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CHARACTERIZATIONS OF INTUITIONISTIC FUZZY SUBSEMIRINGS OF SEMIRINGS AND THEIR HOMOMORPHISMS BY NORMS

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Abstaract − In this paper, we introduce the notion of intuitionistic fuzzy subsemirings, level subsets of of intuitionistic fuzzy subsemirings, intersection and direct sum of intuitionistic fuzzy subsemirings under norms and investigate many properties of them. We also made an attempt to study the characterizations of them under homomorphism and anti-homomorphism.

Keywords − *Ring theory, norms, fuzzy set theory, intuitionistic fuzzy subsemirings, homomorphisms, anti-homomorphisms, direct sum.*

1 Introduction

In abstract algebra, a semiring is an algebraic structure similar to a ring, but without the requirement that each element must have an additive inverse. After the introduction of fuzzy sets by Zadeh [26], a number of generalizations of this fundamental concept have come up. Algebraic structures play a vital role in Mathematics and numerous applications of these structures are seen in many disciplines such as computersciences, information sciences, theoretical physics, control engineering and so on. This inspires researchers to study and carry out research in various concepts of abstract algebra in fuzzy setting. There are natural ways to fuzzify various algebraic structures and it has been done successfully by many mathematicians. For instance, Rosenfeld [23] is the father of fuzzy abstract algebra and the reader may consult the papers [12] or [13] about fuzzy semigroups; [11], [10], [15], [24] or [27] about fuzzy ideals and fuzzy rings; [14] or [17] about fuzzy modules; [16] about fuzyy vector spaces; [7] about fuzzy coalgebras over a field; [25] about Lie algebras, and so on. In 1993, Ahsan et al. [1] introduced the notion of fuzzy semirings. In1994, Dutta and Biswas [8] characterized fuzzy prime ideals of a semiring. Recently, many results of semiring theory are investigated by many researchers in fuzzy context. The notion of intuitionistic fuzzy sets introduced by Atanassov [3] (also see [4], [5]) is one among them. Biswas [6] applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. Norms originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of norms. Later, Hohle [9], Alsina et al. [2] introduced the norms into fuzzy set theory and suggested that norms be used for the intersection of fuzzy sets. The author by using norms, investigated some properties of fuzzy submodules, fuzzy subrings, fuzzy ideals of subtraction semigroups, intuitionistic fuzzy subrings and ideals of a ring, fuzzy Lie algebra (See [18, 19, 20, 21, 22]).

In this paper, we introduce the notions of intuitionistic fuzzy subsemirings of a semiring with respect to norms and establish necessary and sufficient conditions for them. We also investigate the algebraic nature of such type of them under intersection, direct some, homomorphism and anti-homomorphism.

2 Preliminary

Definition 2.1. A semiring is a set R equipped with two binary operations " + " and "." called addition and multiplication, such that:

(1) $(R, +)$ is a commutative monoid with identity element 0:
(a) $(a + b) + c = a + (b + c)$,
(b) $0 + a = a + 0 = a$,
(c) $a + b = b + a$.
(2) $(R, .)$ is a monoid with identity element 1:
(a) $(a.b).c = a.(b.c)$,
(b) $1.a = a.1 = a$.
(3) Multiplication left and right distributes over addition:
(a) $a.(b + c) = (a.b) + (a.c)$,
(b) $(a + b).c = (a.c) + (b.c)$.
(4) Multiplication by 0 annihilates R: $0.a = a.0 = 0$.

This last axiom is omitted from the definition of a ring: it follows from the other ring axioms. Here it does not, and it is necessary to state it in the definition. The difference between rings and semirings, then, is that addition yields only a commutative monoid, not necessarily a commutative group. Specifically, elements in semirings do not necessarily have an inverse for the addition. The symbol . is usually omitted from the notation; that is, $a.b$ is just written ab. Similarly, an order of operations is accepted, according to which . is applied before +; that is, $a+bc$ is $a+(bc)$.

A commutative semiring is one whose multiplication is commutative. An idempotent semiring is one whose addition is idempotent: $a + a = a$, that is, $(R, +, 0)$ is a join-semilattice with zero.

Example 2.2. (1) By definition, any ring is also a semiring. A motivating example of a semiring is the set of natural numbers $\mathbb{N}$ (including zero) under ordinary addition and multiplication. Likewise, the non-negative rational numbers and the non-negative real numbers form semirings. All these semirings are commutative.

