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FOUR NEW OPERATORS OVER THE GENERALIZED INTUITIONISTIC FUZZY SETS

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Abstract – In this paper, newly defined four level operators over generalized intuitionistic fuzzy sets ($GIFS_B$s) are proposed. Some of the basic properties of the new operators are discussed. Geometric interpretation of operators over generalized intuitionistic fuzzy sets is given.

Keywords *– Generalized intuitionistic fuzzy sets, intuitionistic fuzzy sets, level operators.*

1 Introduction

The theory of intuitionistic fuzzy sets (IFSs), proposed by Atanassov [1], and has earned successful applications in various fields. Modal operators, topological operators, level operators, negation operators and aggregation operators are different groups of operators over the IFSs due to Atanassov[1]. Atanassov [2] defined level operators $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ over the IFSs. Lupianez [3] show relations between topological operators and intuitionistic fuzzy topology. In 2008, Atanassov [4] studied some relations between intuitionistic fuzzy negations and intuitionistic fuzzy level operators $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$. Atanassov [5] introduced extended level operators over intuitionistic fuzzy sets. In 2009, Parvathi and Geetha [6] defined some level operators, max-min implication operators and $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ operators on temporal intuitionistic fuzzy sets. Atanassov [7] introduced two new operators that partially extend the intuitionistic fuzzy operators from modal type. Yilmaz and Cuvalcioglu [8] introduced level operators of temporal intuitionistic fuzzy sets. Sheik Dhavudh and Srinivasan [9] proposed level operators on intuitionistic L-fuzzy sets and establish some of their properties. The intuitionistic fuzzy operators are important from the point of view

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application. The intuitionistic fuzzy operators applied in contracting a classifier recognizing imbalanced classes, image recognition, image processing, multi-criteria decision making, deriving the similarity measure, sales analysis, new product marketing, medical diagnosis, financial services, solving optimization problems etc.. Baloui Jamkhaneh and Nadarajah [10] considered a new generalized intuitionistic fuzzy sets ($GIFS_Bs$) and introduced some operators over $GIFS_B$. Baloui Jamkhaneh and Garg [11] considered some new operations over the generalized intuitionistic fuzzy sets and their application to decision making process. In 2017 Baloui Jamkhnaeh [12] defined level operators $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ over the $GIFS_B$s. In this paper we shall introduce the sum of level operators over $GIFS_B$ and we will discuss their properties.

2 Preliminaries

In this section we will briefly remind some of the basic definition and notions of IFS which will be helpful in further study of the paper. Let X be a non-empty set.

Definition 2.1. [1] An intuitionistic fuzzy setsA in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership and non-membership functions of A respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$for each $x \in X$.

Definition 2.2. [10] A generalized intuitionistic fuzzy set ($GIFS_B$) A in X, is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$, denote the degree of membership and degree of non-membership functions of A respectively, and $0 \le \mu_A(x)^\delta + \nu_A(x)^\delta \le 1$ for each $x \in X$ where $\delta = n$ or $\frac{1}{n}$, $n = 1,2,\dots,N$. The collection of all generalized intuitionistic fuzzy sets is denoted by $GIFS_B(\delta, X)$. Let X is a universal set and F is a subset in the Euclidean plane with the Cartesian coordinates. For a $GIFS_B$ A, a function f_A from X to F can be constructed, such that if $x \in X$ then

$$(\nu_A(x), \mu_A(x)) = f_A(x) \in F, \quad 0 \le \nu_A(x),\ \mu_A(x) \le 1$$

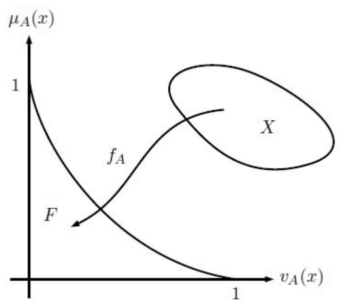

Figure 1.A geometrical interpretation of $GIFS_B$ with $\delta = 0.5$

Definition2.3. Let A and B be two $GIFS_B$s such that

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}\ ,\ B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\},$$

define the following relations and operations on A and B

i. A⊂B if and only if μ_A(x)≤ μ_B(x) and ν_A(x)≥ ν_A(x) , ∀xϵX,
ii. A=B if and only if μ_A(x) =μ_B(x) and ν_A(x)=ν_B(x) , ∀xϵX,
iii. A∪B $=\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \epsilon X\}$,
iv. A∩B $=\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \epsilon X\}$,
v. $\bar{A} = \{\langle x\, , \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

