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A MATRIX REPRESENTATION OF A GENERALIZED FIBONACCI POLYNOMIAL

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Abstaract − The Fibonacci polynomial $F_n(x)$ defined recurrently by $F_{n+1}(x) = xF_n(x)+F_{n-1}(x)$, with $F_0(x) = 0$, $F_1(0) = 1$, for $n \geq 1$ is the topic of wide interest for many years. In this article, generalized Fibonacci polynomials $\widehat{F}_{n+1}(x)$ and $\widehat{L}_{n+1}(x)$ are introduced and defined by $\widehat{F}_{n+1}(x) = x\widehat{F}_n(x)+\widehat{F}_{n-1}(x)$ with $\widehat{F}_0(x) = 0$, $\widehat{F}_1(x) = x^2+4$, for $n \geq 1$ and $\widehat{L}_{n+1}(x) = x\widehat{L}_n(x)+\widehat{L}_{n-1}(x)$ with $\widehat{L}_0(x) = 2x^2 + 8$, $\widehat{L}_1(x) = x^3 + 4x$, for $n \geq 1$. Also some basic properties of these polynomials are obtained by matrix methods.

Keywords − *Fibonacci Sequence, Fibonacci Polynomial, Generalised Fibonacci Polynomial.*

1 Introduction

Horadam [9] introduced and studied the generalized Fibonacci sequence $W_n = W_n(a; b; p; q)$ defined by $W_n = pW_{n-1} - qW_{n-2}$ with $W_0 = a$, $W_1 = b$, for $n \geq 1$ where $a; b; p$ and q are arbitrary complex numbers with $q \neq 0$. These numbers were first studied by Horadam and they are called Horadam numbers. In [7] Silvester shows that a number of the properties of the Fibonacci sequence can be derived from a matrix representation.

In [27] Demirturk obtained summation formulae for the Fibonacci and Lucas sequences by matrix methods. In [28] the authors presented some important relationship between k-Jacobsthal matrix sequence and k-Jacobsthal-Lucas matrix sequence. In [22] Godase and Dhakne described some properties of k- Fibonacci and k-Lucas numbers by matrix terminology.

In [18] Catarino and Vasco introduced a 2×2 matrix for the k-Pell sequence with its nth power. The well-known Fibonacci polynomial is studied over several years. Many authors are dedicated to study this polynomial. The most research on Fibonacci polynomials are dedicated to study the generalizations of Fibonacci

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polynomials [4],[5], [20], [19], [21], [28]. The main aim of the present paper is to study other generalized Fibonacci polynomial by matrix methods.

Somnuk Srisawat and Wanna Sriprad [1] investigated generalization of Pell and Pell-Lucas numbers, which is called (s,t)-Pell and (s,t)-Pell-Lucas numbers, also they defined the 2×2 matrix

$$W=\begin{bmatrix} s & 2(s^2+t) \\ \frac{1}{2} & s \end{bmatrix}$$

using this matrix they established many identities of (s,t)-Pell and (s,t)-Pell-Lucas numbers. Hasan Huseyin Gulec, Necati Taskara [2] established a new generalizations for (s,t)-Pell (s,t)- Pell Lucas $\{q_n(s,t)\}n \in N$ sequences for Pell and Pell Lucas numbers. Considering these sequences, they defined the matrix sequences which have elements of $\{p_n(s,t)\}n \in N$ and $\{q_n(s,t)\}n \in N$.

Yuan, Yi, and Wenpeeg Zhang [3] introduced different methods to calculate the summations involving the Fibonacci polynomials. Fikri Koken and Durmus Bozkurt [6] defined the Jacobsthal Lucas E-matrix and R-matrix alike to the Fibonacci Q-matrix. Using this matrix representation they found some equalities and Binet-like formula for the Jacobsthal and Jacobsthal-Lucas numbers.

Falcon and Plaza[10] presented the derivatives of Fibonacci polynomials in the form of convolution of k-Fibonacci polynomials and many relations for the derivatives of these polynomials are proved.

We denote the Fibonacci and Lucas polynomial by $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ respectively. Most of the identities for Fibonacci and Lucas polynomials can be found in the articles [11], [12], [13], [14], [15], [16], [17], [23], [24], [25], [26] on Fibonacci and Lucas sequences and their applications. The ultimate aim of this paper is to introduce new generalization $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ of Fibonacci and Lucas polynomials and establish a collection of identities for the $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ using matrix method.

2 Generalized Fibonacci polynomials $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$

In [8] the Fibonacci polynomial $F_n(x)$ defined recurrently by $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$, with $F_0(x) = 0$, $F_1(0) = 1$, for $n \geq 1$. In this paper we defined generalized Fibonacci polynomials $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$.

Definition 2.1. The generalized Fibonacci polynomial $\widehat{F}_n(x)$ is defined by the recurrence relation

$$\widehat{F}_{n+1}(x) = x\widehat{F}_n(x) + \widehat{F}_{n-1}(x) \quad \text{with} \quad \widehat{F}_0(x) = 0, \quad \widehat{F}_1(x) = x^2+4, \text{ for } n \geq 1 \tag{1}$$

Definition 2.2. The generalized Lucas polynomial $\widehat{L}_n(x)$ is defined by the recurrence relation

$$\widehat{L}_{n+1}(x) = x\widehat{L}_n(x) + \widehat{L}_{n-1}(x) \quad \text{with} \quad \widehat{L}_0(x) = 2x^2+8, \quad \widehat{L}_1(x) = x^3+4x, \text{ for } n \geq 1 \tag{2}$$

Polynomial	Initial value $G_0(x)=a(x)$	Initial value $G_1(x)=b(x)$	Recursive Formula $G_{n+1}(x)=a(x)G_n(x)+b(x)G_{n-1}(x)$
Fibonacci	0	1	$F_{n+1}(x)=F_n(x)+F_{n-1}(x)$
Lucas	2	x	$L_{n+1}(x)=L_n(x)+L_{n-1}(x)$
Pell	0	1	$P_{n+1}(x)=2xP_n(x)+P_{n-1}(x)$
Pell-Lucas	2	2x	$Q_{n+1}(x)=2xQ_n(x)+Q_{n-1}(x)$
Jacobsthal	0	1	$J_{n+1}(x)=J_n(x)+2xJ_{n-1}(x)$
Jacobsthal-Lucas	2	1	$j_{n+1}(x)=j_n(x)+2xj_{n-1}(x)$
Generalized Fibonacci	0	x^2+4	$\widehat{F}_{n+1}(x)=x\widehat{F}_n(x)+\widehat{F}_{n-1}(x)$
Generalized Lucas	$2x^2+8$	x^3+4x	$\widehat{L}_{n+1}(x)=x\widehat{L}_n(x)+\widehat{L}_{n-1}(x)$

Table 1: **Recurrence relation of some GFP.**

Characteristic equation of the initial recurrence relation (1 and 2) is,

$$r^2 - xr - 1 = 0 \tag{3}$$

Characteristic roots of (3) are

$$r_1(x) = \frac{x+\sqrt{x^2+4}}{2}, \quad r_2(x) = \frac{x-\sqrt{x^2+4}}{2} \tag{4}$$

Characteristic roots (4) satisfy the properties

$$r_1(x) - r_2(x) = \sqrt{x^2+4} = \sqrt{\Delta(x)}, \quad r_1(x)+r_2(x) = x, \quad r_1(x)r_2(x) = -1 \tag{5}$$

Binet forms for both $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ are given by

$$\widehat{F}_n(x) = r_1(x)^n - r_2(x)^n \tag{6}$$

$$\widehat{L}_{k,n} = \left[r_1(x)^2 + r_2(x)^2 + 2\right]\left[r_1(x)^n + r_2(x)^n)\right] \tag{7}$$

