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Research Article

New ideals of Bloch mappings which are $\mathcal{I}$-factorizable and Möbius-invariant

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ABSTRACT. In this paper, we introduce a unified method for generating ideals of Möbius-invariant Banach-valued Bloch mappings on the complex open unit disc $\mathbb{D}$, through the composition with the members of a Banach operator ideal $\mathcal{I}$. Using the linearization of derivatives of Banach-valued normalized Bloch mappings on $\mathbb{D}$, this composition method yields the so-called ideals of $\mathcal{I}$-factorizable normalized Bloch mappings $\mathcal{I} \circ \hat{\mathcal{B}}$, where $\hat{\mathcal{B}}$ denotes the class of normalized Bloch mappings on $\mathbb{D}$. We present new examples of them as ideals of separable (Rosenthal, Asplund) normalized Bloch mappings and p-integral (strictly p-integral, p-nuclear) normalized Bloch mappings for any $p \in [1, \infty)$. Moreover, the Bloch dual ideal $\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}$ of an operator ideal $\mathcal{I}$ is introduced and shown that it coincides with the composition ideal $\mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}$.

Keywords: Bloch mapping, linearization, factorization theorems, operator ideal.

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1. INTRODUCTION

Let $\mathbb{D}$ be the complex open unit disc, let X be a complex Banach space and let $\mathcal{H}(\mathbb{D}, X)$ be the space of all holomorphic mappings from $\mathbb{D}$ into X. The Bloch space $\mathcal{B}(\mathbb{D}, X)$ is the linear space of all mappings $f \in \mathcal{H}(\mathbb{D}, X)$ such that

$$\rho_{\mathcal{B}}(f) := \sup\{(1 - |z|^2)\|f'(z)\| : z \in \mathbb{D}\} < \infty,$$

under the Bloch seminorm $\rho_{\mathcal{B}}$, and the normalized Bloch space $\hat{\mathcal{B}}(\mathbb{D}, X)$ is the closed subspace of $\mathcal{B}(\mathbb{D}, X)$ formed by all those maps f for which $f(0) = 0$, under the Bloch norm $\rho_{\mathcal{B}}$. For simplicity, it is usual to write $\mathcal{B}(\mathbb{D}) := \mathcal{B}(\mathbb{D}, \mathbb{C})$ and $\hat{\mathcal{B}}(\mathbb{D}) := \hat{\mathcal{B}}(\mathbb{D}, \mathbb{C})$. These spaces of Bloch mappings have been studied by some authors and we know a lot of their properties (see, e.g., [2]).

A useful procedure for constructing new X-valued Bloch mappings on $\mathbb{D}$ consists of composing the X-valued Bloch mappings on $\mathbb{D}$ with operators of some distinguished Banach operator ideal $\mathcal{I}$. This process was used in [13] to characterize X-valued Bloch mappings on $\mathbb{D}$ that have a Bloch range which is relatively (weakly) compact in X or such that the linear hull of its Bloch range is a finite-dimensional subspace of X. Furthermore, these latter function spaces enjoy the property of being invariant under the action of Möbius transformations of the unit disc.

Motivated by these results, our main purpose here is to present a unified method of composition for generating ideals of Möbius-invariant $\mathcal{I}$-factorizable Bloch mappings $\mathcal{I} \circ \mathcal{B}$, where

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$\mathcal{I}$ is a Banach operator ideal and $\mathcal{B}$ is the class of Bloch mappings on $\mathbb{D}$. For a first study of Möbius invariant function spaces, we refer the reader to [3].

In the literature, we can find some interesting papers where this composition method has been applied for constructing new classes of functions in different contexts as, for example, the Lipschitz setting in [1] by Achour, Rueda, Sánchez-Pérez and Yahi and [18] by Saadi; the polynomial and holomorphic contexts in [4] by Aron, Botelho, Pellegrino and Rueda; the polynomial and multilinear settings in [5] by Belaada, Saadi and Tiaiba and [7] by Botelho, Pellegrino and Rueda; and the bounded holomorphic setting in [8] by Cabrera-Padilla and the same authors of this paper.

