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VECTOR APPROACH TO A NEW GENERALIZATION OF FIBONACCI POLYNOMIAL

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Abstaract − In this paper we introduce a new generalization of Fibonacci polynomial and vectors of length d are defined for these Polynomials. Using these vectors some properties for Generalized Fibonacci Polynomial (GFP) are established.

Keywords − *Fibonacci polynomial, Lucas polynomial, Pell polynomial, Pell-Lucas polynomial, Jacobsthal polynomial, Jacobsthal-Lucas polynomial.*

1 Introduction

The Generalized Fibonacci Polynomial (GFP) is a natural generalization of the Fibonacci polynoial $F_n(x)$ which defined recurrently by $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$, with $F_0(x) = 0$, $F_1(0) = 1$, for $n \geq 1$. Fibonacci Polynomial is the topic of wide interest, for the literature of these Polynomials one can refer to articles by many authors like [1, 5, 6, 7, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 25, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 2, 3, 4, 10, 9]. The study of properties for Fibonacci polynomials and Lucas polynomials have received less attention than numerical sequences. The identities involving Fibonacci and Lucas sequences extend naturally to the GFP that satisfy closed formulas similar to the Binet formulas satisfied by Fibonacci and Lucas sequences. We denote the GFP of Lucas type and Fibonacci type by $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ respectively. We adapt some known identities given for Fibonacci or Lucas polynomials to the $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$. Most of the identities for Fibonacci and Lucas polynomials that we extend to GFP can be found in the books, articles on Fibonacci and Lucas sequences and their applications, [2, 3, 4, 8, 9, 10, 22, 23, 24, 26, 12] The ultimate aim of this paper is to introduce new generalization $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ of Fibonacci and Lucas polynomials and establish a collection of identities for the $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ using vector method.

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2 Generalized Fibonacci Polynomial (GFP)

In this section we introduce the new generalized Fibonacci polynomials $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$.

Definition 2.1. The generalized Fibonacci polynomial $\widehat{F}_n(x)$ is defined by the recurrence relation

$$\widehat{F}_{n+1}(x) = x\widehat{F}_n(x) + \widehat{F}_{n-1}(x) \quad \text{with} \quad \widehat{F}_0(x) = 0, \quad \widehat{F}_1(x) = x^2 + 4, \text{ for } n \geq 1 \tag{1}$$

Definition 2.2. The generalized Lucas polynomial $\widehat{L}_n(x)$ is defined by the recurrence relation

$$\widehat{L}_{n+1}(x) = x\widehat{L}_n(x) + \widehat{L}_{n-1}(x) \quad \text{with} \quad \widehat{L}_0(x) = 2x^2 + 8, \quad \widehat{L}_1(x) = x^3 + 4x, \text{ for } n \geq 1 \tag{2}$$

Polynomial	Initial value $G_0(x) = a(x)$	Initial value $G_1(x) = b(x)$	Recursive Formula $G_{n+1}(x) = a(x)G_n(x) + b(x)G_{n-1}(x)$
Fibonacci	0	1	$F_{n+1}(x) = F_n(x) + F_{n-1}(x)$
Lucas	2	x	$L_{n+1}(x) = L_n(x) + L_{n-1}(x)$
Pell	0	1	$P_{n+1}(x) = 2xP_n(x) + P_{n-1}(x)$
Pell-Lucas	2	2x	$Q_{n+1}(x) = 2xQ_n(x) + Q_{n-1}(x)$
Jacobsthal	0	1	$J_{n+1}(x) = J_n(x) + 2xJ_{n-1}(x)$
Jacobsthal-Lucas	2	1	$j_{n+1}(x) = j_n(x) + 2xj_{n-1}(x)$
Generalized Fibonacci	0	$x^2 + 4$	$\widehat{F}_{n+1}(x) = x\widehat{F}_n(x) + \widehat{F}_{n-1}(x)$
Generalized Lucas	$2x^2 + 8$	$x^3 + 4x$	$\widehat{L}_{n+1}(x) = x\widehat{L}_n(x) + \widehat{L}_{n-1}(x)$