Let X is a non-empty finite set, and A ∈ $GIFS_B$, as $A = \{\langle x\, , \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Baloui Jamkhaneh and Nadarajah [10] and Baloui Jamkhaneh [12] introduced following operators of $GIFS_B$ and investigated some their properties.

i. $\Box A = \left\{\langle\, x, \mu_A(x), (1 - \mu_A(x)^{\delta})^{\frac{1}{\delta}}\rangle : x \epsilon\, X\right\}$,(modal logic: the necessity measure),
ii. $\Diamond\, A = \left\{\langle x, (1 - \nu_A(x)^{\delta})^{\frac{1}{\delta}}, \nu_A(x)\rangle : x \epsilon\, X\right\}$,(modal logic: the possibility measure),
iii. $P_{\alpha,\beta}(A) = \left\{\langle x, \max\,(\alpha^{\frac{1}{\delta}}, \mu_A(x))\,, \min\,(\beta^{\frac{1}{\delta}}, \nu_A(x))\rangle : x \epsilon\, X\right\}$, where $\alpha + \beta \leq 1$,
iv. $Q_{\alpha,\beta}(A) = \left\{\langle x, \min\,(\alpha^{\frac{1}{\delta}}, \mu_A(x))\,, \max\,(\beta^{\frac{1}{\delta}}, \nu_A(x))\rangle : x \epsilon\, X\right\}$, where $\alpha + \beta \leq 1$.

The geometrical interpretations of operators of $GIFS_B$ are shown on Fig. 2-4.

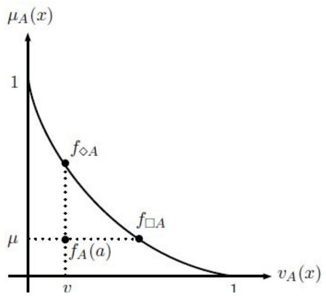

Figure 2.A geometrical interpretation of $\Diamond\, A$ and $\Box A$ with $\delta = 0.5$

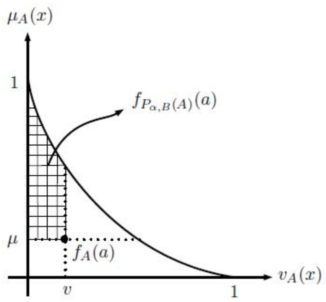

Figure 3.A geometrical interpretation of $P_{\alpha,\beta}(A)$ with $\delta = 0.5$

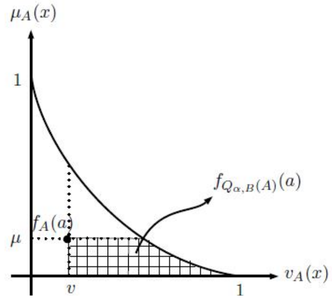

Figure 4. A geometrical interpretation of $Q_{\alpha,\beta}(A)$ with $\delta = 0.5$

3 Main Results

Here, we will introduce new operators over the $GIFS_B$, which extend some operators in the research literature related to IFSs. Let X is a non-empty finite set.

Definition 3.1. Letting $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$. For every $GIFS_B$ as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, we define the level operators as follows:

i. $P^{(1)}_{\alpha,\beta}(A) = \left\{\langle x, \min\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1 - \nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \min\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right)\rangle : x \epsilon X\right\}$,

ii. $P^{(2)}_{\alpha,\beta}(A) = \left\{\langle x, \max\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1 - \nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \min\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right)\rangle : x \epsilon X\right\}$,

iii. $Q^{(1)}_{\alpha,\beta}(A) = \left\{\langle x, \min\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \min\left(\max\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right), \left(1 - \mu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \rangle : x \epsilon X\right\}$,

iv. $Q^{(2)}_{\alpha,\beta}(A) = \left\{\langle x, \min\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \max\left(\max\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right), \left(1 - \mu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \rangle : x \epsilon X\right\}$.