This note is organized as follows: Section 2 presents the composition method for generating Banach (normalized) Bloch ideals for a given Banach operator ideal $\mathcal{I}$. In the normalized case, our approach is based on the characterization of normalized $\mathcal{I}$-factorizable Bloch mappings in terms of their linearizations on a suitable Banach predual space of $\hat{\mathcal{B}}(\mathbb{D})$ and, towards this end, we will first recall some definitions and results of the theory of Bloch maps.

Section 3 contains a complete study on some new ideals of Banach (normalized) Bloch ideals which can be generated by composition with a Banach operator ideal. This is the case of separable (Rosenthal, Asplund) Bloch mappings and p-integral (strictly p-integral, p-nuclear) Bloch mappings where $p \in [1, \infty)$.

Section 4 deals the notion of Bloch dual ideal $\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}$ of an operator ideal $\mathcal{I}$ with the aid of the transpose of a Bloch mapping. To this end, let us recall that the dual $\mathcal{I}^{\mathrm{dual}}$ of an operator ideal $\mathcal{I}$ is an operator ideal, and it consists for any normed spaces X, Y of those $T \in \mathcal{L}(X, Y)$ such that $T^* \in \mathcal{I}(Y^*, X^*)$. Thus, we will be able to prove that $\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}$ is justly the composition ideal generated by $\mathcal{I}^{\mathrm{dual}}$. Analogous studies have been done in some other nonlinear contexts such as the Lipschitz setting in [1, 18], or the polynomial and multilinear settings in [6, 11, 16].

Notation. Throughout this paper, X and Y will denote complex Banach spaces. As usual, B_X and X^* stand for the closed unit ball and the dual space of X, respectively. κ_X denotes the canonical isometric linear embedding from X into X^{**}. $\mathbb{T}$ represents the set of all complex numbers with modulus 1. For a set $A \subseteq X$, $\mathrm{lin}(A)$, $\overline{\mathrm{lin}}(A)$ and $\overline{\mathrm{aco}}(A)$ denote the linear hull, the norm-closed linear hull and the norm-closed absolutely convex hull of A in X, respectively. We denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators from X to Y endowed with the operator canonical norm, and by $\mathcal{F}(X, Y)$ the linear space of all finite-rank bounded linear operators from X to Y. We denote by $\mathcal{B}(\mathbb{D}, \mathbb{D})$ the set of all Bloch functions $h \colon \mathbb{D} \to \mathbb{D}$, and by $\hat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$ its subset formed by all those h so that $h(0) = 0$.

2. $\mathcal{I}$-FACTORIZABLE BLOCH MAPPINGS

We begin by introducing these types of Bloch mappings. A complete information on operator ideals may be seen in the monograph [15] by Pietsch.

Definition 2.1. *Let X be a complex Banach space and let $\mathcal{I}$ be an operator ideal. A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be $\mathcal{I}$-factorizable Bloch if there exist a complex Banach space Y, an operator $T \in \mathcal{I}(Y, X)$ and a mapping $g \in \mathcal{B}(\mathbb{D}, Y)$ such that $f = T \circ g$, that is, the following diagram commutes*

$$\begin{tikzcd} \mathbb{D} \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y \arrow[ur, "T"'] & \end{tikzcd}$$

The set of all $\mathcal{I}$-factorizable Bloch mappings from $\mathbb{D}$ into X is denoted by $\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$. If in the preceding factorization of f, the mapping g satisfies in addition that $g(0)=0$, then the set of all such mappings f is denoted by $\mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$.

If $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is a normed operator ideal and $f\in\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$, we set

$$\gamma_{\mathcal{I}\circ\mathcal{B}}(f)=\inf\left\{\|T\|_{\mathcal{I}}\,\rho_{\mathcal{B}}(g)\right\},$$

where the infimum extends over all possible factorizations of f as above. In particular, we will write $\|f\|_{\mathcal{I}\circ\mathcal{B}}$ instead of $\gamma_{\mathcal{I}\circ\mathcal{B}}$ whenever $f\in\mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$.