The most commonly used matrix in relation to the recurrence relation(3) is

$$M(x) = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \tag{8}$$

Using Principle of Mathematical induction the matrix M can be generalized to

$$M(x)^n = \begin{bmatrix} \dfrac{\widehat{F}_{n+1}(x)}{\Delta(x)} & \dfrac{\widehat{F}_n(x)}{\Delta(x)} \\ \dfrac{\widehat{F}_n(x)}{\Delta(x)} & \dfrac{\widehat{F}_{n-1}(x)}{\Delta(x)} \end{bmatrix} \quad \text{where, } n \text{ is an integer}$$

3 Auxiliary Results

Several identities for $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$ can be established using (6), (7). Some of these are listed below

$$\widehat{L}_n(x) = \widehat{F}_{n+1}(x) + \widehat{F}_{n-1}(x) \tag{9}$$

$$\Delta(x)\widehat{F}_n(x) = \widehat{F}_{n+1}(x) + \widehat{L}_{n-1}(x) \tag{10}$$

$$\Delta(x)\widehat{F}_n^2(x) = \widehat{F}_{2n}(x) - 2\Delta(x)(-1)^n \tag{11}$$

$$\widehat{F}_m(x)\widehat{L}_n(x) = \Delta\widehat{F}_{m+n}(x) - \Delta(x)(-1)^m\widehat{L}_{n-m}(x) \tag{12}$$

$$\widehat{F}_m(x)\widehat{F}_n(x) = \widehat{L}_{m+n}(x) - (-1)^m\widehat{L}_{n-m}(x) \tag{13}$$

$$(-1)^{n-m+1}\widehat{F}_m(x)^2 = \widehat{F}_{m+n}(x)\widehat{F}_{n-m}(x) - \widehat{F}_n(x)^2 \tag{14}$$

$$(-1)^{n-m}\widehat{F}_m(x)^2 = \widehat{L}_{m+n}(x)\widehat{L}_{n-m}(x) - \widehat{L}_n(x)^2 \tag{15}$$

$$\widehat{F}_m(x)\widehat{F}_{n+r+m}(x) = \widehat{F}_{m+n}(x)\widehat{F}_{r+m}(x) - (-1)^m\widehat{F}_n(x)\widehat{F}_r(x) \tag{16}$$

$$2\Delta(x)\widehat{L}_{(m+1)n}(x) = \widehat{L}_{mn}(x)\widehat{L}_n(x) + \Delta(x)\widehat{F}_{mn}(x)\widehat{F}_n(x) \tag{17}$$

$$2\Delta(x)\widehat{F}_{(m+1)n}(x) = \widehat{F}_{mn}(x)\widehat{L}_n(x) + \widehat{L}_{mn}(x)\widehat{F}_n(x) \tag{18}$$

$$\Delta(x)\widehat{F}_{2n+m}(x)\widehat{F}_m(x) = \widehat{L}_{m+n}(x)^2 + (-1)^{m-1}\widehat{L}_n(x)^2 \tag{19}$$

$$\Delta(x)\widehat{F}_{2n}(x)\widehat{F}_m(x) = \widehat{L}_{m+n}(x)\widehat{L}_n(x) + (-1)^{m+1}\widehat{L}_{n-m}(x)\widehat{L}_n(x) \tag{20}$$

$$\widehat{F}_{2(r+1)n+m}(x)\widehat{F}_m(x) = \widehat{F}_{m+2rn}(x)\widehat{F}_{2n+m}(x) + (-1)^{m+1}\widehat{F}_{2rn}(x)\widehat{F}_{2n}(x) \tag{21}$$

$$\widehat{F}_{-n}(x) = (-1)^{n+1}\widehat{F}_n(x) \tag{22}$$

$$\widehat{F}_{-n-1}(x) = (-1)^n\widehat{F}_{n+1}(x) \tag{23}$$

4 Main Results

Lemma 4.1. If X is a square matrix with $\Delta(x)X^2 = xX + I$, then $\Delta(x)X^n = \widehat{F}_n(x)X + \widehat{F}_{n-1}(x)I$, for all $n \in Z$

Proof. For $n = 0$ the result is true
For $n = 1$

$$\begin{aligned}\Delta(x)(X)^1 &= \widehat{F}_1(x)X + \widehat{F}_0(x)I \\ &= \Delta(x)X + 0I \\ &= \Delta(x)X\end{aligned}$$

Hence result is true for $n = 1$.
Assume that, $\Delta(x)X^n = \widehat{F}_n(x)X + \widehat{F}_{n-1}(x)I$, and prove that, $\Delta(x)X^{n+1} = \widehat{F}_{n+1}(x)X + \widehat{F}_n(x)I$

Consider,

$$\begin{aligned}
\widehat{F}_{n+1}(x)X + \widehat{F}_n(x)I &= (\widehat{F}_n(x)x + \widehat{F}_{n-1}(x)I)X + \widehat{F}_n(x)I \\
&= (xX + I)\widehat{F}_n(x) + X\widehat{F}_{n-1}(x) \\
&= X^2\widehat{F}_n(x) + X\widehat{F}_{n-1}(x) \\
&= X(X\widehat{F}_n(x) + \widehat{F}_{n-1}(x)) \\
&= X(\Delta(x)X^n) \\
&= \Delta(x)X^{n+1} \\
\text{Hence,} \quad \Delta(x)X^{n+1} &= \widehat{F}_{n+1}(x)X + \widehat{F}_n(x)I
\end{aligned}$$

We now show that, $\Delta(x)X^{-n} = \widehat{F}_{-n}(x)X + \widehat{F}_{-n-1}(x)I$, for all $n \in Z^+$
Let, $Y = xI - X$, then

$$\begin{aligned}
Y^2 &= (xI - X)^2 \\
&= x^2I - 2xX + X^2 \\
&= x^2I - 2xX + xX + I \\
&= x^2I - xX + I \\
&= x(xI - X) + X + I \\
&= xY + I \\
\text{This shows that} \quad \Delta Y^n &= \widehat{F}_n(x)Y + \widehat{F}_{n-1}(x)I \\
\Delta(x)(-X^{-1})^n &= \widehat{F}_n(x)(xI - X) + \widehat{F}_{n-1}(x)I \\
\Delta(x)(-1)^nX^{-n} &= -\widehat{F}_n(x)X + \widehat{F}_{n+1}(x)I \\
\Delta(x)X^{-n} &= (-1)^{n+1}\widehat{F}_n(x)X + (-1)^n\widehat{F}_{n+1}(x)I
\end{aligned}$$

Using equations (22 and 23), it gives that $\Delta(x)X^{-n} = \widehat{F}_{-n}(x)X + \widehat{F}_{-n-1}(x)I$, for all $n \in Z^+$ □

Corollary 4.2. Let, $S(x) = \begin{bmatrix} \frac{x}{2} & \frac{\Delta(x)}{2} \\ \frac{1}{2} & \frac{x}{2} \end{bmatrix}$, then $\Delta(x)S(x)^n = \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix}$, for every $n \in Z$

Proof.

$$\begin{aligned}
\text{Since} \quad S(x)^2 &= \begin{bmatrix} \frac{x^2+2}{2} & \frac{x\Delta(x)}{2} \\ \frac{x}{2} & \frac{x^2+2}{2} \end{bmatrix} \\
&= xS(x) + I
\end{aligned}$$

Using Lemma (4.1), it is clear that

$$\begin{aligned}
\Delta(x)S(x)^n &= \widehat{F}_n(x)S(x) + \widehat{F}_{n-1}(x)I \\
&= \begin{bmatrix} \frac{x\widehat{F}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{x\widehat{F}_n(x)}{2} \end{bmatrix} + \begin{bmatrix} \widehat{F}_{n-1}(x) & 0 \\ 0 & \widehat{F}_{n-1}(x) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix}
\end{aligned}$$

□

Lemma 4.3.

$$\widehat{L}_n^2(x) - \Delta(x)\widehat{F}_n^2(x) = 4\Delta(x)^2(-1)^n \qquad \text{for all,} \quad n \in Z \tag{24}$$