Table 1: **Recurrence relation of some GFP.**

Characteristic equation of the initial recurrence relation (1 and 2) is,

$$r^2 - xr - 1 = 0 \tag{3}$$

Characteristic roots of (3) are

$$r_1(x) = \frac{x + \sqrt{x^2 + 4}}{2}, \quad r_2(x) = \frac{x - \sqrt{x^2 + 4}}{2} \tag{4}$$

Characteristic roots (4) satisfy the properties

$$r_1(x) - r_2(x) = \sqrt{x^2 + 4} = \sqrt{\Delta(x)}, \quad r_1(x) + r_2(x) = x, \quad r_1(x)r_2(x) = -1 \tag{5}$$

Binet forms for both $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ are given by

$$\widehat{F}_n(x) = r_1(x)^n - r_2(x)^n \tag{6}$$

$$\widehat{L}_n(x) = \left[r_1(x)^2 + r_2(x)^2 + 2\right] \left[r_1(x)^n + r_2(x)^n)\right] \tag{7}$$

Definition 2.3. (Generalized Fibonacci polynomial vector of length d)
For all integers n, Generalized Fibonacci polynomial vector $\overrightarrow{F_n^d}(x)$ and Generalized Lucas polynomial vector $\overrightarrow{L_n^d}(x)$ of length d are defined as follows

$$\overrightarrow{F_n^d}(x)=\begin{bmatrix}\widehat{F}_n(x)\\\widehat{F}_{n+1}(x)\\\widehat{F}_{n+2}(x)\\\vdots\\\widehat{F}_{n+d-1}(x)\end{bmatrix}\text{ and }\overrightarrow{L_n^d}(x)=\begin{bmatrix}\widehat{L}_n(x)\\\widehat{L}_{n+1}(x)\\\widehat{L}_{n+2}(x)\\\vdots\\\widehat{L}_{n+d-1}(x)\end{bmatrix}$$

Definition 2.4. (Reverse Generalized Fibonacci polynomial vector of length d)
For all integers n, Reverse Generalized Fibonacci polynomial vector $\overrightarrow{f_n^d}(x)$ and Reverse Generalized Lucas polynomial vector $\overrightarrow{l_n^d}(x)$ of length d are defined as follows

$$\overrightarrow{f_n^d}(x)=\begin{bmatrix}\widehat{L}_{n+d-1}(x)\\\widehat{L}_{n+d-2}(x)\\\widehat{L}_{n+d-3}(x)\\\vdots\\\widehat{L}_n(x)\end{bmatrix}\text{ and }\overrightarrow{l_n^d}(x)=\begin{bmatrix}\widehat{L}_{n+d-1}(x)\\\widehat{L}_{n+d-2}(x)\\\widehat{L}_{n+d-3}(x)\\\vdots\\\widehat{L}_n(x)\end{bmatrix}$$

Definition 2.5. (Vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ and $\overrightarrow{d}$)
For all integers n, the vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ and $\overrightarrow{d}$ of length d are defined as follows

$$\overrightarrow{a}=\begin{bmatrix}1\\r_1(x)\\r_1^2(x)\\\vdots\\r_1^{d-2}(x)\\r_1^{d-1}(x)\end{bmatrix},\ \overrightarrow{b}=\begin{bmatrix}1\\r_2(x)\\r_2^2(x)\\\vdots\\r_2^{d-2}(x)\\r_2^{d-1}(x)\end{bmatrix},\ \overrightarrow{c}=\begin{bmatrix}r_1^{d-1}(x)\\r_1^{d-2}(x)\\r_1^{d-3}(x)\\\vdots\\r_1(x)\\1\end{bmatrix}\text{ and }\overrightarrow{d}=\begin{bmatrix}r_2^{d-1}(x)\\r_2^{d-2}(x)\\r_2^{d-3}(x)\\\vdots\\r_2(x)\\1\end{bmatrix}$$