An easy argument will show that the functions introduced are in fact Bloch and by the way we will see that they are invariant by Möbius transformations of $\mathbb{D}$.

The Möbius group of $\mathbb{D}$ is formed by all one-to-one holomorphic maps ϕ that send $\mathbb{D}$ onto $\mathbb{D}$. This set is denoted by $\mathrm{Aut}(\mathbb{D})$, and each $\phi\in\mathrm{Aut}(\mathbb{D})$ has the form $\phi=\eta\phi_a$ for some $\eta\in\mathbb{T}$ and $a\in\mathbb{D}$, where $\phi_a(z)=(a-z)/(1-\overline{a}z)$ for all $z\in\mathbb{D}$.

Let us recall that a seminormed space $(\mathcal{A}(\mathbb{D},X),\rho_{\mathcal{A}})$ of holomorphic maps from $\mathbb{D}$ into X is said to be Möbius-invariant if the following two conditions are satisfied:

(1) $\mathcal{A}(\mathbb{D},X)\subseteq\mathcal{B}(\mathbb{D},X)$ and there exists $k>0$ such that $\rho_{\mathcal{B}}(f)\leq k\rho_{\mathcal{A}}(f)$ for all $f\in\mathcal{A}(\mathbb{D},X)$,

(2) $f\circ\phi\in\mathcal{A}(\mathbb{D},X)$ with $\rho_{\mathcal{A}}(f\circ\phi)=\rho_{\mathcal{A}}(f)$ for every $f\in\mathcal{A}(\mathbb{D},X)$ and $\phi\in\mathrm{Aut}(\mathbb{D})$.

Proposition 2.1. *Let X be a complex Banach space and let $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ be a normed operator ideal. Then $(\mathcal{I}\circ\mathcal{B}(\mathbb{D},X),\gamma_{\mathcal{I}\circ\mathcal{B}})$ is a Möbius-invariant space.*

Proof. Let $f\in\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$ and assume that $f=T\circ g$ for some complex Banach space Y, an operator $T\in\mathcal{I}(Y,X)$ and a mapping $g\in\mathcal{B}(\mathbb{D},Y)$.

(1) Then $f'=T\circ g'$. Therefore

$$(1-|z|^2)\,\|f'(z)\|=(1-|z|^2)\,\|T(g'(z))\|\leq(1-|z|^2)\,\|T\|\,\|g'(z)\|\leq\|T\|\,\rho_{\mathcal{B}}(g)$$

for all $z\in\mathbb{D}$, and so $f\in\mathcal{B}(\mathbb{D},X)$ with $\rho_{\mathcal{B}}(f)\leq\|T\|\,\rho_{\mathcal{B}}(g)$. Taking the infimum over all such factorizations of f, we conclude that $\rho_{\mathcal{B}}(f)\leq\gamma_{\mathcal{I}\circ\mathcal{B}}(f)$. If $\lambda\in\mathbb{C}$, then $\lambda f=\lambda T\circ g$, hence $\lambda f\in\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$ with $\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)\leq\|\lambda T\|\,\rho_{\mathcal{B}}(g)=|\lambda|\,\|T\|\,\rho_{\mathcal{B}}(g)$ and taking the infimum over all the factorizations of f yields $\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)\leq|\lambda|\,\gamma_{\mathcal{I}\circ\mathcal{B}}(f)$. Conversely, if $\lambda\neq0$, this implies that $\gamma_{\mathcal{I}\circ\mathcal{B}}(f)=\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda^{-1}(\lambda f))\leq|\lambda|^{-1}\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)$ and thus $|\lambda|\,\gamma_{\mathcal{I}\circ\mathcal{B}}(f)\leq\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)$, while if $\lambda=0$, it is clear that $\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)=0=|\lambda|\,\gamma_{\mathcal{I}\circ\mathcal{B}}(f)$.