(2) The set of all ideals of a given ring form a semiring under addition and multiplication of ideals.

(3) Any unital quantale is an idempotent semiring.

(4) Any bounded, distributive lattice is a commutative, idempotent semiring under join and meet.

Definition 2.3. Let R be a semiring. A nonempty subset S of R is a subsemiring of R if and only if $x+y \in S$ and $xy \in S$ for all $x, y \in S$.

Definition 2.4. A t-norm T is a function $T : [0,1] \times [0,1] \rightarrow [0,1]$ having the following four properties: For all $x, y, z \in [0,1]$;

(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1,y_1) \geq T(x_2,y_2)$.

Example 2.5. (1) Standard intersection T-norm $T_m(x,y) = \min\{x,y\}$.

(2) Bounded sum T-norm $T_b(x,y) = \max\{0, x+y-1\}$.

(3) algebraic product T-norm $T_p(x,y) = xy$.

(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x, y \in [0,1]$.

Definition 2.6. A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties: For all $x, y, z \in [0,1]$;

(C1) $C(x,0) = x$,
(C2) $C(x,y) \le C(x,z)$ if $y \le z$,
(C3) $C(x,y) = C(y,x)$,
(C4) $C(x, C(y,z)) = C(C(x,y), z)$,

Example 2.7. (1) Standard union t-conorm $C_m(x,y) = \max\{x,y\}$.
(2) Bounded sum t-conorm $C_b(x,y) = \min\{1, x+y\}$.
(3) Algebraic sum t-conorm $C_p(x,y) = x + y - xy$.
(4) Drastic T-conorm

$$C_D(x,y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic T-norm.
(5) Nilpotent maximum T-conorm , dual to the nilpotent minimum T-norm:

$$C_{nM}(x,y) = \begin{cases} \max\{x,y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x,y) = \dfrac{x+y}{1+xy}$ is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x,y) \le C(x,y) \le C_D(x,y)$ for any t-conorm C and all $x, y \in [0,1]$.

Recall that t-norm T(t-conorm C) is idempotent if for all $x \in [0,1]$, $T(x,x) = x(C(x,x) = x)$.

Definition 2.8. For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 2.9. Let Let φ be a function from set X into set Y such that $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in X and Y respectively.
For all $x \in X, y \in Y$, we define

$$\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y)) =$$

$$= \begin{cases} (\sup\{\mu_A(x) \mid x \in R, \varphi(x) = y\}, \inf\{\nu_A(x) \mid x \in R, \varphi(x) = y\}), & \text{if } \varphi^{-1}(y) \ne \emptyset \\ (0,1), & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x)))$.

Lemma 2.10. Let T be a t-norm. Then

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Lemma 2.11. Let C be a t-conorm. Then

$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z))$$

for all $x, y, w, z \in [0,1]$

Definition 2.12. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$, respectively.

We will denote the set of all $IFSs$ in X as $IFS(X)$.

Definition 2.13. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IFSs$ in X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.

Definition 2.14. If A is intuitionistic fuzzy subset of R, then the sets $\{< x, \mu_A(x) > \mid x \in R\}$ and $\{< x, \nu_A(x) > \mid x \in R\}$, are called fuzzy subset and anti-fuzzy subset of R with respect to intuitionistic fuzzy set A. For $\alpha, \beta \in [0,1]$, we define the following sets

(1) $U_1(A,\alpha) = \{x \in R \mid \mu_A(x) \geq \alpha\}$,
(2) $U_2(A,\alpha) = \{x \in R \mid \nu_A(x) \geq \alpha\}$,
(3) $L_1(A,\beta) = \{x \in R \mid \mu_A(x) \leq \beta\}$,
(4) $L_2(A,\beta) = \{x \in R \mid \nu_A(x) \leq \beta\}$
(5) $C_{\alpha,\beta} = \{x \in R \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$.

The sets $U_1(A,\alpha)$ and $L_1(A,\beta)$ are respectively called the upper α-level cut and lower β-level cut of the fuzzy subset of R w.r.t. $IFSA$ and the sets $U_2(A,\alpha)$ and $L_2(A,\beta)$ are respectively called the upper α-level cut and lower β-level cut of the anti-fuzzy subset of R w.r.t. $IFSA$.

Definition 2.15. Let R and S be any two semirings and $f : R \to S$ be a function:

(1) f is called a homomorphism if $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$.