The geometrical interpretations of new operators of $GIFS_B$ are shown on Fig. 5-8.

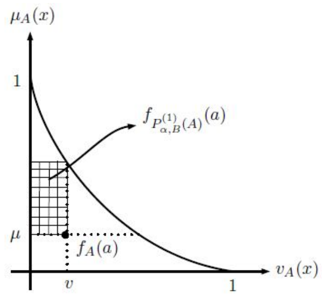

Figure 5. A geometrical interpretation of $P^{(1)}_{\alpha,\beta}(A)$ with $\delta = 0.5$

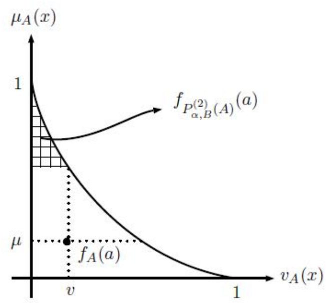

Figure 6. A geometrical interpretation of $P^{(2)}_{\alpha,\beta}(A)$ with $\delta = 0.5$

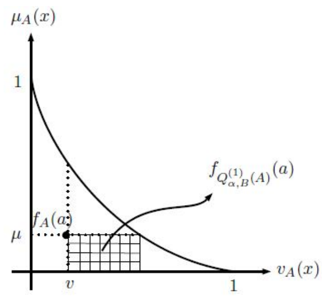

Figure 7.A geometrical interpretation of $Q^{(1)}_{\alpha,\beta}(A)$ with $\delta = 0.5$

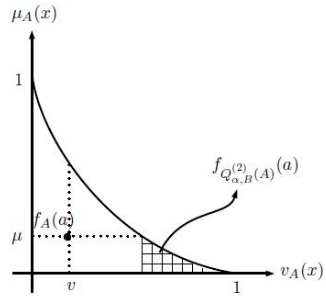

Figure 8.A geometrical interpretation of $Q^{(2)}_{\alpha,\beta}(A)$ with $\delta = 0.5$

Corollary 3.2. According to definition of new operators and geometrical interpretations of them, we have

i. If $A = \{\langle x, 1, 0\rangle : x \in X\}$, then $P_{\alpha,\beta}(A) = P^{(1)}_{\alpha,\beta}(A) = P^{(2)}_{\alpha,\beta}(A) = \{\langle x, 1, 0\rangle : x \in X\}$,

ii. If $A = \{\langle x, 0, 1\rangle : x \in X\}$, then $Q_{\alpha,\beta}(A) = Q^{(1)}_{\alpha,\beta}(A) = Q^{(2)}_{\alpha,\beta}(A) = \{\langle x, 0, 1\rangle : x \in X\}$.

Remark 3.3. If $\mu_A(x) \geq \alpha^{\frac{1}{\delta}}$ and $\nu_A(x) \leq \beta^{\frac{1}{\delta}}$ then $P^{(1)}_{\alpha,\beta}(A) = A$, $P^{(2)}_{\alpha,\beta}(A) = \Diamond A$ and if $\mu_A(x) \leq \alpha^{\frac{1}{\delta}}$ and $\nu_A(x) \geq \beta^{\frac{1}{\delta}}$ then $Q^{(1)}_{\alpha,\beta}(A) = A$, $Q^{(2)}_{\alpha,\beta}(A) = \Box A$.

Remark 3.4. If $\alpha_1 \leq \alpha_2$ then $P^{(i)}_{\alpha_1,\beta}(A) \subseteq P^{(i)}_{\alpha_2,\beta}(A)$ and $Q^{(i)}_{\alpha_1,\beta}(A) \subseteq Q^{(i)}_{\alpha_2,\beta}(A)$, also if $\beta_1 \leq \beta_2$ then $P^{(i)}_{\alpha,\beta_2}(A) \subseteq P^{(i)}_{\alpha,\beta_1}(A)$ and $Q^{(i)}_{\alpha,\beta_2}(A) \subseteq Q^{(i)}_{\alpha,\beta_1}(A)$, i=1,2.