Proof.

$$\begin{aligned}
\text{Since,} \quad \det(S(x)) &= -1 \\
\det(S(x)^n) &= (-1)^n \\
\text{Moreover since,} \quad \Delta(x)S(x)^n &= \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix} \\
\text{We get,} \quad \det(\Delta(x)S(x)^n) &= \frac{\widehat{L}_n^2(x)}{4} - \frac{\Delta(x)\widehat{F}_n^2(x)}{4} \\
\text{Thus it follows that,} \quad \widehat{L}_n^2(x) - \Delta(x)\widehat{F}_n^2(x) &= 4\Delta(x)^2(-1)^n \qquad \text{for all,} \quad n \in Z
\end{aligned}$$

□

Lemma 4.4.

$$2\Delta(x)\widehat{L}_{n+m}(x) = \widehat{L}_n(x)\widehat{L}_m(x) + \Delta(x)\widehat{F}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \tag{25}$$

$$2\Delta(x)\widehat{F}_{n+m}(x) = \widehat{F}_n(x)\widehat{L}_m(x) + \widehat{L}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \tag{26}$$

Proof.

$$\begin{aligned}
\text{Since,} \quad \Delta(x)^2 S(x)^{n+m} &= \Delta(x)S(x)^n.\Delta(x)S(x)^m \\
&= \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix} . \begin{bmatrix} \frac{\widehat{L}_m(x)}{2} & \frac{\Delta(x)\widehat{F}_m(x)}{2} \\ \frac{\widehat{F}_m(x)}{2} & \frac{\widehat{L}_m(x)}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\widehat{L}_n(x)\widehat{L}_m(x)+\Delta(x)\widehat{F}_n(x)\widehat{F}_m(x)}{4} & \frac{\Delta(x)[\widehat{L}_n(x)\widehat{F}_m(x)+\widehat{F}_n(x)\widehat{L}_m(x)}{4} \\ \frac{\widehat{L}_n(x)\widehat{F}_m(x)+\widehat{F}_n(x)\widehat{L}_m(x)}{4} & \frac{\widehat{L}_n(x)\widehat{L}_m(x)+\Delta(x)\widehat{F}_n(x)\widehat{F}_m(x)}{4} \end{bmatrix} \\
\text{But,} \quad \Delta(x)S(x)^{n+m} &= \begin{bmatrix} \frac{\widehat{L}_{n+m}(x)}{2} & \frac{\Delta(x)\widehat{F}_{n+m}(x)}{2} \\ \frac{\widehat{F}_{n+m}(x)}{2} & \frac{\widehat{L}_{n+m}(x)}{2} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{It gives that} \quad 2\Delta(x)\widehat{L}_{n+m}(x) &= \widehat{L}_n(x)\widehat{L}_m(x) + \Delta(x)\widehat{F}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \\
2\Delta(x)\widehat{F}_{n+m}(x) &= \widehat{F}_n(x)\widehat{L}_m(x) + \widehat{L}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z
\end{aligned}$$

□

Lemma 4.5.

$$2(-1)^m\Delta(x)^2\widehat{L}_{n-m}(x) = \widehat{L}_n(x)\widehat{L}_m(x) - \Delta(x)\widehat{F}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \tag{27}$$

$$2(-1)^m\Delta(x)^2\widehat{F}_{n-m}(x) = \widehat{F}_n(x)\widehat{L}_m(x) - \widehat{L}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z \tag{28}$$

Proof. Since

$$\begin{aligned}
\Delta(x)^2 S(x)^{n-m} &= \Delta(x)S(x)^n.\Delta(x)S(x)^{-m}\\
&= \Delta(x)S(x)^n.\Delta(x)[S(x)^m]^{-1}\\
&= \Delta(x)S(x)^n.(-1)^m \begin{bmatrix} \frac{\widehat{L}_m(x)}{2} & \frac{-\Delta(x)\widehat{F}_m(x)}{2} \\ \frac{-\widehat{F}_m(x)}{2} & \frac{\widehat{L}_m(x)}{2} \end{bmatrix}\\
&= (-1)^m \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix} . \begin{bmatrix} \frac{\widehat{L}_m(x)}{2} & \frac{-\Delta(x)\widehat{F}_m(x)}{2} \\ \frac{-\widehat{F}_m(x)}{2} & \frac{\widehat{L}_m(x)}{2} \end{bmatrix}\\
&= \begin{bmatrix} \frac{\widehat{L}_n(x)\widehat{L}_m(x)-\Delta\widehat{F}_n(x)\widehat{F}_m(x)}{4} & \frac{\Delta(x)[\widehat{L}_n(x)\widehat{F}_m(x)-\widehat{F}_n(x)\widehat{L}_m(x)}{4} \\ \frac{\widehat{L}_n(x)\widehat{F}_m(x)-\widehat{F}_n(x)\widehat{L}_m(x)}{4} & \frac{\widehat{L}_n(x)\widehat{L}_m(x)-\Delta(x)\widehat{F}_n(x)\widehat{F}_m(x)}{4} \end{bmatrix}
\end{aligned}$$

But

$$\Delta(x)^2 S(x)^{n-m} = \begin{bmatrix} \frac{L_{n-m}(x)}{2} & \frac{\Delta(x)F_{n-m}(x)}{2} \\ \frac{F_{n-m}(x)}{2} & \frac{L_{n-m}(x)}{2} \end{bmatrix}$$

It gives

$$2(-1)^m\Delta(x)^2\widehat{L}_{n-m}(x) = \widehat{L}_n(x)\widehat{L}_m(x) - \Delta(x)\widehat{F}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z$$
$$2(-1)^m\Delta(x)^2\widehat{F}_{n-m}(x) = \widehat{F}_n(x)\widehat{L}_m(x) - \widehat{L}_n(x)\widehat{F}_m(x) \quad \text{for all} \quad n, m \in Z$$

□

Lemma 4.6.

$$(-1)^m\Delta(x)\widehat{L}_{n-m}(x) + \Delta(x)\widehat{L}_{n+m}(x) = \widehat{L}_n(x)\widehat{L}_m(x) \tag{29}$$

$$(-1)^m\Delta(x)\widehat{F}_{n-m}(x) + \Delta(x)\widehat{F}_{n+m}(x) = \widehat{F}_n(x)\widehat{L}_m(x) \tag{30}$$

Proof.

$$\Delta(x)^2 S(x)^{n+m} + (-1)^m\Delta(x)^2 S(x)^{n-m}$$
$$= \begin{bmatrix} \frac{\Delta(x)\widehat{L}_{n+m}(x)+(-1)^m\Delta(x)\widehat{L}_{n-m}(x)}{2} & \frac{\Delta(x)[\widehat{F}_{n+m}(x)+(-1)^m\widehat{F}_{n-m}(x)}{2} \\ \frac{\Delta(x)\widehat{F}_{n+m}(x)+(-1)^m\Delta(x)\widehat{F}_{n-m}(x)}{2} & \frac{\Delta(x)\widehat{L}_{n+m}(x)+(-1)^m\Delta(x)\widehat{L}_{n-m}(x)}{2} \end{bmatrix}$$