(2) f is called an anti-homomorphism if $f(x+y) = f(x)+f(y)$ and $f(xy) = f(y)f(x)$ for all $x, y \in R$.

3 Level Subsets of Intuitionistic Fuzzy Subsemiring af a Semiring with Respect to Norms

Definition 3.1. Let R be a semiring. An $A = (\mu_A, \nu_A)$ is said to be intuitionistic fuzzy subsemiring with respect to norms(a t-norm T and a t-conorm C) (in short, $IFSN(R)$) of R if

(1) $\mu_A(x + y) \geq T(\mu_A(x), \mu_A(y))$
(2) $\mu_A(xy) \geq T(\mu_A(x), \mu_A(y))$
(3) $\nu_A(x + y) \leq C(\nu_A(x), \nu_A(y))$
(4) $\nu_A(xy) \leq C(\nu_A(x), \nu_A(y))$,

for all $x, y \in R$.

Example 3.2. Let $R = (\mathbb{Z}, +, .)$ be a semiring of integer. For all $x \in R$ we define a fuzzy subset μ_A and ν_A of R as

$$\mu_A(x) = \begin{cases} 0.75 & \text{if } x \in \{0, \pm2, \pm4, ...\} \\ 0.60 & \text{if } x \in \{\pm1, \pm3, ...\} \end{cases}$$

$$\nu_A(x) = \begin{cases} 0.35 & \text{if } x \in \{0, \pm2, \pm4, ...\} \\ 0.55 & \text{if } x \in \{\pm1, \pm3, ...\} \end{cases}$$

Let $T(x, y) = T_p(x, y) = xy$ and $C(x, y) = C_p(x, y) = x + y - xy$ for all $x, y \in R$, then $A = (\mu_A, \nu_A) \in IFSN(R)$.

Proposition 3.3. Let $A \in IFSN(R)$ and T, C be idempotent. If $\alpha, \beta \in [0, 1]$, then $C_{\alpha,\beta}$ is a subsemiring of R.

Proof. If $x, y \in C_{\alpha,\beta}$, then $\mu_A(x), \mu_A(y) \geq \alpha$ and $\nu_A(x), \nu_A(y) \leq \beta$. Now

(1) $\mu_A(x + y) \geq T(\mu_A(x), \mu_A(y)) \geq T(\alpha, \alpha) = \alpha$.
(2) $\mu_A(xy) \geq T(\mu_A(x), \mu_A(y)) \geq T(\alpha, \alpha) = \alpha$.
(3) $\nu_A(x + y) \leq C(\nu_A(x), \nu_A(y)) \leq C(\beta, \beta) = \beta$.
(4) $\nu_A(xy) \leq C(\nu_A(x), \nu_A(y)) \leq C(\beta, \beta) = \beta$.

Thus $x + y, xy \in C_{\alpha,\beta}$ and therefore $C_{\alpha,\beta}$ is a subsemiring of R.

Proposition 3.4. Let R be a semiring and $A \in IFS(R)$. If T, C be idempotent and $C_{\alpha,\beta}$ be a subsemiring of R for all $\alpha, \beta \in [0, 1]$, then $A \in IFSN(R)$.

Proof. Let $x, y \in R$ and for C_{α_i,β_i} with $i = 1, 2$ we have $\mu_A(x) = \alpha_1$, $\mu_A(y) = \alpha_2$, $\nu_A(x) = \beta_1$ and $\nu_A(y) = \beta_2$ such that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$. Since $x, y \in C_{\alpha_i,\beta_i}$ and C_{α_i,β_i} is a subsemiring of R so $x + y, xy \in C_{\alpha_i,\beta_i}$. Now we prove that $A \in IFSN(R)$ in the following conditions.

(a) Let $\alpha_1 > \alpha_2$ and $\beta_1 < \beta_2$ such that $x, y \in C_{\alpha_1,\beta_1}$. Then

(1) $\mu_A(x + y) \geq \alpha_1 = T(\alpha_1, \alpha_1) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) \geq \alpha_1 = T(\alpha_1, \alpha_1) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x+y) \leq \beta_1 = C(\beta_1, \beta_1) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) \leq \beta_1 = C(\beta_1, \beta_1) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(b) Let $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ such that $x, y \in C_{\alpha_2, \beta_1}$. Then

(1) $\mu_A(x+y) \geq \alpha_2 = T(\alpha_2, \alpha_2) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) \geq \alpha_2 = T(\alpha_2, \alpha_2) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x+y) \leq \beta_1 = C(\beta_1, \beta_1) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) \leq \beta_1 = C(\beta_1, \beta_1) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(c) Let $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$ such that $x, y \in C_{\alpha_1, \beta_2}$. Then