If $f_1,f_2\in\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$, given $\varepsilon>0$, for each $i=1,2$, we can find a complex Banach space Y_i, an operator $T_i\in\mathcal{I}(Y_i,X)$ and a mapping $g_i\in\mathcal{B}(\mathbb{D},Y_i)$ with $\rho_{\mathcal{B}}(g_i)=1$ and $\|T_i\|_{\mathcal{I}}\leq\gamma_{\mathcal{I}\circ\mathcal{B}}(f_i)+\varepsilon/2$ such that $f_i=T_i\circ g_i$. Consider the Banach space $Y=Y_1\oplus_\infty Y_2$ and define the mappings $T\colon Y\to X$ and $g\colon\mathbb{D}\to Y$ by $T(y_1,y_2)=T_1(y_1)+T_2(y_2)$ for all $(y_1,y_2)\in Y$ and $g(z)=(g_1(z),g_2(z))$ for all $z\in\mathbb{D}$, respectively. An easy calculation shows that $T \in \mathcal{I}(Y, X)$ with $\|T\|_{\mathcal{I}} \leq \|T_1\|_{\mathcal{I}} + \|T_2\|_{\mathcal{I}}$ and $g \in \mathcal{B}(\mathbb{D}, Y)$ with $\rho_{\mathcal{B}}(g) \leq 1$. Clearly, $T \circ g = f_1 + f_2$, and so $f_1 + f_2 \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, X)$ with

$$\gamma_{\mathcal{I}\circ\mathcal{B}}(f_1 + f_2) \leq \|T\|_{\mathcal{I}}\, \rho_{\mathcal{B}}(g) \leq \|T_1\|_{\mathcal{I}} + \|T_2\|_{\mathcal{I}} \leq \gamma_{\mathcal{I}\circ\mathcal{B}}(f_1) + \gamma_{\mathcal{I}\circ\mathcal{B}}(f_2) + \varepsilon.$$

The arbitrariness of $\varepsilon > 0$ ensures that $\gamma_{\mathcal{I}\circ\mathcal{B}}(f_1 + f_2) \leq \gamma_{\mathcal{I}\circ\mathcal{B}}(f_1) + \gamma_{\mathcal{I}\circ\mathcal{B}}(f_2)$. So we have proved that $(\mathcal{I} \circ \mathcal{B}(\mathbb{D}, X), \gamma_{\mathcal{I}\circ\mathcal{B}})$ is a seminormed space.

(2) Let $\phi \in \mathrm{Aut}(\mathbb{D})$. Then $f \circ \phi = T \circ g \circ \phi$, where $g \circ \phi \in \mathcal{B}(\mathbb{D}, Y)$ with $\rho_{\mathcal{B}}(g \circ \phi) = \rho_{\mathcal{B}}(g)$. Hence $f \circ \phi \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, X)$ and $\gamma_{\mathcal{I}\circ\mathcal{B}}(f \circ \phi) \leq \|T\|\, \rho_{\mathcal{B}}(g)$. Passing to the infimum over all the factorizations of f yields $\gamma_{\mathcal{I}\circ\mathcal{B}}(f \circ \phi) \leq \gamma_{\mathcal{I}\circ\mathcal{B}}(f)$. The converse inequality follows from what we have proved.

□

Motivated by [13, Definition 5.11] (see the definition that follows Theorem 2.2), we say that a seminormed ideal of Bloch mappings (or simply, a seminormed Bloch ideal) is a subclass $\mathcal{I}^{\mathcal{B}}$ of the class of all Bloch mappings $\mathcal{B}$, equipped with a function $\rho_{\mathcal{I}^{\mathcal{B}}} : \mathcal{I}^{\mathcal{B}} \to \mathbb{R}$, such that for each complex Banach space X, the components

$$\mathcal{I}^{\mathcal{B}}(\mathbb{D}, X) := \mathcal{I}^{\mathcal{B}} \cap \mathcal{B}(\mathbb{D}, X)$$

satisfy the properties:

(P1) $(\mathcal{I}^{\mathcal{B}}(\mathbb{D}, X), \rho_{\mathcal{I}^{\mathcal{B}}})$ is a seminormed space with $p_{\mathcal{B}}(f) \leq \rho_{\mathcal{I}^{\mathcal{B}}}(f)$ for all $f \in \mathcal{I}^{\mathcal{B}}(\mathbb{D}, X)$,

