underline{X}$ containing $\underline{x}_t$, which is the $qD-$component $\underline{Cq}(\underline{X}, \underline{x}_t)$ of X with respect to $\underline{x}_t$ and containing $\underline{x}_t$ from Definition 3.58.

Now, suppose that $\underline{Cq}^*(\underline{X}, \underline{x}_t)$ be another $qD-$component containing $\underline{x}_t$, thus $\underline{Cq}^*(\underline{X}, \underline{x}_t)$ is $qD-$connected subset of X containing $\underline{x}_t$. Since $\underline{Cq}(\underline{X}, \underline{x}_t)$ is $qD-$component containing $\underline{x}_t$, then $\underline{Cq}^*(\underline{X}, \underline{x}_t) \subseteq \underline{Cq}(\underline{X}, \underline{x}_t)$. Again, since $\underline{Cq}^*(\underline{X}, \underline{x}_t)$ is $qD-$component containing $\underline{x}_t$, then $\underline{Cq}(\underline{X}, \underline{x}_t) \subseteq \underline{Cq}^*(\underline{X}, \underline{x}_t)$. Therefore, $\underline{Cq}(\underline{X}, \underline{x}_t) = \underline{Cq}^*(\underline{X}, \underline{x}_t)$. Hence, $\underline{x}_t$ is contained in exactly one $qD-$component of X.

Theorem 3.64. Let (X, η) be a DTS. Then, any two D-components with respect to two disjoint D-points in X are either disjoint or identical.

Proof. Let $\underline{C}(\underline{X}, \underline{x}_t)$ and $\underline{C}(\underline{X}, \underline{y}_r)$ be two D-components of the DTS (X, η) with respect to the D-points $\underline{x}_t, \underline{y}_r$ in X. and $\underline{x}_t \underline{\cap} \underline{y}_r = \underline{\emptyset}$. Then, if $\underline{C}(\underline{X}, \underline{x}_t) \sqcap \underline{C}(\underline{X}, \underline{y}_r) = \underline{\emptyset}$, then we are done. So let $\underline{C}(\underline{X}, \underline{x}_t) \underline{\sqcap}\ \underline{C}(\underline{X}, \underline{y}_r) \neq \underline{\emptyset}$. We may choose $\underline{z}_s \underline{\in} \underline{C}(\underline{X}, \underline{x}_t) \underline{\sqcap}\ \underline{C}(\underline{X}, \underline{y}_r)$. Clearly, $\underline{z}_s \underline{\in}\ \underline{C}(\underline{X}, \underline{x}_t)$ and $\underline{z}_s \underline{\in}\ \underline{C}(\underline{X}, \underline{y}_r)$. Thus, $\underline{C}(\underline{X}, \underline{x}_t) = \underline{C}(\underline{X}, \underline{y}_r)$ [from Theorem 3.62]. Therefore, $\underline{C}(\underline{X}, \underline{x}_t)$ and $\underline{C}(\underline{X}, \underline{y}_r)$ are identical. This completes the proof.

Theorem 3.65. Let (X, η) be a DTS. Then, any two $qD-$components with respect to two disjoint D-points in X are either not quasi-coincident or identical.

Proof. Let $\underline{Cq}(\underline{X}, \underline{x}_t)$ and $\underline{Cq}(\underline{X}, \underline{y}_r)$ be two $qD-$components of the DTS (X, η) with respect to the D-points $\underline{x}_t, \underline{y}_r$ in X, $\underline{x}_{\frac{1}{2}}\ q\ cl_\eta(\underline{x}_{\frac{1}{2}})$, $\underline{y}_{\frac{1}{2}}\ q\ cl_\eta(\underline{y}_{\frac{1}{2}})$ and $\underline{x}_t \underline{\cap} \underline{y}_r = \underline{\emptyset}$. Then, if
$\underline{Cq}(\underline{X}, \underline{x}_t)\ \not{q}\ \underline{Cq}(\underline{X}, \underline{y}_r)$, then we are done. So let $\underline{Cq}(\underline{X}, \underline{x}_t)\ q\ \underline{Cq}(\underline{X}, \underline{y}_r)$. Then, $\underline{Cq}(\underline{X}, \underline{x}_t) \underline{\sqcap}$
$\underline{Cq}(\underline{X}, \underline{y}_r) \neq \underline{\emptyset}$. We may choose $\underline{z}_s \underline{\in} \underline{Cq}(\underline{X}, \underline{x}_t) \underline{\sqcap}\ \underline{Cq}(\underline{X}, \underline{y}_r)$. Clearly, $\underline{z}_s \underline{\in}\ \underline{Cq}(\underline{X}, \underline{x}_t)$ and $\underline{z}_s \underline{\in}\ \underline{Cq}(\underline{X}, \underline{y}_r)$, thus $\underline{Cq}(\underline{X}, \underline{x}_t) = \underline{Cq}(\underline{X}, \underline{y}_r)$ [from Theorem 3.63]. Therefore, $\underline{Cq}(\underline{X}, \underline{x}_t)$ and $\underline{Cq}(\underline{X}, \underline{y}_r)$ are identical. This completes the proof.