(1) $\mu_A(x+y) \geq \alpha_1 = T(\alpha_1, \alpha_1) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) \geq \alpha_1 = T(\alpha_1, \alpha_1) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x+y) \leq \beta_2 = C(\beta_2, \beta_2) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) \leq \beta_2 = C(\beta_2, \beta_2) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(d) Let $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$ such that $x, y \in C_{\alpha_2, \beta_2}$. Then

(1) $\mu_A(x+y) \geq \alpha_2 = T(\alpha_2, \alpha_2) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) \geq \alpha_2 = T(\alpha_2, \alpha_2) \geq T(\alpha_1, \alpha_2) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x+y) \leq \beta_2 = C(\beta_2, \beta_2) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) \leq \beta_2 = C(\beta_2, \beta_2) \leq C(\beta_1, \beta_2) = C(\nu_A(x), \nu_A(y))$.

Thus from (a) to (d) we get that $A \in IFSN(R)$.

Proposition 3.5. Let R be a semiring and $A \in IFS(R)$ defined by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0 & \text{if } x \in H \\ 1 & \text{if } x \notin H. \end{cases}$$

If H is a subsemiring of R and T, C be idempotent, then $A \in IFSN(R)$.

Proof. Let $x, y \in R$ and H is a subsemiring of R. Then

(a) If $x, y \in H$, then $x + y, xy \in H$ and we have:

(1) $\mu_A(x + y) = 1 \geq 1 = T(1, 1) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) = 1 \geq 1 = T(1, 1) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x + y) = 0 \leq= 0 = C(0, 0) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) = 0 \leq 0 = C(0, 0) = C(\nu_A(x), \nu_A(y))$.

(b) If $x \in H$ and $y \notin H$, then $x + y, xy \notin H$ and then:

(1) $\mu_A(x + y) = 0 \geq 0 = T(1, 0) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) = 0 \geq 0 = T(1, 0) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x + y) = 1 \leq 1 = C(0, 1) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) = 1 \leq 1 = C(0, 1) = C(\nu_A(x), \nu_A(y))$.

(c) If $x, y \notin H$, then $x + y, xy \notin H$ and so:

(1) $\mu_A(x + y) = 0 \geq 0 = T(0, 0) = T(\mu_A(x), \mu_A(y))$.

(2) $\mu_A(xy) = 0 \geq 0 = T(0, 0) = T(\mu_A(x), \mu_A(y))$.

(3) $\nu_A(x + y) = 1 \leq 1 = C(1, 1) = C(\nu_A(x), \nu_A(y))$.

(4) $\nu_A(xy) = 1 \leq 1 = C(1, 1) = C(\nu_A(x), \nu_A(y))$.

Now fram (a) to (c) we obtain that $A \in IFSN(R)$.

Definition 3.6. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in semiring R. Define $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B})$ as $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = C(\nu_A(x), \nu_B(y))$ for all $x \in R$.

Proposition 3.7. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in semiring R. If $A, B \in IFSN(R)$, then $(A \cap B) \in IFSN(R)$.

Proof. Let $x, y \in R$. Then

$$\begin{aligned}(1)\ \mu_{A\cap B}(x + y) &= T(\mu_A(x + y), \mu_B(x + y)) \\ &\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))) \\ &= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y))) \\ &= T(\mu_{A\cap B}(x), \mu_{A\cap B}(y))\end{aligned}$$

$$\begin{aligned}(2)\ \mu_{A\cap B}(xy) &= T(\mu_A(xy),\mu_B(xy))\\ &\geq T(T(\mu_A(x),\mu_A(y)),T(\mu_B(x),\mu_B(y)))\\ &= T(T(\mu_A(x),\mu_B(x)),T(\mu_A(y),\mu_B(y)))\\ &= T(\mu_{A\cap B}(x),\mu_{A\cap B}(y))\end{aligned}$$

$$\begin{aligned}(3)\ \nu_{A\cap B}(x+y) &= C(\nu_A(x+y),\nu_B(x+y))\\ &\leq T(T(\mu_A(x),\mu_A(y)),T(\mu_B(x),\mu_B(y)))\\ &= C(C(\nu_A(x),\nu_A(y)),C(\nu_B(x),\nu_B(y)))\\ &= C(\nu_{A\cap B}(x),\nu_{A\cap B}(y))\end{aligned}$$

$$\begin{aligned}(4)\ \nu_{A\cap B}(xy) &= C(\nu_A(xy),\nu_B(xy))\\ &\leq T(T(\mu_A(x),\mu_A(y)),T(\mu_B(x),\mu_B(y)))\\ &= C(C(\nu_A(x),\nu_A(y)),C(\nu_B(x),\nu_B(y)))\\ &= C(\nu_{A\cap B}(x),\nu_{A\cap B}(y))\end{aligned}$$

Therefore $(A\cap B)\in IFSN(R)$.