Theorem 3.5. For every $A \in GIFS_B$ and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, we have

i. $P^{(i)}_{\alpha,\beta}(A) \in GIFS_B$, i=1,2,

ii. $Q^{(i)}_{\alpha,\beta}(A) \in GIFS_B$, i=1,2,

iii. $\overline{P^{(i)}_{\alpha,\beta}(\bar{A})} = Q^{(i)}_{\beta,\alpha}(A)$, i=1,2.

Proof. (i) Let i=1

$$\mu_{P^{(i)}_{\alpha,\beta}(A)}(x)^\delta + \nu_{P^{(i)}_{\alpha,\beta}(A)}(x)^\delta =$$

$$(\min(\max(\alpha^{\frac{1}{\delta}}, \mu_A(x)), (1-\nu_A(x)^\delta)^{\frac{1}{\delta}}))^\delta + (\min(\beta^{\frac{1}{\delta}}, \nu_A(x)))^\delta =$$

$$\min(\max(\alpha, \mu_A(x)^\delta), (1-\nu_A(x)^\delta)) + (\min(\beta, \nu_A(x)^\delta) = I.$$

If $\max(\alpha, \mu_A(x)^\delta) \leq (1-\nu_A(x)^\delta)$ then

(1) If $\max\left(\alpha, \mu_A(x)^\delta\right) = \alpha$ and $\min\left(\beta, \nu_A(x)^\delta\right) = \beta$ then

$$I = \alpha + \beta \leq 1.$$

(2) If $\max\left(\alpha, \mu_A(x)^\delta\right) = \alpha$ and $\min\left(\beta, \nu_A(x)^\delta\right) = \nu_A(x)^\delta$ then

$$I = \alpha + \nu_A(x)^\delta \leq \alpha + \beta \leq 1,$$

(3) If $\max\left(\alpha, \mu_A(x)^\delta\right) = \mu_A(x)^\delta$ and $\min\left(\beta, \nu_A(x)^\delta\right) = \beta$ then

$$I = \mu_A(x)^\delta + \beta \leq \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1,$$

(4) If $\max\left(\alpha, \mu_A(x)^\delta\right) = \mu_A(x)^\delta$ and $\min\left(\beta, \nu_A(x)^\delta\right) = \nu_A(x)^\delta$ then

$$I = \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1.$$

If $\max\left(\alpha, \mu_A(x)^\delta\right) \geq \left(1 - \nu_A(x)^\delta\right)$ then

(1) If $\min\left(\beta, \nu_A(x)^\delta\right) = \beta$ then $I = 1 - \nu_A(x)^\delta + \beta \leq 1 - \nu_A(x)^\delta + \nu_A(x)^\delta = 1,$

(2) If $\min\left(\beta, \nu_A(x)^\delta\right) = \nu_A(x)^\delta$ then $I = 1 - \nu_A(x)^\delta + \nu_A(x)^\delta = 1.$

The proof is completed. Proof of (ii) is similar to that of (i).

(iii) Let i=1

$\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$

$$P^{(1)}_{\alpha,\beta}(\overline{A}) = \left\{\langle x, \min\left(\max\left(\alpha^{\frac{1}{\delta}}, \nu_A(x)\right), \left(1 - \mu_A(x)^\delta\right)^{\frac{1}{\delta}}\right), \min\left(\beta^{\frac{1}{\delta}}, \mu_A(x)\right)\rangle : x \in X\right\},$$

$$\overline{P^{(1)}_{\alpha,\beta}(\overline{A})} = \left\{\langle x, \min\left(\beta^{\frac{1}{\delta}}, \mu_A(x)\right), \min\left(\max\left(\alpha^{\frac{1}{\delta}}, \nu_A(x)\right), \left(1 - \mu_A(x)^\delta\right)^{\frac{1}{\delta}}\right)\rangle : x \in X\right\} = Q^{(1)}_{\beta,\alpha}(A).$$

The proof is completed.