Definition 2.6. Define $d \times d$ matrix T by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 & 1 & x \end{bmatrix}_{d \times d}$$

Definition 2.7. Define $d \times d$ matrix S by

$$\begin{bmatrix} x & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{d \times d}$$

The matrices T and S have characteristic polynomial $r^{(m-2)}(r^2 - xr - 1)$. The non zero eigenvalues of both are $r_1(x)$ and $r_2(x)$ and eigenspace associated with $r_1(x)$ and $r_2(x)$ for T are spanned respectively by $\vec{a}$, $\vec{b}$ and for S are spanned respectively by $\vec{c}$ and $\vec{d}$.

3 Auxiliary Results

Theorem 3.1. (Binet form for $\vec{F^d_n}(x)$, $\vec{L^d_n}(x)$, $\vec{f^d_n}(x)$ and $\vec{l^d_n}(x)$) For all integers n,

$$\vec{F^d_n}(x) = [r_1(x) - r_2(x)][r_1^n(x)\,\vec{a} - r_2^n(x)\,\vec{b}] \tag{8}$$

$$\vec{L^d_n}(x) = [r_1(x) - r_2(x)]^2[r_1^n(x)\,\vec{a} + r_2^n(x)\,\vec{b}] \tag{9}$$

$$\vec{f^d_n}(x) = [r_1(x) - r_2(x)][r_1^n(x)\,\vec{c} - r_2^n(x)\,\vec{d}] \tag{10}$$

$$\vec{l^d_n}(x) = [r_1(x) - r_2(x)]^2[r_1^n(x)\,\vec{c} + r_2^n(x)\,\vec{d}] \tag{11}$$

Theorem 3.2. For all integers n

$$\vec{F^d_{n+1}}(x) = T\,\vec{F^d_n}(x)$$
$$\vec{L^d_{n+1}}(x) = T\,\vec{L^d_n}(x)$$
$$\vec{f^d_{n+1}}(x) = S\,\vec{F^d_n}(x)$$
$$\vec{l^d_{n+1}}(x) = S\,\vec{F^d_n}(x)$$

Theorem 3.3. For all integers $t \geq n+1$

$$\overrightarrow{F_{n+1}^{d}}(x) = T^{(n-t+1)} \overrightarrow{F_{n}^{d}}(x)$$
$$\overrightarrow{L_{n+1}^{d}}(x) = T^{(n-t+1)} \overrightarrow{L_{n}^{d}}(x)$$
$$\overrightarrow{f_{n+1}^{d}}(x) = S^{(n-t+1)} \overrightarrow{F_{n}^{d}}(x)$$
$$\overrightarrow{l_{n+1}^{d}}(x) = S^{(n-t+1)} \overrightarrow{F_{n}^{d}}(x)$$

Theorem 3.4. For $k > 0$

$$\overrightarrow{a} \cdot \overrightarrow{a} = \overrightarrow{c} \cdot \overrightarrow{c} = \| \overrightarrow{a} \|^2 = \| \overrightarrow{c} \|^2 = \begin{cases} \dfrac{\widehat{F}_n(x) r_1^{d-1}(x)}{(r_1(x) - r_2(x))}, & \text{If } d \text{ is even;} \\ \dfrac{\widehat{L}_n(x) r_1^{d-1}(x)}{(r_1(x) - r_2(x))^2}, & \text{If } d \text{ is odd} \end{cases}$$

$$\overrightarrow{b} \cdot \overrightarrow{b} = \overrightarrow{d} \cdot \overrightarrow{d} = \| \overrightarrow{b} \|^2 = \| \overrightarrow{d} \|^2 = \begin{cases} \dfrac{-\widehat{F}_n(x) r_2^{d-1}(x)}{(r_1(x) - r_2(x))}, & \text{If } d \text{ is even;} \\ \dfrac{\widehat{L}_n(x) r_2^{d-1}(x)}{(r_1(x) - r_2(x))^2}, & \text{If } d \text{ is odd} \end{cases}$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{c} \cdot \overrightarrow{d} = \begin{cases} 0, & \text{If } d \text{ is even;} \\ 1, & \text{If } d \text{ is odd} \end{cases}$$