Corollary 3.8. Let $\{A_i=(\mu_{A_i},\nu_{A_i})\mid i=1,2,3,...,n\}\subseteq IFSN(R)$. Then so does $\cap_{A_i}=(\mu_{\cap A_i},\nu_{\cap A_i})$.

Proposition 3.9. Let $A\in IFSN(R)$ and T,C be idempotent.

(1) For all $\alpha\in[0,1]$, the μ-level α-cut $U(\mu_A,\alpha)=\{x\in R\mid \mu_A\geq\alpha\}$ is a subsemiring of R.

(2) For all $\beta\in[0,1]$, the ν-level β-cut $L(\nu_A,\beta)=\{x\in R\mid \nu_A\leq\beta\}$ is a subsemiring of R.

Proof. (1) Let $x,y\in U(\mu_A,\alpha)$. Since $A\in IFSN(R)$ so $\mu_A(x+y)\geq T(\mu_A(x),\mu_A(y))\geq T(\alpha,\alpha)=\alpha$ and $\mu_A(xy)\geq T(\mu_A(x),\mu_A(y))\geq T(\alpha,\alpha)=\alpha$. Thus $x+y,xy\in U(\mu_A,\alpha)$ and then $U(\mu_A,\alpha)$ is a subsemiring of R.

(2) Let $x,y\in L(\nu_A,\beta)$. As $A\in IFSN(R)$ then $\nu_A(x+y)\leq C(\nu_A(x),\nu_A(y))\leq C(\beta,\beta)=\beta$ and $\nu_A(xy)\leq C(\nu_A(x),\nu_A(y))\leq C(\beta,\beta)=\beta$. Therefore $x+y,xy\in L(\nu_A,\beta)$ and $L(\nu_A,\beta)$ is a subsemiring of R.

Definition 3.10. Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be two intuitionistic fuzzy sets in R and S, respectively. The direct som of A and B, denoted by $A\oplus B=(\mu_A\oplus\mu_B,\nu_A\oplus\nu_B)$, is an intuitionistic fuzzy set in $R\oplus S$ such that for all x in R and y in S,$(\mu_A\oplus\mu_B)(x,y)=T(\mu_A(x),\mu_B(y))$ and $(\nu_A\oplus\nu_B)(x,y)=C(\nu_A(x),\nu_B(y))$

Proposition 3.11. If $A_i=(\mu_{A_i},\nu_{A_i})\in IFSN(R_i)$ for $i=1,2$, then $A_1\oplus A_2\in IFSN(R_1\oplus R_2)$.

Proof. Let $(x_1,y_1),(x_2,y_2)\in R_1\oplus R_2$. Then

$$\begin{aligned}
(1)\ \ (\mu_{A_1} \oplus \mu_{A_2})((x_1, y_1) + (x_2, y_2)) &= (\mu_{A_1} \oplus \mu_{A_2})(x_1 + x_2, y_1 + y_2) \\
&= T(\mu_{A_1}(x_1 + x_2), \mu_{A_2}(y_1 + y_2)) \\
&\geq T(T(\mu_{A_1}(x_1), \mu_{A_1}(x_2)), T(\mu_{A_2}(y_1), \mu_{A_2}(y_2))) \\
&= T(T(\mu_{A_1}(x_1), \mu_{A_2}(y_1)), T(\mu_{A_1}(x_2), \mu_{A_2}(y_2))) \\
&= T((\mu_{A_1} \oplus \mu_{A_2})(x_1, y_1), (\mu_{A_1} \oplus \mu_{A_2})(x_2, y_2))
\end{aligned}$$