On the other hand

$$\begin{aligned}
&\Delta(x)^2 S(x)^{n+m} + (-1)^m\Delta(x)^2 S(x)^{n-m}\\
&= \Delta(x)S(x)^n\Delta(x)S(x)^m + (-1)^m\Delta(x)S(x)^n\Delta(x)S(x)^{-m}\\
&= \Delta(x)S(x)^n[\Delta(x)S(x)^m + (-1)^m\Delta(x)S(x)^{-m}]
\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} \frac{\widehat{L}_n}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix} \left\{ \begin{bmatrix} \frac{\widehat{L}_m(x)}{2} & \frac{\Delta(x)\widehat{F}_m(x)}{2} \\ \frac{\widehat{F}_m(x)}{2} & \frac{\widehat{L}_m(x)}{2} \end{bmatrix} + \begin{bmatrix} \frac{\widehat{L}_m(x)}{2} & \frac{-\Delta(x)\widehat{F}_m(x)}{2} \\ \frac{-\widehat{F}_m}{2} & \frac{\widehat{L}_m(x)}{2} \end{bmatrix} \right\}\\
&= \begin{bmatrix} \frac{\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)}{2} \\ \frac{\widehat{F}_n(x)}{2} & \frac{\widehat{L}_n(x)}{2} \end{bmatrix} . \begin{bmatrix} \widehat{L}_m(x) & 0 \\ 0 & \widehat{L}_m(x) \end{bmatrix}\\
&= \begin{bmatrix} \frac{\widehat{L}_m(x)\widehat{L}_n(x)}{2} & \frac{\Delta(x)\widehat{F}_n(x)\widehat{L}_m(x)}{2} \\ \frac{\widehat{F}_n(x)\widehat{L}_m(x)}{2} & \frac{\widehat{L}_m(x)\widehat{L}_n(x)}{2} \end{bmatrix}
\end{aligned}$$

It gives

$$(-1)^m\Delta(x)\widehat{L}_{n-m}(x)+\Delta(x)\widehat{L}_{n+m}(x)=\widehat{L}_n(x)\widehat{L}_m(x)$$
$$(-1)^m\Delta(x)\widehat{F}_{n-m}(x)+\Delta(x)\widehat{F}_{n+m}(x)=\widehat{F}_n(x)\widehat{L}_m(x)$$

□

Lemma 4.7.

$$8\Delta(x)^2\widehat{F}_{t+y+z}(x)=\widehat{L}_t(x)\widehat{L}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{L}_y(x)\widehat{L}_z(x) \quad (31)$$
$$+\widehat{L}_t(x)\widehat{F}_y(x)\widehat{L}_z(x)+\Delta(x)\widehat{F}_t(x)\widehat{F}_y(x)\widehat{F}_z(x) \quad (32)$$

$$8\Delta(x)^2\widehat{L}_{t+y+z}(x)=\widehat{L}_t(x)\widehat{L}_y(x)\widehat{L}_z(x) \quad (33)$$
$$+\Delta(x)\left[\widehat{L}_t(x)\widehat{F}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{L}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{F}_y(x)\widehat{L}_z(x)\right] \quad (34)$$

Proof. Since

$$\Delta(x)^2S(x)^{t+y+z}=\begin{bmatrix}\frac{\Delta(x)L_{t+y+z}(x)}{2} & \frac{\Delta(x)^2F_{t+y+z}(x)}{2}\\ \frac{\Delta(x)F_{t+y+z}(x)}{2} & \frac{\Delta(x)L_{t+y+z}(x)}{2}\end{bmatrix}$$

On the other hand

$$\begin{aligned}\Delta(x)^2S(x)^{t+y+z}&=\Delta(x)S(x)^{t+y}\Delta(x)S(x)^z\\&=\begin{bmatrix}\frac{\widehat{L}_{t+y}(x)}{2} & \frac{\Delta(x)\widehat{F}_{t+y}(x)}{2}\\ \frac{\widehat{F}_{t+y}(x)}{2} & \frac{\widehat{L}_{t+y}(x)}{2}\end{bmatrix}\cdot\begin{bmatrix}\frac{\widehat{L}_z(x)}{2} & \frac{\Delta(x)\widehat{F}_z(x)}{2}\\ \frac{\widehat{F}_z(x)}{2} & \frac{\widehat{L}_z(x)}{2}\end{bmatrix}\\&=\begin{bmatrix}\frac{\widehat{L}_{t+y}(x)\widehat{L}_z(x)+\Delta(x)\widehat{F}_{t+y}(x)\widehat{F}_z(x)}{4} & \frac{\Delta(x)[\widehat{L}_{t+y}(x)\widehat{F}_z(x)+\widehat{F}_{t+y}(x)\widehat{L}_z(x)}{4}\\ \frac{\widehat{L}_z(x)\widehat{F}_{t+y}(x)+\widehat{F}_z(x)\widehat{L}_{t+y}(x)}{4} & \frac{\widehat{L}_{t+y}(x)\widehat{L}_z(x)+\Delta(x)\widehat{F}_{t+y}(x)\widehat{F}_z(x)}{4}\end{bmatrix}\end{aligned}$$

Using equations (25) and (26), it gives that

$$8\Delta(x)^2\widehat{F}_{t+y+z}(x)=\widehat{L}_t(x)\widehat{L}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{L}_y(x)\widehat{L}_z(x)$$
$$+\widehat{L}_t(x)\widehat{F}_y(x)\widehat{L}_z(x)+\Delta(x)\widehat{F}_t(x)\widehat{F}_y(x)\widehat{F}_z(x)$$

$$8\Delta(x)^2\widehat{L}_{t+y+z}(x)=\widehat{L}_t(x)\widehat{L}_y(x)\widehat{L}_z(x)$$
$$+\Delta(x)\left[\widehat{L}_t(x)\widehat{F}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{L}_y(x)\widehat{F}_z(x)+\widehat{F}_t(x)\widehat{F}_y(x)\widehat{L}_z(x)\right]$$

□

Theorem 4.8.

$$\widehat{L}^2_{t+y}(x)-(-1)^{t+y+1}\widehat{F}_{z-t}(x)\widehat{L}_{t+y}(x)\widehat{F}_{y+z}(x)-\Delta(x)(-1)^{t+z}(x)\widehat{F}^2_{y+z}(x)$$
$$=(-1)^{y+z}\widehat{L}^2_{z-t}(x) \quad (35)$$

$$\Delta(x)\widehat{L}^2_{t+y}(x) - (-1)^{x+z}\widehat{L}_{z-t}(x)\widehat{L}_{t+y}(x)\widehat{L}_{y+z}(x) + (-1)^{x+z}\Delta(x)\widehat{L}^2_{y+z}(x) \\ = (-1)^{y+z+1}\Delta(x)^2\widehat{F}^2_{z-t}(x) \tag{36}$$

$$\Delta(x)\widehat{F}^2_{t+y}(x) - \widehat{L}_{t-z}(x)\widehat{F}_{t+y}(x)\widehat{F}_{y+z}(x) + \Delta(x)(-1)^{x+z}\widehat{F}^2_{y+z}(x) \\ = (-1)^{y+z}\Delta(x)\widehat{F}^2_{z-t}, \tag{37}$$

for all t, y, $z \in Z$, $t \neq z$

Proof.

Consider the matrix multiplication

$$\begin{bmatrix} \frac{\widehat{L}_t(x)}{2} & \frac{\Delta(x)\widehat{F}_t(x)}{2} \\ \frac{\widehat{F}_z(x)}{2} & \frac{\widehat{L}_z(x)}{2} \end{bmatrix} \begin{bmatrix} \widehat{L}_y(x) \\ \widehat{F}_y(x) \end{bmatrix} = \begin{bmatrix} \Delta(x)\widehat{L}_{t+y}(x) \\ \Delta(x)\widehat{F}_{y+z}(x) \end{bmatrix}$$

$$\det \begin{bmatrix} \frac{\widehat{L}_t(x)}{2} & \frac{\Delta(x)\widehat{F}_t(x)}{2} \\ \frac{\widehat{F}_z(x)}{2} & \frac{\widehat{L}_z(x)}{2} \end{bmatrix} = \frac{\widehat{L}_t(x)\widehat{L}_z(x) - \Delta(x)\widehat{F}_t\widehat{F}_z(x)}{4} = \frac{(-1)^t\Delta(x)\widehat{L}_{z-t}(x)}{2} = R \neq 0$$