(P2) for any $g \in \mathcal{B}(\mathbb{D})$ and $x \in X$, the mapping $g \cdot x\colon z \mapsto g(z)x$ from $\mathbb{D}$ to X is in $\mathcal{I}^{\mathcal{B}}(\mathbb{D}, X)$ with $\rho_{\mathcal{I}^{\mathcal{B}}}(g \cdot x) = p_{\mathcal{B}}(g)\, \|x\|$,

(P3) the ideal property: if $f \in \mathcal{I}^{\mathcal{B}}(\mathbb{D}, X)$, $h \in \mathcal{B}(\mathbb{D}, \mathbb{D})$ and $T \in \mathcal{L}(X, Y)$ where Y is a complex Banach space, then $T \circ f \circ h$ is in $\mathcal{I}^{\mathcal{B}}(\mathbb{D}, Y)$ with $\rho_{\mathcal{I}^{\mathcal{B}}}(T \circ f \circ h) \leq \|T\|\, \rho_{\mathcal{I}^{\mathcal{B}}}(f)$.

Proposition 2.2. *Let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be a normed operator ideal. Then $[\mathcal{I} \circ \mathcal{B}, \gamma_{\mathcal{I}\circ\mathcal{B}}]$ is a seminormed Bloch ideal.*

Proof. (P1): It has been proved in Proposition 2.1.

(P2): Take g and x as in (P2), note that $g \cdot x = M_x \circ g$, where $M_x \in \mathcal{F}(\mathbb{C}, X) \subseteq \mathcal{I}(\mathbb{C}, X)$ is the operator defined by $M_x(\lambda) = \lambda x$ for all $\lambda \in \mathbb{C}$, and so $g \cdot x \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, X)$ with $\gamma_{\mathcal{I}\circ\mathcal{B}}(g \cdot x) \leq \|M_x\|_{\mathcal{I}}\, \rho_{\mathcal{B}}(g) = \|M_x\|\, \rho_{\mathcal{B}}(g) = \|x\|\, \rho_{\mathcal{B}}(g)$ and, conversely, $\rho_{\mathcal{B}}(g)\, \|x\| = \rho_{\mathcal{B}}(g \cdot x) \leq \gamma_{\mathcal{I}\circ\mathcal{B}}(g \cdot x)$ by the inequality in (P1).

(P3): Take $f \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, X)$, and h and T as in (P3). We can write $f = S \circ g$ for some complex Banach space Z, an operator $S \in \mathcal{I}(Z, X)$ and a mapping $g \in \mathcal{B}(\mathbb{D}, Z)$. Hence $T \circ f \circ h = T \circ S \circ g \circ h$. Therefore $T \circ f \circ h \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, Y)$ with $\gamma_{\mathcal{I}\circ\mathcal{B}}(T \circ f \circ h) \leq \|T \circ S\|_{\mathcal{I}}\, \rho_{\mathcal{B}}(g \circ h) \leq \|T\|\, \|S\|_{\mathcal{I}}\, \rho_{\mathcal{B}}(g)$. Note that $\rho_{\mathcal{B}}(g \circ h) \leq \rho_{\mathcal{B}}(g)$ since $(1 - |z|^2)|h'(z)| \leq 1 - |h(z)|^2$ for all $z \in \mathbb{D}$ by the Pick–Schwarz Lemma. Taking the infimum over all the factorizations of f gives $\gamma_{\mathcal{I}\circ\mathcal{B}}(T \circ f \circ h) \leq \|T\|\, \gamma_{\mathcal{I}\circ\mathcal{B}}(f)$. □

Our next goal is to characterize $\mathcal{I}$-factorizable normalized Bloch mappings by means of their linearizations on a strongly unique predual of $\hat{\mathcal{B}}(\mathbb{D})$, called Bloch-free Banach space over $\mathbb{D}$. With this purpose, we now recall some basic concepts of the theory initiated in [13] on this predual. A similar technique of linearization of Bloch mappings has been applied in the recent paper [17] by Quang.