Definition 3.66. A DTS (X,η) is said to be a quasi-coincident hyperconnected (q-hyperconnected, for short) if every pair of non null proper open D-sets $\underline{A}$ and $\underline{B}$ are quasi-coincident, i. e.,
(X,η) is said to be q-hyperconnected if $\forall\ \underline{A},\underline{B}\in\eta$, we have $\underline{A}\ q\ \underline{B}$.

Theorem 3.67. Every q-hyperconnected DTS is $qD-$connected.

Proof. Suppose that (X,η) is is $qD-$disconnected, then there exist two open D-sets $\underline{A}$ and $\underline{B}$ such that $\underline{A}\ \not q\ \underline{B}$ [Theorem 3.20]. Hence, (X,η) is not q-hyperconnected DTS, which a contradiction. Thus, (X,η) is $qD-$connected.

Remark 3.68. The converse of Theorem 3.67 is not true in general, as shown in the following Examples.

Example 3.69. In Example3.7, we see that: The DTS (X,η) is a $qD-$connected, but it is not a q-hyperconnected, for $(\{a\},\{a\})\ \not q\ (\emptyset,\{b\})$.

Example 3.70. Let $X=\{a,b\}$ and $\eta=\{\underline{\emptyset},\underline{X},(\emptyset,\{a\}),(\emptyset,\{b\}),(\emptyset,X)\}$. Then, η defines a stratified double topology on X. Hence, the DTS (X,η) is $qD-$connected, but it is not q-hyperconnected, for $(\emptyset,\{a\})\ \not q\ (\emptyset,\{b\})$.

Theorem 3.71. Let (X,η) be a DTS. Then, the following are equivalent:

1. (X,η) is q-hyperconnected,
2. $\underline{A}$ is D-dense, $\forall\ \underline{A}\in\eta$,
3. $\underline{A}$ is D-dense or D-nowhere dense, $\forall\ \underline{A}\in D(X)$.

Proof. (1) $\rightarrow$ (2) Suppose that $\exists\ \underline{B}\in\eta$ such that $\underline{B}$ is not D-dense in X, thus $cl_\eta(\underline{B})\neq\underline{X}$. Hence, $\underline{X}\setminus cl_\eta(\underline{B})$ and $\underline{B}$ are not quasi-coincident [by Proposition 3.3], which a contradiction with q-hyperconnected of (X,η).
(2) $\rightarrow$ (3) Suppose that $\underline{B}$ is not D-nowhere dense, then $int_\eta(cl_\eta(\underline{B}))\neq\underline{\emptyset}$. So by (2), $cl_\eta(int_\eta(cl_\eta(\underline{B})))=\underline{X}$. Since $cl_\eta(int_\eta(cl_\eta(\underline{B})))\subseteq cl_\eta(\underline{B})$, then $cl_\eta(\underline{B})=\underline{X}$. Hence, $\underline{B}$ is D-dense.
(3) $\rightarrow$ (1) Suppose that $\underline{A}\ \not q\ \underline{B}$, for some non-empty open D-subsets $\underline{A},\underline{B}$ of X, then $cl_\eta(\underline{A})\ \not q\ \underline{B}$ [by Theorem 2.7], and $\underline{A}$ is not D-dense. Since $\underline{A}\in\eta$, then $\underline{\emptyset}\neq\underline{A}\subseteq int_\eta(cl_\eta(\underline{A}))$, which a contradiction with (3). Hence, the result.

Remark 3.72. If the DTS (X,η) is a stratified, then the Theorems 3.19, 3.20, ..., 3.71 and Propositions 3.2, ..., 3.53 are satisfied.

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