Hence

$$\begin{bmatrix} \widehat{L}_y(x) \\ \widehat{F}_y(x) \end{bmatrix} = \begin{bmatrix} \frac{\widehat{L}_t(x)}{2} & \frac{\Delta(x)\widehat{F}_t(x)}{2} \\ \frac{\widehat{F}_z(x)}{2} & \frac{\widehat{L}_z(x)}{2} \end{bmatrix}^{-1} . \begin{bmatrix} \Delta(x)\widehat{L}_{t+y}(x) \\ \Delta(x)\widehat{F}_{y+z}(x) \end{bmatrix} = \frac{1}{R}\begin{bmatrix} \frac{\widehat{L}_z(x)}{2} & \frac{-\Delta(x)\widehat{F}_t(x)}{2} \\ \frac{-\widehat{F}_z(x)}{2} & \frac{\widehat{L}_t(x)}{2} \end{bmatrix} \begin{bmatrix} \Delta(x)\widehat{L}_{t+y}(x) \\ \Delta(x)\widehat{F}_{y+z}(x) \end{bmatrix}$$

$$\widehat{L}_y(x) = \frac{(-1)^t[\widehat{L}_z(x)\widehat{L}_{t+y}(x) - \Delta(x)\widehat{F}_t\widehat{F}_{y+z}(x)]}{\widehat{L}_{z-t}(x)}$$

$$\widehat{F}_y(x) = \frac{(-1)^t[\widehat{L}_t(x)\widehat{F}_{z+y}(x) - \widehat{F}_z(x)\widehat{L}_{y+t}(x)]}{\widehat{L}_{z-t}(x)}$$

Since

$$\widehat{L}^2_y(x) - \Delta(x)\widehat{F}^2_y(x) = 4\Delta(x)^2(-1)^y$$

We get

$$[\widehat{L}_z(x)\widehat{L}_{t+y}(x) - \Delta(x)\widehat{F}_t(x)\widehat{F}_{y+z}(x)]^2 - \Delta(x)[\widehat{L}_t(x)\widehat{F}_{z+y}(x) - \widehat{F}_z(x)\widehat{L}_{y+t}(x)]^2 \\ = 4(-1)^y\Delta(x)^2\widehat{L}^2_{z-t}$$

Using equations (27), (28), (31) and (33)

$$(\widehat{L}_z^2(x)\widehat{L}_{t+y}^2(x) - 2\Delta(x)\widehat{L}_z(x)\widehat{F}_{t+y}(x)\widehat{F}_{y+z}(x) + \Delta(x)^2\widehat{F}_t^2(x)\widehat{F}_{y+z}^2(x))-$$
$$\Delta(x)(\widehat{L}_t^2(x)\widehat{F}_{y+z}^2(x) - 2\widehat{L}_t(x)\widehat{F}_z(x)\widehat{F}_{y+z}(x)\widehat{L}_{t+y}(x) + \widehat{F}_z^2(x)\widehat{L}_{t+y}^2(x))$$
$$= 4(-1)^y\Delta(x)^2L_{z-t}^2(x)$$

It gives that

$$\widehat{L}_{t+y}^2(x) - (-1)^{t+y+1}\widehat{F}_{z-t}(x)\widehat{L}_{t+y}(x)\widehat{F}_{y+z}(x) - \Delta(x)(-1)^{t+z}\widehat{F}_{y+z}^2(x) = (-1)^{y+z}\widehat{L}_{z-t}^2(x)$$
$$\text{for all} \quad t, y, z \in Z$$

□

Proof of (36) and (37) is similar to (35).

Theorem 4.9. For $n \in N$ and $m, t \in Z$ with $m \neq 0$

$$\sum_{j=0}^{j=n}\widehat{L}_{mj+t}(x) = \frac{\Delta(x)\widehat{L}_t(x) - \Delta(x)\widehat{L}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{L}_{mn+t}(x) - \widehat{L}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)} \tag{38}$$

$$\sum_{j=0}^{j=n}\widehat{F}_{mj+t}(x) = \frac{\Delta(x)\widehat{F}_t(x) - \Delta(x)\widehat{F}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{F}_{mn+t}(x) - \widehat{F}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)} \tag{39}$$

Proof.

Since $\quad \Delta(x)^2I - \Delta(x)^2(S(x)^m)^{n+1} = (\Delta(x)I - \Delta(x)S(x)^m)\sum_{j=0}^{j=n}\Delta(x)(S(x)^m)^j$

From Lemma (4.3) it is clear that $\det(\Delta(x)I - \Delta(x)S(x)^m) \neq 0$, therefore we get

$$(\Delta(x)I - \Delta(x)(S(x)^m))^{-1}(\Delta(x)I - \Delta(x)(S(x)^m)^{n+1})\Delta(x)S(x)^t$$
$$= \sum_{j=0}^{j=n}(\Delta(x)S(x)^{mj+t})$$
$$= \begin{bmatrix} \frac{1}{2}\sum_{j=0}^{j=n}(\widehat{L}_{mj+t}(x)) & \frac{\Delta(x)}{2}\sum_{j=0}^{j=n}(\widehat{F}_{mj+t}(x)) \\ \frac{1}{2}\sum_{j=0}^{j=n}(\widehat{F}_{mj+t}(x)) & \frac{1}{2}\sum_{j=0}^{j=n}(\widehat{L}_{mj+t}(x)) \end{bmatrix}$$

For $m \neq 0$, take $D = \Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)$, then we can write

$$(\Delta(x)I - \Delta(x)S(x)^m)^{-1} = \frac{1}{D}\begin{bmatrix} \Delta(x) - \frac{(\widehat{L}_m(x))}{2} & \Delta(x)\frac{(\widehat{F}_m(x))}{2} \\ \frac{(\widehat{F}_m(x))}{2} & \Delta(x) - \frac{(\widehat{L}_m(x))}{2} \end{bmatrix}$$
$$= \frac{1}{D}\left[(\Delta(x) - \frac{(\widehat{L}_m(x))}{2})I + \frac{(\widehat{F}_m(x))}{2}R\right]$$

It gives that,

$$(\Delta(x)I - \Delta(x)S(x)^m)^{-1}(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})$$
$$= \frac{1}{D}\left[(\Delta(x) - \frac{(\widehat{L}_m(x))}{2})I + \frac{(\widehat{F}_m(x))}{2}R\right](\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})$$

$$(\Delta(x)I - \Delta(x)S(x)^m)^{-1}(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})$$
$$= \frac{1}{D}\left[(\Delta(x) - \frac{(\widehat{L}_m(x))}{2})(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})\right]$$
$$+\frac{1}{D}\left[\frac{(\widehat{F}_m(x))}{2}R(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t})\right]$$

From Corollary (4.2) and (4.3), we get

$$R(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t}) =$$
$$\Delta(x)\begin{bmatrix} \Delta(x)\frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} \end{bmatrix}$$

Hence, we get

$$(\Delta(x)I - \Delta(x)S(x)^m)^{-1}(\Delta(x)^2S(x)^t - \Delta(x)^2S(x)^{mn+m+t}) = \frac{\Delta(x)}{D}(\Delta(x) - \frac{\widehat{L}_m(x)}{2})$$
$$\begin{bmatrix} \Delta(x)\frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} & \frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} \end{bmatrix} + \frac{\Delta(x)\widehat{F}_m(x)}{2D}$$
$$\begin{bmatrix} \frac{\Delta(x)(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{L}_t(x)-\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)-\widehat{F}_{mn+m+t}(x))}{2} \end{bmatrix}$$

Hence, it gives that

$$\sum_{j=0}^{j=n}\widehat{L}_{mj+t}(x)$$
$$= \frac{\Delta(x)^2}{D}\left[(\Delta(x) - \frac{\widehat{L}_m(x)}{2})(\widehat{L}_t(x) - \widehat{L}_{mn+m+t}(x)) + \frac{\widehat{F}_m(x)}{2}(\widehat{F}_t(x) - \widehat{F}_{mn+m+t}(x)\right]$$
$$\sum_{j=0}^{j=n}\widehat{F}_{mj+t}(x)$$
$$= \frac{\Delta(x)^2}{D}\left[(\Delta(x) - \frac{\widehat{L}_m(x)}{2})(\widehat{F}_t(x) - \widehat{F}_{mn+m+t}(x)) + \frac{\widehat{F}_m(x)}{2}(\widehat{L}_t(x) - \widehat{L}_{mn+m+t}(x)\right]$$