$$\begin{aligned}
(2)\ \ (\mu_{A_1} \oplus \mu_{A_2})((x_1, y_1)(x_2, y_2)) &= (\mu_{A_1} \oplus \mu_{A_2})(x_1x_2, y_1y_2) \\
&= T(\mu_{A_1}(x_1x_2), \mu_{A_2}(y_1y_2)) \\
&\geq T(T(\mu_{A_1}(x_1), \mu_{A_1}(x_2)), T(\mu_{A_2}(y_1), \mu_{A_2}(y_2))) \\
&= T(T(\mu_{A_1}(x_1), \mu_{A_2}(y_1)), T(\mu_{A_1}(x_2), \mu_{A_2}(y_2))) \\
&= T((\mu_{A_1} \oplus \mu_{A_2})(x_1, y_1), (\mu_{A_1} \oplus \mu_{A_2})(x_2, y_2))
\end{aligned}$$

$$\begin{aligned}
(3)\ \ (\nu_{A_1} \oplus \mu_{A_2})((x_1, y_1) + (x_2, y_2)) &= (\nu_{A_1} \oplus \nu_{A_2})(x_1 + x_2, y_1 + y_2) \\
&= C(\nu_{A_1}(x_1 + x_2), \nu_{A_2}(y_1 + y_2)) \\
&\leq C(C(\nu_{A_1}(x_1), \nu_{A_1}(x_2)), C(\nu_{A_2}(y_1), \nu_{A_2}(y_2))) \\
&= C(C(\nu_{A_1}(x_1), \nu_{A_2}(y_1)), C(\nu_{A_1}(x_2), \nu_{A_2}(y_2))) \\
&= C((\nu_{A_1} \oplus \nu_{A_2})(x_1, y_1), (\nu_{A_1} \oplus \nu_{A_2})(x_2, y_2))
\end{aligned}$$

$$\begin{aligned}
(4)\ \ (\nu_{A_1} \oplus \mu_{A_2})((x_1, y_1)(x_2, y_2)) &= (\nu_{A_1} \oplus \nu_{A_2})(x_1x_2, y_1y_2) \\
&= C(\nu_{A_1}(x_1x_2), \nu_{A_2}(y_1y_2)) \\
&\leq C(C(\nu_{A_1}(x_1), \nu_{A_1}(x_2)), C(\nu_{A_2}(y_1), \nu_{A_2}(y_2))) \\
&= C(C(\nu_{A_1}(x_1), \nu_{A_2}(y_1)), C(\nu_{A_1}(x_2), \nu_{A_2}(y_2))) \\
&= C((\nu_{A_1} \oplus \nu_{A_2})(x_1, y_1), (\nu_{A_1} \oplus \nu_{A_2})(x_2, y_2))
\end{aligned}$$

Corollary 3.12. Let $A_i = (\mu_{A_i}, \nu_{A_i}) \in IFSN(R_i)$ for $i = 1, 2, ..., n$. Then

$$A_1 \oplus A_2 \oplus ... \oplus A_n \in IFSN(R_1 \oplus R_2 \oplus ... \oplus R_n).$$

4 Homomorphisms and Anti-Homomorphisms of Intuitionistic Fuzzy Subsemirings of Semirings Under Norms

Proposition 4.1. Let φ be an epihomomorphism from semiring R into semiring S. If $A = (\mu_A, \nu_A) \in IFSN(R)$, then $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in IFSN(S)$.

Proof. Let $y_1, y_2 \in S$. Then

$$\begin{aligned}
(1)\quad & \varphi(\mu_A)(y_1 + y_2) \\
= \ & \sup\{\mu_A(x_1 + x_2) \mid x_1, x_2 \in R, \varphi(x_1)y_1, \varphi(x_2) = y_2\} \\
\geq \ & \sup\{T(\mu_A(x_1), \mu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\
= \ & T(\sup\{\mu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \sup\{\mu_A(x_2) \mid x_2 \in R, \varphi(x_2))y_2\}) \\
= \ & T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2))
\end{aligned}$$

$$\begin{array}{rl}(2) & \varphi(\mu_A)(y_1y_2)\\ = & \sup\{\mu_A(x_1x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ \geq & \sup\{T(\mu_A(x_1), \mu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ = & T(\sup\{\mu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \sup\{\mu_A(x_2) \mid x_2 \in R, \varphi(x_2) = y_2\})\\ = & T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2))\end{array}$$

$$\begin{array}{rl}(3) & \varphi(\nu_A)(y_1 + y_2)\\ = & \inf\{\nu_A(x_1 + x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ \leq & \inf\{C(\nu_A(x_1), \nu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ = & C(\inf\{\nu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \inf\{\nu_A(x_2) \mid x_2 \in R, \varphi(x_2) = y_2\})\\ = & C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2))\end{array}$$

$$\begin{array}{rl}(4) & \varphi(\nu_A)(y_1y_2)\\ = & \inf\{\nu_A(x_1x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ \leq & \inf\{C(\nu_A(x_1), \nu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ = & C(\inf\{\nu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \inf\{\nu_A(x_2) \mid x_2 \in R, \varphi(x_2) = y_2\})\\ = & C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2))\end{array}$$

Hence $\varphi(A) \in IFSN(S)$.