Theorem 3.5. For all positive integer d and for all integers n_1 and n_2

$$\begin{aligned} \overrightarrow{F_{n_1}^{d}}(x) \cdot \overrightarrow{F_{n_2}^{d}}(x) &= \overrightarrow{f_{n_1}^{d}}(x) \cdot \overrightarrow{f_{n_2}^{d}}(x) \\ &= \begin{cases} \dfrac{\widehat{F}_d(x) \widehat{F}_{d+n_1+n_2-1}(x)}{(r_1(x) - r_2(x))^2}, & \text{If } d \text{ is even;} \\ \widehat{L}_d(x) \widehat{L}_{d+n_1+n_2-1}(x) - \dfrac{(-1)^{n_1} \widehat{L}_{n_2-n_1}(x)}{(r_1(x) - r_2(x))}, & \text{If } d \text{ is odd} \end{cases} \end{aligned}$$

$$\begin{aligned} \overrightarrow{L_{n_1}^{d}}(x) \cdot \overrightarrow{L_{n_2}^{d}}(x) &= \overrightarrow{l_{n_1}^{d}}(x) \cdot \overrightarrow{l_{n_2}^{d}}(x) \\ &= \begin{cases} \widehat{F}_d(x) \widehat{F}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;} \\ \dfrac{\widehat{L}_d(x) \widehat{L}_{d+n_1+n_2-1}(x)}{(r_1(x) - r_2(x))^2} + \dfrac{(-1)^{n_1} \widehat{L}_{n_2-n_1}(x)}{(r_1(x) - r_2(x))}, & \text{If } d \text{ is odd} \end{cases} \end{aligned}$$

$$\begin{aligned} \overrightarrow{F_{n_1}^{d}}(x) \cdot \overrightarrow{L_{n_2}^{d}}(x) &= \overrightarrow{f_{n_1}^{d}}(x) \cdot \overrightarrow{l_{n_2}^{d}}(x) \\ &= \begin{cases} \widehat{F}_d(x) \widehat{L}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;} \\ \dfrac{\widehat{L}_d(x) \widehat{F}_{d+n_1+n_2-1}(x)}{(r_1(x) - r_2(x))^2} + \dfrac{(-1)^{n_1} \widehat{F}_{n_2-n_1}(x)}{(r_1(x) - r_2(x))}, & \text{If } d \text{ is odd} \end{cases} \end{aligned}$$

Theorem 3.6. For all positive integer d and for all integer n

$$\| \overrightarrow{F_{n_1}^d}(x)\|^2 = \| \overrightarrow{f_{n_1}^d}(x)\|^2 = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+2n-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{L}_{d+2n-1}(x) - \dfrac{2(-1)^n}{(r_1(x)-r_2(x))^2}, & \text{If } d \text{ is odd} \end{cases}$$

$$\| \overrightarrow{L_{n_1}^d}(x)\|^2 = \| \overrightarrow{l_{n_1}^d}(x)\|^2 = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+2n-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{L}_{d+2n-1}(x) + \dfrac{2(-1)^n}{(r_1(x)-r_2(x))^2}, & \text{If } d \text{ is odd} \end{cases}$$