Using (31) and (33), we get

$$\sum_{j=0}^{j=n} \widehat{L}_{mj+t}(x) = \frac{\Delta(x)\widehat{L}_t(x) - \Delta(x)\widehat{L}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{L}_{mn+t}(x) - \widehat{L}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)}$$

$$\sum_{j=0}^{j=n} \widehat{F}_{mj+t}(x) = \frac{\Delta(x)\widehat{F}_t(x) - \Delta(x)\widehat{F}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{F}_{mn+t}(x) - \widehat{F}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)}$$

□

Theorem 4.10. For $n \in N$ and $m, t \in Z$

$$\sum_{j=0}^{j=n} (-1)^j \widehat{L}_{mj+t}(x) = \frac{\Delta(x)\widehat{L}_t(x) - \Delta(x)\widehat{L}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{L}_{mn+t}(x) - \widehat{L}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)} \tag{40}$$

$$\sum_{j=0}^{j=n} (-1)^j \widehat{F}_{mj+t}(x) = \frac{\Delta(x)\widehat{F}_t(x) - \Delta(x)\widehat{F}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{F}_{mn+t}(x) - \widehat{F}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)} \tag{41}$$

Proof. **Case:** 1 If n is an even natural number then we have

$$\Delta(x)^2 I + \Delta(x)^2 (S(x)^m)^{n+1} = (\Delta(x)I + \Delta(x)S(x)^m) \sum_{j=0}^{j=n} (\Delta(x)S(x)^m)^j (-1)^j$$

From Lemma (4.3) it is clear that $\det(\Delta(x)I + \Delta(x)S(x)^m) \neq 0$, therefore we get

$$\begin{aligned}\Delta(x)I + \Delta(x)(S(x)^m)^{-1}(\Delta(x)I + \Delta(x)(S(x)^m)^{n+1})\Delta(x)S(x)^t \\ = \sum_{j=0}^{j=n} (-1)^j \Delta(x)(S(x)^{mj+t})\end{aligned}$$

$$= \begin{bmatrix} \frac{1}{2}\sum_{j=0}^{j=n}(-1)^j(\widehat{L}_{mj+t}(x)) & \frac{\Delta(x)}{2}\sum_{j=0}^{j=n}(-1)^j(\widehat{F}_{mj+t}(x)) \\ \frac{1}{2}\sum_{j=0}^{j=n}(-1)^j(\widehat{F}_{mj+t}(x)) & \frac{1}{2}\sum_{j=0}^{j=n}(-1)^j(\widehat{L}_{mj+t}(x)) \end{bmatrix}$$

For $m \neq 0$ take $d = \Delta(x) + \Delta(x)(-1)^m + \widehat{L}_m(x)$, then we get

$$\begin{aligned}(\Delta(x)I + \Delta(x)S(x)^m)^{-1} &= \frac{1}{d}\begin{bmatrix} \Delta(x) + \frac{(\widehat{L}_m(x))}{2} & -\Delta(x)\frac{(\widehat{F}_m(x))}{2} \\ \frac{(-\widehat{F}_m(x))}{2} & \Delta(x) + \frac{(\widehat{L}_m(x))}{2} \end{bmatrix} \\ &= \frac{1}{d}\left[(\Delta(x) + \frac{(\widehat{L}_m(x))}{2})I - \frac{(\widehat{F}_m(x))}{2}R\right]\end{aligned}$$

Thus, it is seen that

$$\begin{aligned}(\Delta(x)I + \Delta(x)S(x)^m)^{-1}(\Delta(x)^2 S(x)^t + \Delta(x)^2 S(x)^{mn+m+t}) \\ = \frac{1}{d}\left[(\Delta(x) + \frac{(\widehat{L}_m(x))}{2})I - \frac{(\widehat{F}_m)}{2}R\right] \\ (\Delta(x)^2 S(x)^t + \Delta(x)^2 S(x)^{mn+m+t})\end{aligned}$$

By Corollary (4.2) and (4.3), we get

$$R(\Delta(x)^2 S(x)^t + \Delta(x)^2 S(x)^{mn+m+t})$$
$$= \Delta(x)\begin{bmatrix} \Delta(x)\frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} \end{bmatrix}$$

Thus, it follows that

$$(\Delta(x)I + \Delta(x)S(x)^m)^{-1}(\Delta(x)^2 S(x)^t + \Delta(x)^2 S(x)^{mn+m+t}) = \frac{\Delta(x)}{d}(\Delta(x) + \frac{\widehat{L}_m(x)}{2})$$
$$\begin{bmatrix} \Delta(x)\frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} & \frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} \end{bmatrix}$$
$$-\frac{\Delta(x)\widehat{F}_m(x)}{2d}\begin{bmatrix} \Delta(x)\frac{\Delta(x)(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} \\ \frac{(\widehat{L}_t(x)+\widehat{L}_{mn+m+t}(x))}{2} & \Delta(x)\frac{(\widehat{F}_t(x)+\widehat{F}_{mn+m+t}(x))}{2} \end{bmatrix}$$

Thus, it gives that

$$\sum_{j=0}^{j=n}(-1)^j \widehat{L}_{mj+t}(x)$$
$$= \frac{\Delta(x)^2}{d}\left[(\Delta(x) + \frac{\widehat{L}_m(x)}{2})(\widehat{L}_t(x) + \widehat{L}_{mn+m+t}(x)) - \frac{\widehat{F}_m(x)\Delta(x)}{2}(\widehat{F}_t(x) + \widehat{F}_{mn+m+t}(x)\right]$$
$$\sum_{j=0}^{j=n}(-1)^j \widehat{F}_{mj+t}(x)$$
$$= \frac{\Delta(x)^2}{d}\left[(\Delta(x) + \frac{\widehat{L}_m(x)}{2})(\widehat{F}_t(x) + \widehat{F}_{mn+m+t}(x)) - \frac{\widehat{F}_m(x)\Delta(x)}{2}(\widehat{L}_t(x) + \widehat{L}_{mn+m+t}(x)\right]$$

Using (31) and (33), we get

$$\sum_{j=0}^{j=n}(-1)^j \widehat{L}_{mj+t}(x)$$
$$= \frac{\Delta(x)\widehat{L}_t(x) - \Delta(x)\widehat{L}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{L}_{mn+t}(x) - \widehat{L}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)}$$

$$\sum_{j=0}^{j=n}(-1)^j \widehat{F}_{mj+t}(x)$$
$$= \frac{\Delta(x)\widehat{F}_t(x) - \Delta(x)\widehat{F}_{mn+m+t}(x) + (-1)^m\Delta(x)(\widehat{F}_{mn+t}(x) - \widehat{F}_{t-m}(x))}{\Delta(x) + (-1)^m\Delta(x) - \widehat{L}_m(x)}$$

Case: 2 If n is an odd natural number, we get

$$\sum_{j=0}^{j=n}(-1)^j \widehat{L}_{mj+t}(x) = \sum_{j=0}^{j=n-1}(-1)^j \widehat{L}_{mj+t}(x) - \widehat{L}_{mn+t}(x)$$

Since n is an odd natural number then $(n-1)$ is an even number, hence using case-I, it follows that,

$$\sum_{j=0}^{j=n}(-1)^j\widehat{L}_{mj+t}(x)=\frac{\Delta(x)\widehat{L}_t(x)+\Delta(x)\widehat{L}_{mn+t}(x)+(-1)^m\Delta(x)(\widehat{L}_{mn-m+t}(x)+\widehat{L}_{t-m}(x))}{\Delta(x)+(-1)^m\Delta(x)+\widehat{L}_m(x)}-\widehat{L}_{mn+t}(x)$$
$$=\frac{\Delta(x)\widehat{L}_t(x)+(-1)^m\Delta(x)(\widehat{L}_{mn-m+t}(x)+\widehat{L}_{t-m}(x))-(-1)^m\Delta(x)\widehat{L}_{mn+t}(x)-\Delta(x)\widehat{L}_m(x)\widehat{L}_{mn+t}(x)}{\Delta(x)+(-1)^m\Delta(x)+\widehat{L}_m(x)}$$