For each $z \in \mathbb{D}$, a Bloch atom of $\mathbb{D}$ is the functional $\gamma_z \in \hat{\mathcal{B}}(\mathbb{D})^*$ defined by $\gamma_z(f) = f'(z)$ for all $f \in \hat{\mathcal{B}}(\mathbb{D})$. The elements of $\mathrm{lin}(\{\gamma_z : z \in \mathbb{D}\}) \subseteq \hat{\mathcal{B}}(\mathbb{D})^*$ are referred to as Bloch molecules of $\mathbb{D}$, and the Bloch-free Banach space over $\mathbb{D}$ is the space $\mathcal{G}(\mathbb{D}) := \overline{\mathrm{lin}}(\{\gamma_z : z \in \mathbb{D}\}) \subseteq \hat{\mathcal{B}}(\mathbb{D})^*$.

We collect some basic properties of $\mathcal{G}(\mathbb{D})$ in the following result.

Theorem 2.1 ([13]).

(i) *The mapping* $\Gamma\colon \mathbb{D} \to \mathcal{G}(\mathbb{D})$, *defined by* $\Gamma(z) = \gamma_z$ *for all* $z \in \mathbb{D}$, *is holomorphic with* $\|\gamma_z\| = 1/(1-|z|^2)$.

(ii) *The space* $\hat{\mathcal{B}}(\mathbb{D})$ *is isometrically isomorphic to* $\mathcal{G}(\mathbb{D})^*$, *via* $\Lambda\colon \hat{\mathcal{B}}(\mathbb{D}) \to \mathcal{G}(\mathbb{D})^*$ *given by*

$$\Lambda(f)(\gamma) = \sum_{k=1}^{n} \lambda_k f'(z_k) \qquad \left(f \in \hat{\mathcal{B}}(\mathbb{D}),\ \gamma = \sum_{k=1}^{n} \lambda_k \gamma_{z_k} \in \operatorname{lin}(\Gamma(\mathbb{D})) \right).$$

(iii) *The closed unit ball of* $\mathcal{G}(\mathbb{D})$ *coincides with* $\overline{\operatorname{aco}}(\mathcal{M}(\mathbb{D}))$, *where* $\mathcal{M}(\mathbb{D}) = \{(1-|z|^2)\gamma_z : z \in \mathbb{D}\}$.

(iv) *For each function* $h \in \hat{\mathcal{B}}(\mathbb{D},\mathbb{D})$, *the composition operator* $C_h\colon \hat{\mathcal{B}}(\mathbb{D}) \to \hat{\mathcal{B}}(\mathbb{D})$, *defined by* $C_h(f) = f \circ h$ *for all* $f \in \hat{\mathcal{B}}(\mathbb{D})$, *is linear and continuous with* $||C_h|| \leq 1$.

(v) *For each function* $h \in \hat{\mathcal{B}}(\mathbb{D},\mathbb{D})$, *there exists a unique operator* $\hat{h} \in \mathcal{L}(\mathcal{G}(\mathbb{D}),\mathcal{G}(\mathbb{D}))$ *such that* $\hat{h} \circ \Gamma = h' \cdot (\Gamma \circ h)$. *Furthermore,* $||\hat{h}|| = ||C_h||$.

(vi) *For every complex Banach space* X *and every mapping* $f \in \hat{\mathcal{B}}(\mathbb{D},X)$, *there exists a unique operator* $S_f \in \mathcal{L}(\mathcal{G}(\mathbb{D}),X)$ *such that* $S_f \circ \Gamma = f'$. *Furthermore,* $\|S_f\| = \rho_{\mathcal{B}}(f)$.

(vii) *The mapping* $f \mapsto S_f$ *is an isometric isomorphism from* $\hat{\mathcal{B}}(\mathbb{D},X)$ *onto* $\mathcal{L}(\mathcal{G}(\mathbb{D}),X)$.