4 Main Results

Theorem 4.1. For all positive integer d and for all integers n_1 and n_2

$$\sum_{i=0}^{i=d-1} \widehat{F}_{n_1+i}(x)\widehat{F}_{n_2+i}(x) = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{L}_{d+n_1+n_2-1}(x) - \dfrac{(-1)^{n_1}\widehat{L}_{n_2-n_1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd} \end{cases}$$

$$\sum_{i=0}^{i=d-1} \widehat{L}_{n_1+i}(x)\widehat{L}_{n_2+i}(x) = \begin{cases} (r_1(x)-r_2(x))^2\widehat{F}_d(x)\widehat{F}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{L}_{d+n_1+n_2-1}(x) + \dfrac{(-1)^{n_1}\widehat{L}_{n_2-n_1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd} \end{cases}$$

$$\sum_{i=0}^{i=d-1} \widehat{F}_{n_1+i}(x)\widehat{L}_{n_2+i}(x) = \begin{cases} (r_1(x)-r_2(x))^2\widehat{F}_d(x)\widehat{L}_{d+n_1+n_2-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d(x)\widehat{F}_{d+n_1+n_2-1}(x) + \dfrac{(-1)^{n_1}\widehat{F}_{n_2-n_1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd} \end{cases}$$

Corollary 4.2. For all positive integer d and for all integer n

$$\sum_{i=0}^{i=d-1} \widehat{F}_n(x)^2 = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+2n-1}(x), & \text{If } d \text{ is even;} \\ \dfrac{\widehat{L}_d(x)\widehat{L}_{d+2n-1}(x)}{(r_1(x)-r_2(x))^2} - 2(-1)^n, & \text{If } d \text{ is odd} \end{cases}$$

$$\sum_{i=0}^{i=d-1} \widehat{L}_n(x)^2 = \begin{cases} \widehat{F}_d(x)\widehat{F}_{d+2n-1}(x), & \text{If } d \text{ is even;} \\ \widehat{L}_d\widehat{L}_{d+2n-1}(x) + 2(-1)^{n_1})(r_1(x)-r_2(x))^2, & \text{If } d \text{ is odd} \end{cases}$$

Corollary 4.3. For all positive integer d and for all integer t and n

$$\widehat{F}_{n+d-2}(x)\widehat{F}_{n+d-t-1}(x) + \widehat{F}_{n-1}(x)\widehat{F}_{n-t}(x) =$$

$$\begin{cases} (r_1(x)+r_2(x))\widehat{F}_{d-1}(x)\widehat{F}_{d+2n-t-2}(x), & \text{If } d \text{ is even;} \\ \dfrac{(r_1(x)+r_2(x))^2}{(r_1(x)^2-r_2(x)^2)}\widehat{L}_{d-1}(x)\widehat{L}_{d+2n-t}(x) - \dfrac{2(-1)^{n-t}\widehat{L}_{t-1}(x)}{(r_1(x)-r_2(x))}, & \text{If } d \text{ is odd} \end{cases}$$

$$\widehat{L}_{n+d-2}(x)\widehat{L}_{n+d-t-1}(x) + \widehat{L}_{n-1}(x)\widehat{L}_{n-t}(x) = \begin{cases} (r_1(x)+r_2(x))(r_1(x)-r_2(x))^2\widehat{L}_{d-1}(x)\widehat{L}_{d+2n-t-2}(x), & \text{If } d \text{ is even;} \\ (r_1(x)+r_2(x))\widehat{L}_{d-1}(x)\widehat{L}_{d+2n-t}(x) + 2(-1)^{n-1}\widehat{L}_{t+1}(x)(r_1(x)-r_2(x)), & \text{If } d \text{ is odd} \end{cases}$$

$$\widehat{F}_{n+d-2}(x)\widehat{L}_{n+d-t-1}(x) + \widehat{F}_{n-1}(x)\widehat{L}_{n-t}(x) = \begin{cases} (r_1(x)+r_2(x))(r_1(x)-r_2(x))^2\widehat{F}_{d-1}(x)\widehat{L}_{d+2n-t-2}(x), & \text{If } d \text{ is even;} \\ (r_1(x)+r_2(x))\widehat{L}_{d-1}(x)\widehat{F}_{d+2n-t}(x) + 2(-1)^{n-1}\widehat{F}_{t+1}(x)(r_1(x)-r_2(x)), & \text{If } d \text{ is odd} \end{cases}$$

5 Concluding Remarks

In this paper, we have generalized and derived some identities of $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ using matrix method. In the future, we would like to continue working on the more generalizations of these type of the polynomials.