Using equations (31) and (33), we get

$$\sum_{j=0}^{j=n}(-1)^j\widehat{L}_{mj+t}(x)=\frac{\Delta(x)\widehat{L}_t(x)+\Delta(x)\widehat{L}_{mn-m+t}(x)+(-1)^m\Delta(x)(\widehat{L}_{t-m}(x)-\widehat{L}_{mn+t}(x))}{\Delta(x)+(-1)^m\Delta(x)+\widehat{L}_m(x)}$$
$$\sum_{j=0}^{j=n}(-1)^j\widehat{F}_{mj+t}(x)=\frac{\Delta(x)\widehat{F}_t(x)-\Delta(x)\widehat{L}_{mn+m+t}(x)+(-1)^m\Delta(x)(\widehat{F}_{t-m}(x)-\widehat{F}_{mn+t}(x))}{\Delta(x)+(-1)^m\Delta(x)+\widehat{L}_m(x)}$$

□

5 Tables

n	$\widehat{F}_n(x)$	$\widehat{L}_n(x)$
1	$4+x^2$	$4x+x^3$
2	$4x+x^3$	$8+6x^2+x^4$
3	$4+5x^2+x^4$	$12x+7x^3+x^5$
4	$8x+6x^3+x^5$	$8+18x^2+8x^4+x^6$
5	$4+13x^2+7x^4+x^6$	$20x+25x^3+9x^5+x^7$
6	$12x+19x^3+8x^5+x^7$	$8+38x^2+33x^4+10x^6+x^8$
7	$4+25x^2+26x^4+9x^6+x^8$	$28x+63x^3+42x^5+11x^7+x^9$
8	$16x+44x^3+34x^5+10x^7+x^9$	$8+66x^2+96x^4+52x^6+12x^8+x^{10}$
9	$4+41x^2+70x^4+43x^6+11x^8+x^{10}$	$36x+129x^3+138x^5+63x^7+13x^9+x^{11}$
10	$20x+85x^3+104x^5+53x^7+12x^9+x^{11}$	$8+102x^2+225x^4+190x^6+75x^8+14x^{10}+x^{12}$
11	$4+61x^2+155x^4+147x^6+64x^8+13x^{12}+x^{12}$	$44x+231x^3+363x^5+253x^7+88x^9+15x^{11}+x^{13}$
12	$24x+146x^3+259x^5+200x^7+76x^9+14x^{11}+x^{13}$	$8+146x^2+456x^4+553x^6+328x^8+102x^{10}+16x^{12}+x^{14}$

Table 2: **First 12 terms of $\widehat{F}_n(x)$ and $\widehat{L}_n(x)$**

$\widehat{F}_n(x)$	$\widehat{F}_1(x)$	$\widehat{F}_2(x)$	$\widehat{F}_3(x)$	$\widehat{F}_4(x)$	$\widehat{F}_5(x)$	$\widehat{F}_6(x)$	$\widehat{F}_7(x)$	$\widehat{F}_8(x)$	$\widehat{F}_9(x)$	$\widehat{F}_{10}(x)$
$x=1$	5	5	10	15	25	40	65	105	170	275
$x=2$	8	16	40	96	232	560	1352	3264	7880	19024
$x=3$	13	39	130	429	1417	4680	15457	51051	168610	556881
$x=4$	20	80	340	1440	6100	25840	109460	463680	1964180	8320400
$x=5$	29	145	754	3915	20329	105560	548129	2846205	14779154	76741975
$x=6$	40	240	1480	9120	56200	346320	2134120	13151040	81040360	499393200
$x=7$	53	371	2650	18921	135097	964600	6887297	49175679	351117050	2506995029
$x=8$	68	544	4420	35904	291652	2369120	19244612	156326016	1269852740	10315147936
$x=9$	85	765	6970	63495	578425	5269320	48002305	437290065	3983612890	36289806075
$x=10$	104	1040	10504	106080	1071304	10819120	109262504	1103444160	11143704104	112540485200
$x=11$	125	1375	15250	169125	1875625	20801000	230686625	2558353875	28372579250	314656725625
$x=12$	148	1776	21460	259296	3133012	37855440	457398292	5526634944	66777017620	806850846384
$x=13$	173	2249	29410	384579	5028937	65760760	859918817	11244705381	147041088770	1922778859391
$x=14$	200	2800	39400	554400	7801000	109768400	1544558600	21733588800,	305814801800	4303140814000
$x=15$	229	3435	51754	779745	11747929	176998680	2666728129	40177920615,	605335537354	9120210980925
$x=16$	260	4160	66820	1073280	17239300	276902080	4447672580	71439663360	1147482286340	18431156244800
$x=17$	293	4981	84970	1449471	24725977	421791080	7195174337	122739754809	2093771006090	35716846858339
$x=18$	328	5904	106600	1924704	34751272	627447600	11328808072	204545992896	3693156680200	66681366236496
$x=19$	365	6935	132130	2517405	47962825	913811080	17410373345	331710904635	6319917561410	120410144571425
$x=20$	404	8080	162004	3248160	65125204	1305752240	26180170004	524909152320	10524363216404	211012173480400

Table 3: **First few terms of** $\widehat{F}_n$

$\widehat{L}_n(x)$	$\widehat{L}_1(x)$	$\widehat{L}_2(x)$	$\widehat{L}_3(x)$	$\widehat{L}_4(x)$	$\widehat{F}_5(x)$	$\widehat{L}_6(x)$	$\widehat{L}_7(x)$	$\widehat{L}_8(x)$	$\widehat{L}_9(x)$	$\widehat{L}_{10}(x)$
$x=1$	5	15	20	35	55	90	145	235	380	615
$x=2$	16	48	112	272	656	1584	3824	9232	22288	53808
$x=3$	39	143	468	1547	5109	16874	55731	184067	607932	2007863
$x=4$	80	360	1520	6440	27280	115560	489520	2073640	8784080	3720996
$x=5$	145	783	4060	21083	109475	568458	2951765	15327283	79588180	413268183
$x=6$	240	1520	9360	57680	355440	2190320	13497360	83174480	512544240	3158439920
$x=7$	371	2703	19292	137747	983521	7022394	50140279	358004347	2556170708	18251199303
$x=8$	544	4488	36448	296072	2405024	19536264	158695136	1289097352	10471473952	85060888968
$x=9$	765	7055	64260	585395	5332815	48580730	442559385	4031615195	36727096140	334575480455
$x=10$	1040	10608	107120	1081808	10925200	110333808	1114263280	11252966608	113643929360	1147692260208
$x=11$	1375	15375	170500	1890875	20970125	232562250	2579154875	28603265875	317215079500	3517969140375
$x=12$	1776	21608	261072	3154472	38114736	460531304	5564490384	67234415912	812377481328	9815764191848
$x=13$	2249	29583	386828	5058347	66145339	864947754	11310466141	147901007587	1934023564772	25290207349623
$x=14$	2800	39600	557200	7840400	110322800	1552359600	21843357200	307359360400	4324874402800	60855600999600
$x=15$	3435	51983	783180	11799683	177778425	2678476058	40354919295	608002265483	9160388901540	138013835788583
$x=16$	4160	67080	1077440	17306120	277975360	4464911880	71716565440	1151929958920	18502595908160	297193464489480
$x=17$	4981	85263	1454452	24810947	423240551	7219900314	123161545889	2100966180427	35839586613148	611373938603943
$x=18$	5904	106928	1930608	34857872	629372304	11363559344	205173440496	3704485488272	66885912229392	1207650905617328
$x=19$	6935	132495	2524340	48094955	916328485	17458336170	332624715715	6337327934755	120741855476060	2300432581979895
$x=20$	8080	162408	3256240	65287208	1309000400	26245295208	526214904560	10550543386408	211537082632720	4241292196040808