(viii) *Given* $f \in \hat{\mathcal{B}}(\mathbb{D},X)$, *the mapping* $f^t\colon X^* \to \hat{\mathcal{B}}(\mathbb{D})$, *defined by* $f^t(x^*) = x^* \circ f$ *for all* $x^* \in X^*$ *and called Bloch transpose of* f, *is linear and continuous with* $\|f^t\| = \rho_{\mathcal{B}}(f)$. *Furthermore,* $f^t = \Lambda^{-1} \circ (S_f)^*$, *where* $(S_f)^*\colon X^* \to \mathcal{G}(\mathbb{D})^*$ *is the adjoint operator of* S_f. □

We are now prepared to establish the announced result.

Theorem 2.2. *Let* X *be a complex Banach space and* $f \in \hat{\mathcal{B}}(\mathbb{D},X)$. *Given an operator ideal* $\mathcal{I}$, *the following conditions are equivalent:*

(1) f *belongs to* $\mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D},X)$,

(2) S_f *belongs to* $\mathcal{I}(\mathcal{G}(\mathbb{D}),X)$.

If, in addition, $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ *is a normed operator ideal, we have* $\|f\|_{\mathcal{I}\circ\mathcal{B}} = ||S_f||_{\mathcal{I}}$, *where the infimum* $\|f\|_{\mathcal{I}\circ\mathcal{B}}$ *is attained at* $S_f \circ \Gamma$. *Furthermore, the mapping* $f \mapsto S_f$ *is an isometric isomorphism from* $(\mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D},X), \|\cdot\|_{\mathcal{I}\circ\mathcal{B}})$ *onto* $(\mathcal{I}(\mathcal{G}(\mathbb{D}),X), \|\cdot\|_{\mathcal{I}})$.

Proof. (1) $\Rightarrow$ (2): If $f \in \mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D},X)$, then we can find a complex Banach space Y, an operator $T \in \mathcal{I}(Y,X)$ and a mapping $g \in \hat{\mathcal{B}}(\mathbb{D},Y)$ such that $f = T \circ g$. Since $g' = S_g \circ \Gamma$ by Theorem 2.1, it follows that $f' = T \circ g' = (T \circ S_g) \circ \Gamma$, and since $T \circ S_g \in \mathcal{L}(\mathcal{G}(\mathbb{D}),X)$, we have that $S_f = T \circ S_g$ by Theorem 2.1, and thus $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}),X)$ by the ideal property of $\mathcal{I}$. Further, if the ideal $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is normed, we have

$$\|S_f\|_{\mathcal{I}} = \|T \circ S_g\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}} \|S_g\| = \|T\|_{\mathcal{I}}\, \rho_{\mathcal{B}}(g),$$

and taking the infimum over all factorizations of f as above, we deduce that $||S_f||_{\mathcal{I}} \leq \|f\|_{\mathcal{I}\circ\mathcal{B}}$.

(2) $\Rightarrow$ (1): Assume that $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}),X)$. We can write $f' = S_f \circ \Gamma$ by Theorem 2.1. Since $\Gamma \in \mathcal{H}(\mathbb{D},\mathcal{G}(\mathbb{D}))$ by Theorem 2.1, an application of [13, Lemma 2.9] provides a mapping $\Upsilon \in \mathcal{H}(\mathbb{D},\mathcal{G}(\mathbb{D}))$ with $\Upsilon(0) = 0$ such that $\Upsilon' = \Gamma$. Furthermore, $(1-|z|^2)\|\Upsilon'(z)\| = (1-|z|^2)\|\Gamma(z)\| = 1$ for all $z \in \mathbb{D}$, and thus $\Upsilon \in \hat{\mathcal{B}}(\mathbb{D},\mathcal{G}(\mathbb{D}))$ with $\rho_{\mathcal{B}}(\Upsilon) = 1$. Hence $f' = S_f \circ \Upsilon'$. We claim that $f = S_f \circ \Upsilon$. Indeed, since $f' = (S_f \circ \Upsilon)'$, we have $(x^* \circ f)' = x^* \circ f' = x^* \circ (S_f \circ \Upsilon)' = (x^* \circ S_f \circ \Upsilon)'$ for all $x^* \in X^*$, this implies that $x^* \circ f = x^* \circ S_f \circ \Upsilon$ for all $x^* \in X^*$ since $x^* \circ f, x^* \circ S_f \circ \Upsilon \in \mathcal{H}(\mathbb{D})$ and $x^* \circ f(0) = x^* \circ S_f \circ \Upsilon(0) = 0$, and our claim follows because X^* separates the point of X. Hence $f \in \mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D},X)$. If now $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is normed, we have