Corollary 4.2. Let φ be an anti-epihomomorphism from semiring R into semiring S. If $A = (\mu_A, \nu_A) \in IFSN(R)$, then $\varphi(A) \in IFSN(S)$.

Proposition 4.3. Let Let φ be a homorphism from semiring R into semiring S. If $B = (\mu_B, \nu_B) \in IFSN(S)$, then $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in IFSN(R)$.

Proof. Let $x_1, x_2 \in R$.

$$\begin{array}{lrl}(1)\ \varphi^{-1}(\mu_B)(x_1 + x_2) & = & \mu_B(\varphi(x_1 + x_2))\\ & = & \mu_B(\varphi(x_1) + \varphi(x_2))\\ & \geq & T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2)))\\ & = & T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))\end{array}$$

$$\begin{array}{lrl}(2)\ \varphi^{-1}(\mu_B)(x_1x_2) & = & \mu_B(\varphi(x_1x_2))\\ & = & \mu_B(\varphi(x_1)\varphi(x_2))\\ & \geq & T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2)))\\ & = & T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))\end{array}$$

$$\begin{array}{lrl}(3)\ \varphi^{-1}(\nu_B)(x_1 + x_2) & = & \nu_B(\varphi(x_1 + x_2))\\ & = & \nu_B(\varphi(x_1) + \varphi(x_2))\\ & \leq & C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2)))\\ & = & C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))\end{array}$$

$$\begin{array}{lrl}(4)\ \varphi^{-1}(\nu_B)(x_1x_2) & = & \nu_B(\varphi(x_1x_2))\\ & = & \nu_B(\varphi(x_1)\varphi(x_2))\\ & \leq & C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2)))\\ & = & C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))\end{array}$$

Then $\varphi^{-1}(B) \in IFSN(R)$.

Proposition 4.4. Let Let φ be a anti-homorphism from semiring R into semiring S. If $B = (\mu_B, \nu_B) \in IFSN(S)$, then $\varphi^{-1}(B) \in IFSN(R)$.

Proof. Let $x_1, x_2 \in R$.

$$
\begin{aligned}
(1)\ \varphi^{-1}(\mu_B)(x_1+x_2) &= \mu_B(\varphi(x_1+x_2)) \\
&= \mu_B(\varphi(x_1)+\varphi(x_2)) \\
&\geq T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2))) \\
&= T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \varphi^{-1}(\mu_B)(x_1x_2) &= \mu_B(\varphi(x_1x_2)) \\
&= \mu_B(\varphi(x_2)\varphi(x_1)) \\
&\geq T(\mu_B(\varphi(x_2)), \mu_B(\varphi(x_1))) \\
&= T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2))) \\
&= T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))
\end{aligned}
$$

$$
\begin{aligned}
(3)\ \varphi^{-1}(\nu_B)(x_1+x_2) &= \nu_B(\varphi(x_1+x_2)) \\
&= \nu_B(\varphi(x_1)+\varphi(x_2)) \\
&\leq C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) \\
&= C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))
\end{aligned}
$$

$$
\begin{aligned}
(4)\ \varphi^{-1}(\nu_B)(x_1x_2) &= \nu_B(\varphi(x_1x_2)) \\
&= \nu_B(\varphi(x_2)\varphi(x_1)) \\
&\leq C(\nu_B(\varphi(x_2)), \nu_B(\varphi(x_1))) \\
&= C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) \\
&= C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))
\end{aligned}
$$

Therefore $\varphi^{-1}(B) \in IFSN(R)$.