Table 4: **First few terms of** $\widehat{L}_n(x)$

References

[1] Srisawat, Somnuk, and Wanna Sriprad, On the (s, t)-Pell and (s, t)-Pell Lucas numbers by matrix methods, *Applied Mathematics LettersIn Annales Mathematicae et Informaticae*, Vol. 46, (2016), 195204.
http://ami.ektf.hu/uploads/papers/finalpdf/AMI_46_from195to204

[2] H.H. Gulec, N. Taskara, On the (s, t)-Pell and (s, t)-Pell Lucas sequences and their matrix representations, *Applied Mathematics Letters*, Vol. 25, (2012), 15541559.
https://doi.org/10.1016/j.aml.2012.01.014

[3] Yuan, Yi, and Wenpeeg Zhang, Some identities involving the Fibonacci polynomials. *Fibonacci Quarterly*, Vol. 40(02), (2002), 314–318.
http://www.fq.math.ca/Scanned/40-4/yuan

[4] Wu, Zhengang, and Wenpeng Zhang, Several identities involving the Fibonacci polynomials and Lucas polynomials, *Journal of Inequalities and Applications*, Vol. 2013(1), (2013), 205–218.
http://link.springer.com/article/10.1186/1029-242X-2013-205

[5] Wu, Zhengang, and Wenpeng Zhang, The sums of the reciprocals of Fibonacci polynomials and Lucas polynomials, *Journal of Inequalities and Applications*, Vol. 2012(1), (2012), 134–140.
http://link.springer.com/article/10.1186/1029-242X-2012-134

[6] Koken, F., Bozkurt, D. On the Jacobsthal numbers by matrix methods, *Int. J. Contemp. Math. Sciences*, Vol. 3(13), (2008), 605-614.
http://m-hikari.com/ijcms-password2008/33-36-2008/kokenIJCMS33-36-2008

[7] Silvester, J. R. Fibonacci properties by matrix methods, *The Mathematical Gazette*, Vol. 63(425), (1979), 188-191.
http://www.jstor.org/stable/3617892

[8] Nalli A, Haukkanen P. On generalized Fibonacci and Lucas polynomials, *Chaos, Solitons and Fractals*, Vol. 42(5), (2009), 3179-86.
https://doi.org/10.1016/j.chaos.2009.04.048

[9] A. F. Horadam, Basic Properties of a Certain Generalized Sequence of Numbers, *Fibonacci Quarterly*, Vol. 03(3), (1965), 61 – 76.
http://www.fq.math.ca/Scanned/3-3/horadam.

[10] S. Falcon, A. Plaza, On k -Fibonacci sequences and polynomials and their derivatives, *Chaos, Solitons and Fractals*, Vol. 39(03), (2009), 1005 – 1019.
https://doi.org/10.1016/j.chaos.2007.03.007.

[11] S. Falcon, A. Plaza, On the k Fibonacci numbers, *Chaos, Solitons and Fractals*, Vol. 32(05), (2007), 1615 – 1624.
https://doi.org/10.1016/j.chaos.2006.09.022.

[12] S. Falcon, A. Plaza, The k Fibonacci sequence and the Pascal 2 triangle, *Chaos, Solitons and Fractals*, Vol. 33(1), (2007), 38 - 49 .
https://doi.org/10.1016/j.chaos.2006.10.022.

[13] S. Falcon, A. Plaza, The k Fibonacci hyperbolic functions, *Chaos, Solitons and Fractals*, Vol. 38(2), (2008), 409 – 20.
https://doi.org/10.1016/j.chaos.2006.11.019.

[14] P. Filipponi, A. F. Horadam, A Matrix Approach to Certain Identities, *The Fibonacci Quarterly*, Vol. 26(2), (1988), 115 – 126.
http://www.fq.math.ca/Scanned/26-2/filipponi

[15] Waddill, Marcellus E., Louis Sacks , Another generalized Fibonacci sequence, *The Fibonacci Quarterly*, Vol. 5(03), (1967) 209 – 222.
http://www.fq.math.ca/Scanned/5-3/waddill

[16] D. Kalman, Generalized Fibonacci numbers by matrix methods, *Fibonacci Quarterly*, Vol. 20 (1),(1982), 73–76.
http://www.fq.math.ca/Scanned/20-1/kalman

[17] C. Bolat, Properties and applications of k -Fibonacci, k-Lucas numbers, *M. S. Thesis, Selcuk University*, (2008), Konya, Turkey.
http://www.m-hikari.com/ijcms-2010/21-24-2010/bolatIJCMS21-24-2010

[18] Paula Catarino, Vasco P., ASome basic properties and a two-by-two matrix involving the k-Pell numbers, *Int. Journal of Math. Analysis*, Vol. 7(45),(2013), 2209–2215.
https://doi.org/10.1016/j.chaos.2015.04.017

[19] S. Falcon, A. Plaza, On the 3 - dimensional k - Fibonacci spirals, *Chaos, Solitons and Fractals*, Vol. 38, (2008), 993–1003.
https://doi.org/10.1016/j.chaos.2007.02.009

[20] S. Falcon, A. Plaza, On k-Fibonacci numbers of arithmetic indexes, *Applied Mathematics and Computation*, Vol. 208 (1),(2009), 180–185.
https://doi.org/10.1016/j.amc.2008.11.031

[21] F. Sergio, Generalized Fibonacci sequences generated from a k-Fibonacci sequence, *Journal of Mathematics Research*, Vol. 4 (2),(2012), 97–100.
http://www.ccsenet.org/journal/index.php/jmr/article/download/14516/10822

[22] A. D. Godase, M. B. Dhakne, On the properties of k Fibonacci and k Lucas numbers, *International Journal of Advances in Applied Mathematics and Mechanics*, Vol. 02 (2014), 100 – 106.
http://www.ijaamm.com/uploads/2/1/4/8/21481830/v2n1p12

[23] A. D. Godase, M. B. Dhakne, Summation Identities for k Fibonacci and k Lucas Numbers using Matrix Methods, *International Journal of Mathematics and Scientific Computing* , Vol. 05(2), (2015), 75 – 78.

https://www.veltechuniv.edu.in/wp-content/uploads/2016/04/Paper-03-15

[24] A. D. Godase, M. B. Dhakne, Determinantal Identities for k Lucas Sequence, *Journal of New Theory*, Vol. 12(1), (2015), 01 – 07.
http://www.newtheory.org/jnt-papers/12-2016/JNT-12-2016-01

[25] R. S. Melham, Sums of certain products of Fbonacci and Lucas Numbers, *The Fibonacci Quarterly*, Vol. 37(3) (1999), 248–251.
https://opus.lib.uts.edu.au/bitstream/10453/17904/1/2011004786OK

[26] Y. Yazlik, N. Yilmaz and N. Taskara, On the Sums of Powers of k-Fibonacci and k-Lucas Sequences, *Seluk J. Appl. Math.*, Vol. Special Issue (2012), 47–50.
http://sjam.selcuk.edu.tr/sjam/article/view/327/249

[27] Demirturk B., Fibonacci and Lucas Sums by matrix Methods, *International Mathematical Forum*, Vol. 5(3) (2010), 99–107.
http://m-hikari.com/imf-2010/1-4-2010/demirturkIMF1-4-2010

[28] Ugyun, S. and Eldogan H., k-Jacobsthal and k-Jacobsthal Lucas matrix sequences, *International Mathematical Forum*, Vol. 11(3) (2016), 145–154.
http://m-hikari.com/imf-2010/1-4-2010/demirturkIMF1-4-2010