Theorem 3.6. For every $A \in GIFS_B$, we have

i. $P^{(1)}_{0,1}(A) = A,$
ii. $P^{(2)}_{0,1}(A) = P^{(1)}_{1,1}(A) = \lozenge A,$
iii. $Q^{(1)}_{1,0}(A) = A,$
iv. $Q^{(2)}_{1,0}(A) = Q^{(1)}_{1,1}(A) = \Box A.$

Proof. Proofs are obvious.

Theorem 3.7. For every $A \in GIFS_B$ and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, we have

i. $P^{(1)}_{\alpha,\beta}(A) \subset P_{\alpha,\beta}(A) \subset P^{(2)}_{\alpha,\beta}(A),$
ii. $Q^{(2)}_{\alpha,\beta}(A) \subset Q_{\alpha,\beta}(A) \subset Q^{(1)}_{\alpha,\beta}(A),$

iii. $Q^{(1)}_{\alpha,\beta}(A) \subset P^{(1)}_{\alpha,\beta}(A)$,
iv. $Q^{(i)}_{\alpha,\beta}(A) \subset A$, i=1,2,
v. $A \subset P^{(i)}_{\alpha,\beta}(A)$, i=1,2.

Proof. (i) Since

$$\min\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right) \leq \max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right) \text{then } P^{(1)}_{\alpha,\beta}(A) \subset P_{\alpha,\beta}(A),$$

and

$$\max\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right) \geq \max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right) \text{ then } P_{\alpha,\beta}(A) \subset P^{(2)}_{\alpha,\beta}(A),$$

therefore, we have $P^{(1)}_{\alpha,\beta}(A) \subset P_{\alpha,\beta}(A) \subset P^{(2)}_{\alpha,\beta}(A)$.

The proof is completed. (ii)-(v) are proved analogically.

Corollary 3.8. According to definition of new operators and Theorem3.7 (iv)-(v), the operators of $\mathrm{P}^{(\mathrm{i})}_{\alpha,\beta}(\mathrm{A}), \mathrm{i} = 1,2,$ increases the membership degree of A and reduces non-membership degree of A, the operators of $\mathrm{Q}^{(\mathrm{i})}_{\alpha,\beta}(\mathrm{A}), \mathrm{i} = 1,2,$ reduces the membership degree of A and increases non-membership degree of A.

Theorem 3.9. For every $\mathrm{A}, \mathrm{B} \in \mathrm{GIFS_B}$, we have
i. $P^{(i)}_{\alpha,\beta}(A \cup B) = P^{(i)}_{\alpha,\beta}(A) \cup P^{(i)}_{\alpha,\beta}(B)$, i=1,2,
ii. $P^{(i)}_{\alpha,\beta}(A \cap B) = P^{(i)}_{\alpha,\beta}(A) \cap P^{(i)}_{\alpha,\beta}(B)$, i=1,2,
iii. $Q^{(i)}_{\alpha,\beta}(A \cup B) = Q^{(i)}_{\alpha,\beta}(A) \cup Q^{(i)}_{\alpha,\beta}(B)$, i=1,2,
iv. $Q^{(i)}_{\alpha,\beta}(A \cap B) = Q^{(i)}_{\alpha,\beta}(A) \cap Q^{(i)}_{\alpha,\beta}(B)$. i=1,2.

Proof. (i) Let i=1

$$\begin{aligned}
P^{(1)}_{\alpha,\beta}(\mathrm{A} \cup \mathrm{B}) &= \Big\{\langle x, \min\big(\max\big(\alpha^{\frac{1}{\delta}}, \max(\mu_A(x), \mu_B(x))\big), (1 \\
&\qquad - \min(\nu_A(x), \nu_B(x))^{\delta})^{\frac{1}{\delta}}, \min\big(\beta^{\frac{1}{\delta}}, \min(\nu_A(x), \nu_B(x))\big)\rangle : x \epsilon X\Big\}, \\
&= \left\{\langle x, \min\left(\max\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \max\left(\alpha^{\frac{1}{\delta}}, \mu_B(x)\right)\right), max\left(\left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(1-\nu_B(x)^{\delta}\right)^{\frac{1}{\delta}}\right)\right), \min\left(\min\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right), \min\left(\beta^{\frac{1}{\delta}}, \nu_B(x)\right)\right)\rangle : x \epsilon X\right\}, \\
&= \left\{\langle x, \min\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \min\left(\beta^{\frac{1}{\delta}}, \nu_A(x)\right), \rangle : x \epsilon X\right\} \cup \\
&\quad \left\{\langle x, \min\left(\max\left(\alpha^{\frac{1}{\delta}}, \mu_B(x)\right), \left(1-\nu_B(x)^{\delta}\right)^{\frac{1}{\delta}}\right), \min\left(\beta^{\frac{1}{\delta}}, \nu_B(x)\right), \rangle : x \epsilon X\right\}, \\
&= P^{(1)}_{\alpha,\beta}(A) \cup P^{(1)}_{\alpha,\beta}(B).
\end{aligned}$$