6 Tables

n	$\widehat{F}_n(x)$	$\widehat{L}_n(x)$
1	$4 + x^2$	$4x + x^3$
2	$4x + x^3$	$8 + 6x^2 + x^4$
3	$4 + 5x^2 + x^4$	$12x + 7x^3 + x^5$
4	$8x + 6x^3 + x^5$	$8 + 18x^2 + 8x^4 + x^6$
5	$4 + 13x^2 + 7x^4 + x^6$	$20x + 25x^3 + 9x^5 + x^7$
6	$12x + 19x^3 + 8x^5 + x^7$	$8 + 38x^2 + 33x^4 + 10x^6 + x^8$
7	$4 + 25x^2 + 26x^4 + 9x^6 + x^8$	$28x + 63x^3 + 42x^5 + 11x^7 + x^9$
8	$16x + 44x^3 + 34x^5 + 10x^7 + x^9$	$8 + 66x^2 + 96x^4 + 52x^6 + 12x^8 + x^{10}$
9	$4 + 41x^2 + 70x^4 + 43x^6 + 11x^8 + x^{10}$	$36x + 129x^3 + 138x^5 + 63x^7 + 13x^9 + x^{11}$
10	$20x + 85x^3 + 104x^5 + 53x^7 + 12x^9 + x^{11}$	$8 + 102x^2 + 225x^4 + 190x^6 + 75x^8 + 14x^{10} + x^{12}$
11	$4 + 61x^2 + 155x^4 + 147x^6 + 64x^8 + 13x^{12} + x^{12}$	$44x + 231x^3 + 363x^5 + 253x^7 + 88x^9 + 15x^{11} + x^{13}$
12	$24x + 146x^3 + 259x^5 + 200x^7 + 76x^9 + 14x^{11} + x^{13}$	$8 + 146x^2 + 456x^4 + 553x^6 + 328x^8 + 102x^{10} + 16x^{12} + x^{14}$

Table 2: **First 12 terms of $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$**

$\widehat{F}_n(x)$	$\widehat{F}_1(x)$	$\widehat{F}_2(x)$	$\widehat{F}_3(x)$	$\widehat{F}_4(x)$	$\widehat{F}_5(x)$	$\widehat{F}_6(x)$	$\widehat{F}_7(x)$	$\widehat{F}_8(x)$	$\widehat{F}_9(x)$	$\widehat{F}_{10}(x)$
$x = 1$	5	5	10	15	25	40	65	105	170	275
$x = 2$	8	16	40	96	232	560	1352	3264	7880	19024
$x = 3$	13	39	130	429	1417	4680	15457	51051	168610	556881
$x = 4$	20	80	340	1440	6100	25840	109460	463680	1964180	8320400
$x = 5$	29	145	754	3915	20329	105560	548129	2846205	14779154	76741975
$x = 6$	40	240	1480	9120	56200	346320	2134120	13151040	81040360	499393200
$x = 7$	53	371	2650	18921	135097	964600	6887297	49175679	351117050	2506995029
$x = 8$	68	544	4420	35904	291652	2369120	19244612	156326016	1269852740	10315147936
$x = 9$	85	765	6970	63495	578425	5269320	48002305	437290065	3983612890	36289806075
$x = 10$	104	1040	10504	106080	1071304	10819120	109262504	1103444160	11143704104	112540485200
$x = 11$	125	1375	15250	169125	1875625	20801000	230686625	2558353875	28372579250	314656725625
$x = 12$	148	1776	21460	259296	3133012	37855440	457398292	5526634944	66777017620	806850846384
$x = 13$	173	2249	29410	384579	5028937	65760760	859918817	11244705381	147041088770	1922778859391
$x = 14$	200	2800	39400	554400	7801000	109768400	1544558600	21733588800,	305814801800	4303140814000
$x = 15$	229	3435	51754	779745	11747929	176998680	2666728129	40177920615,	605335537354	9120210980925
$x = 16$	260	4160	66820	1073280	17239300	276902080	4447672580	71439663360	1147482286340	18431156244800
$x = 17$	293	4981	84970	1449471	24725977	421791080	7195174337	122739754809	2093771006090	35716846858339
$x = 18$	328	5904	106600	1924704	34751272	627447600	11328808072	204545992896	3693156680200	66681366236496
$x = 19$	365	6935	132130	2517405	47962825	913811080	17410373345	331710904635	6319917561410	120410144571425
$x = 20$	404	8080	162004	3248160	65125204	1305752240	26180170004	524909152320	10524363216404	211012173480400