Proposition 4.5. Let φ be an epihomomorphism from semiring R into semiring S and T, C be idempotent. If $A = (\mu_A, \nu_A) \in IFSN(R)$ and $C_{\alpha,\beta} = \{x \in R \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ be subsemiring of A, then $\varphi(C_{\alpha,\beta}) = C_{\acute{\alpha},\acute{\beta}} = \{\varphi(x) = y \in S \mid \mu_{\varphi(A)}(y) \geq \acute{\alpha}, \nu_{\varphi(A)}(y) \leq \acute{\beta}\}$ will be a subsemiring of $\varphi(A)$.

Proof. Since $A = (\mu_A, \nu_A) \in IFSN(R)$ so from Proposition 4.1 $\varphi(A) = (\mu_{\varphi(A)}, \nu_{\varphi(A)}) \in IFSN(S)$. Let $y_1, y_2 \in C_{\acute{\alpha},\acute{\beta}}$. Then

(1) $\mu_{\varphi(A)}(y_1+y_2) \geq T(\mu_{\varphi(A)}(y_1), \mu_{\varphi(A)}(y_2)) \geq T(\acute{\alpha}, \acute{\alpha}) = \acute{\alpha}$.
(2) $\mu_{\varphi(A)}(y_1y_2) \geq T(\mu_{\varphi(A)}(y_1), \mu_{\varphi(A)}(y_2)) \geq T(\acute{\alpha}, \acute{\alpha}) = \acute{\alpha}$.
(3) $\nu_{\varphi(A)}(y_1+y_2) \leq C(\nu_{\varphi(A)}(y_1), \nu_{\varphi(A)}(y_2)) \leq C(\acute{\beta}, \acute{\beta}) = \acute{\beta}$.
(4) $\nu_{\varphi(A)}(y_1y_2) \leq C(\nu_{\varphi(A)}(y_1), \nu_{\varphi(A)}(y_2)) \leq C(\acute{\beta}, \acute{\beta}) = \acute{\beta}$.

Then $y_1+y_2, y_1y_2 \in C_{\acute{\alpha},\acute{\beta}}$ and $\varphi(C_{\alpha,\beta}) = C_{\acute{\alpha},\acute{\beta}}$ is a subsemiring of $\varphi(A)$.

Proposition 4.6. Let φ be a homorphism from semiring R into semiring S and T, C be idempotent. If $B = (\mu_B, \nu_B) \in IFSN(S)$ and $C_{\alpha,\beta} = \{y \in S \mid \mu_B(y) \geq \alpha, \nu_B(y) \leq \beta\}$ be a subsemiring of B, then $\varphi^{-1}(C_{\alpha,\beta}) = C_{\acute{\alpha},\acute{\beta}} = \{\varphi^{-1}(y) = x \in R \mid \mu_{\varphi^{-1}(B)}(x) \geq \acute{\alpha}, \nu_{\varphi^{-1}(B)}(x) \leq \acute{\beta}\}$ be a subsemiring of $\varphi^{-1}(B)$.

Proof. Let $x_1, x_2 \in C_{\acute{\alpha},\acute{\beta}}$. As Proposition 4.3 $\varphi^{-1}(B) \in IFSN(R)$ and then

(1) $\mu_{\varphi^{-1}(B)}(x_1 + x_2) \geq T(\mu_{\varphi^{-1}(B)}(x_1), \mu_{\varphi^{-1}(B)}(x_2)) \geq T(\acute{\alpha}, \acute{\alpha}) = \acute{\alpha}$.
(2) $\mu_{\varphi^{-1}(B)}(x_1 x_2) \geq T(\mu_{\varphi^{-1}(B)}(x_1), \mu_{\varphi^{-1}(B)}(x_2)) \geq T(\acute{\alpha}, \acute{\alpha}) = \acute{\alpha}$.
(3) $\nu_{\varphi^{-1}(B)}(x_1 + x_2) \leq C(\nu_{\varphi^{-1}(B)}(x_1), \nu_{\varphi^{-1}(B)}(x_2)) \leq C(\acute{\beta}, \acute{\beta}) = \acute{\beta}$.
(4) $\nu_{\varphi^{-1}(B)}(x_1 x_2) \leq C(\nu_{\varphi^{-1}(B)}(x_1), \nu_{\varphi^{-1}(B)}(x_2)) \leq C(\acute{\beta}, \acute{\beta}) = \acute{\beta}$.

Thus $x_1 + x_2, x_1 x_2 \in C_{\acute{\alpha},\acute{\beta}}$ and so $\varphi^{-1}(C_{\alpha,\beta}) = C_{\acute{\alpha},\acute{\beta}}$ is a subsemiring of $\varphi^{-1}(B)$.

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