(ii) Let i=1

$P_{\alpha,\beta}^{(1)}(\mathrm{A} \cap \mathrm{B}) =$
$\left\{\langle x, \min\left(\max\left(\alpha^{\frac{1}{\delta}}, min\left(\mu_A(x), \mu_B(x)\right)\right), (1 - \max(\nu_A(x), \nu_B(x))^{\delta})^{\frac{1}{\delta}}\right), \min\left(\beta^{\frac{1}{\delta}}, \max(\nu_A(x), \nu_B(x))\right)\rangle : x\epsilon\, X\right\},$

$= \left\{\langle x, \min(\min(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \max\left(\alpha^{\frac{1}{\delta}}, \mu_B(x)\right)), min\left(\left(1 - \nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(1 - \nu_B(x)^{\delta}\right)^{\frac{1}{\delta}}\right)\ ), \max(\min(\beta^{\frac{1}{\delta}}, \nu_A(x)), \min(\beta^{\frac{1}{\delta}}, \nu_B(x))\rangle : x\epsilon\, X\right\},$

$= \left\{\langle x, \min(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1 - \nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}), \min(\beta^{\frac{1}{\delta}}, \nu_A(x)), \rangle : x\epsilon\, X\right\} \cap$
$\left\{\langle x, \min(\max\left(\alpha^{\frac{1}{\delta}}, \mu_B(x)\right), \min(\beta^{\frac{1}{\delta}}, \nu_B(x))), \min(\beta^{\frac{1}{\delta}}, \nu_B(x)), \rangle : x\epsilon\, X\right\},$

$= P_{\alpha,\beta}^{(1)}(A) \cap P_{\alpha,\beta}^{(1)}(B),$

The proof is completed. Proofs of (iii) and (iv) are similar to that of (i) and (ii).

Corollary 3.10. For every $\mathrm{A_j} \in \mathrm{GIFS_B}$, $\mathrm{j} = 1, \ldots \mathrm{n}$, we have

i. $P_{\alpha,\beta}^{(i)}\left(\bigcup_{j=1}^{n} A_j\right) = \bigcup_{j=1}^{n} P_{\alpha,\beta}^{(i)}\left(A_j\right)$, i=1,2,
ii. $P_{\alpha,\beta}^{(i)}\left(\bigcap_{j=1}^{n} A_j\right) = \bigcap_{j=1}^{n} P_{\alpha,\beta}^{(i)}\left(A_j\right)$, i=1,2,
iii. $Q_{\alpha,\beta}^{(i)}\left(\bigcup_{j=1}^{n} A_j\right) = \bigcup_{j=1}^{n} Q_{\alpha,\beta}^{(i)}\left(A_j\right)$, i=1,2,
iv. $Q_{\alpha,\beta}^{(i)}\left(\bigcap_{j=1}^{n} A_j\right) = \bigcap_{j=1}^{n} Q_{\alpha,\beta}^{(i)}\left(A_j\right)$, i=1,2.

Theorem 3.11. For every $\mathrm{A}, \mathrm{B} \in \mathrm{GIFS_B}$, where $\mathrm{A} \subseteq \mathrm{B}$ we have

i. $P_{\alpha,\beta}^{(i)}(A) \subseteq P_{\alpha,\beta}^{(i)}(B)$,
ii. $Q_{\alpha,\beta}^{(i)}(A) \subseteq Q_{\alpha,\beta}^{(i)}(B)$.