Table 3: **First few terms of** $\widehat{F}_n(x)$

$\widehat{L}_n(x)$	$\widehat{L}_1(x)$	$\widehat{L}_2(x)$	$\widehat{L}_3(x)$	$\widehat{L}_4(x)$	$\widehat{F}_5(x)$	$\widehat{L}_6(x)$	$\widehat{L}_7(x)$	$\widehat{L}_8(x)$	$\widehat{L}_9(x)$	$\widehat{L}_{10}(x)$
$x=1$	5	15	20	35	55	90	145	235	380	615
$x=2$	16	48	112	272	656	1584	3824	9232	22288	53808
$x=3$	39	143	468	1547	5109	16874	55731	184067	607932	2007863
$x=4$	80	360	1520	6440	27280	115560	489520	2073640	8784080	3720996
$x=5$	145	783	4060	21083	109475	568458	2951765	15327283	79588180	413268183
$x=6$	240	1520	9360	57680	355440	2190320	13497360	83174480	512544240	3158439920
$x=7$	371	2703	19292	137747	983521	7022394	50140279	358004347	2556170708	18251199303
$x=8$	544	4488	36448	296072	2405024	19536264	158695136	1289097352	10471473952	85060888968
$x=9$	765	7055	64260	585395	5332815	48580730	442559385	4031615195	36727096140	334575480455
$x=10$	1040	10608	107120	1081808	10925200	110333808	1114263280	11252966608	113643929360	1147692260208
$x=11$	1375	15375	170500	1890875	20970125	232562250	2579154875	28603265875	317215079500	3517969140375
$x=12$	1776	21608	261072	3154472	38114736	460531304	5564490384	67234415912	812377481328	9815764191848
$x=13$	2249	29583	386828	5058347	66145339	864947754	11310466141	147901007587	1934023564772	25290207349623
$x=14$	2800	39600	557200	7840400	110322800	1552359600	21843357200	307359360400	4324874402800	60855600999600
$x=15$	3435	51983	783180	11799683	177778425	2678476058	40354919295	608002265483	9160388901540	138013835788583
$x=16$	4160	67080	1077440	17306120	277975360	4464911880	71716565440	1151929958920	18502595908160	297193464489480
$x=17$	4981	85263	1454452	24810947	423240551	7219900314	123161545889	2100966180427	35839586613148	611373938603943
$x=18$	5904	106928	1930608	34857872	629372304	11363559344	205173440496	3704485488272	66885912229392	1207650905617328
$x=19$	6935	132495	2524340	48094955	916328485	17458336170	332624715715	6337327934755	120741855476060	2300432581979895
$x=20$	8080	162408	3256240	65287208	1309000400	26245295208	526214904560	10550543386408	211537082632720	4241292196040808

Table 4: **First few terms of** $\widehat{L}_n(x)$

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