Proof. (i) Let i=1, since $\mathrm{A} \subseteq \mathrm{B}$ then $\mu_\mathrm{A}(\mathrm{x}) \leq \mu_\mathrm{B}(\mathrm{x})$ and $\nu_\mathrm{A}(\mathrm{x}) \geq \nu_\mathrm{B}(\mathrm{x})$ therefore

$\max(\alpha^{\frac{1}{\delta}}, \mu_A(x) \leq \max(\alpha^{\frac{1}{\delta}}, \mu_B(x))$, $\left(1 - \nu_B(x)^{\delta}\right)^{\frac{1}{\delta}} \geq \left(1 - \nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}$

and

$\min(\beta^{\frac{1}{\delta}}, \nu_A(x)) \geq \min(\beta^{\frac{1}{\delta}}, \nu_B(x)).$

Finally

$\mu_{P_{\alpha,\beta}^{(1)}(A)}(x) = \min(\max\left(\alpha^{\frac{1}{\delta}}, \mu_A(x)\right), \left(1 - \nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}),$
$\leq \mu_{P_{\alpha,\beta}^{(1)}(B)}(x) = \min(\max\left(\alpha^{\frac{1}{\delta}}, \mu_B(x)\right), \min(\beta^{\frac{1}{\delta}}, \nu_B(x))),$

and similarly, we have$\nu_{P^{(1)}_{\alpha,\beta}(B)}(x) \leq \nu_{P^{(1)}_{\alpha,\beta}(A)}(x)$.

Proof is complete. Proof of (ii) is similar to that of (i).

Example 3.12. Let $A = \{\langle x, 0.36, 0.09\rangle\}, \delta = 0.5$, then

$$\mu_{P_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36 \ , & \alpha \leq 0.6 \\ \alpha^2 \ , & \alpha > 0.6 \end{cases}, \upsilon_{P_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09 \ , & \beta \geq 0.3 \\ \beta^2 \ , & \beta < 0.3 \end{cases},$$

$$\mu_{Q_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36 \ , & \alpha > 0.6 \\ \alpha^2 \ , & \alpha \leq 0.6 \end{cases}, \upsilon_{Q_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09 \ , & \beta < 0.3 \\ \beta^2 \ , & \beta \geq 0.3 \end{cases},$$

$$\mu_{P^{(1)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36 \ , & \alpha \leq 0.6 \\ \alpha^2 \ , & 0.6 < \alpha < 0.7 \\ 0.49 \ , & 0.7 \leq \alpha \end{cases}, \upsilon_{P^{(1)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09 \ , & \beta \geq 0.3 \\ \beta^2 \ , & \beta < 0.3 \end{cases},$$

$$\mu_{P^{(2)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.49 \ , & \alpha < 0.7 \\ \alpha^2 \ , & \alpha \geq 0.7 \end{cases}, \upsilon_{P^{(2)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09 \ , & \beta \geq 0.3 \\ \beta^2 \ , & \beta < 0.3 \end{cases},$$

$$\mu_{Q^{(1)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36 \ , & \alpha > 0.6 \\ \alpha^2 \ , & \alpha \leq 0.6 \end{cases}, \upsilon_{Q^{(1)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.09 \ , & \beta < 0.3 \\ \beta^2 \ , & 0.3 < \beta \leq 0.4, \\ 0.16 \ , & 0.4 < \beta \end{cases}$$

$$\mu_{Q^{(2)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.36 \ , & \alpha > 0.6 \\ \alpha^2 \ , & \alpha \leq 0.6 \end{cases}, \upsilon_{Q^{(2)}_{\alpha,\beta}(A)}(x) = \begin{cases} 0.16 \ , & \beta < 0.4 \\ \beta^2 \ , & \beta \geq 0.4 \end{cases},$$

where $\alpha + \beta \leq 1$.

4 Conclusions

We have introduced four level operators over $GIFS_B$s and showed geometrical interpretation of new operators in the generalized intuitionistic fuzzy sets. Also we proved their relationships and showed that these operators are $GIFS_B$. These operators are well defined since, if $\delta = 1$, the results agree with IFS.

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