$$\|f\|_{\mathcal{I}\circ\mathcal{B}} \leq \|S_f\|_{\mathcal{I}}\, \rho_{\mathcal{B}}(\Upsilon) = \|S_f\|_{\mathcal{I}}\,.$$

The last assertion of the statement follows readily from Theorem 2.1 and the proof above. □

We will now show that the ideal property of $\mathcal{I}$ is inherited by $\mathcal{I}\circ\hat{\mathcal{B}}$. Towards this end, we first recall the concept of normalized Bloch ideal introduced in [13] and some of its properties.

A normed (Banach) normalized Bloch ideal is a subclass $\mathcal{I}^{\hat{\mathcal{B}}}$ of the class of all normalized Bloch mappings $\hat{\mathcal{B}}$, endowed with a function $\|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}}\colon \mathcal{I}^{\hat{\mathcal{B}}} \to \mathbb{R}$, such that for every complex Banach space X, the components

$$\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X) := \mathcal{I}^{\hat{\mathcal{B}}} \cap \hat{\mathcal{B}}(\mathbb{D},X),$$

satisfy the following properties:

(Q1) $(\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X), \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}})$ is a normed (Banach) space with $\rho_{\mathcal{B}}(f) \leq \|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ for all $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$.

(Q2) $g\cdot x \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$ with $\|g\cdot x\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = \rho_{\mathcal{B}}(g)\,\|x\|$ for every $g \in \hat{\mathcal{B}}(\mathbb{D})$ and $x \in X$.

(Q3) The ideal property: $T\circ f\circ h \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},Y)$ with $\|T\circ f\circ h\|_{\mathcal{I}^{\hat{\mathcal{B}}}} \leq \|T\|\,\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ whenever $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$, $h \in \hat{\mathcal{B}}(\mathbb{D},\mathbb{D})$ and $T \in \mathcal{L}(X,Y)$ where Y is a complex Banach space.

Let us recall (see [15, Chapter B.3]) that if X and Y are Banach spaces and $T \in \mathcal{L}(X,Y)$, then T is called a metric injection if $\|T(x)\| = \|x\|$ for all $x \in X$, and T is called a metric surjection if T is surjective and $\|T(x)\| = \inf\{\|y\| : T(y) = T(x)\}$ for all $x \in X$.

A normed normalized Bloch ideal $(\mathcal{I}^{\hat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}})$ is said to be:

(I) Injective if for any $f \in \hat{\mathcal{B}}(\mathbb{D},X)$, any complex Banach space Y and any metric injection $\iota\colon X \to Y$, one has that $f \in \mathcal{I}^{\hat{\mathcal{B}}}(D,X)$ with $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = \|\iota\circ f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ whenever $\iota\circ f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},Y)$.

(S) Surjective if for any complex Banach space X, any $f \in \hat{\mathcal{B}}(\mathbb{D},X)$ and any $\pi \in \hat{\mathcal{B}}(\mathbb{D},\mathbb{D})$ such that $\hat{\pi}$ is a metric surjection, it holds that $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$ with $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = \|f\circ\pi\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ if $f\circ\pi \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$.

(R) Regular if for any $f \in \hat{\mathcal{B}}(\mathbb{D},X)$, we have that $f \in \mathcal{I}^{\hat{\mathcal{B}}}(D,X)$ with $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = \|\kappa_X\circ f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ whenever $\kappa_X\circ f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X^{**})$.

Corollary 2.1. *We have:*

(1) If $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is a normed (Banach) operator ideal, then $[\mathcal{I}\circ\hat{\mathcal{B}},\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is a normed (Banach) normalized Bloch ideal.

(2) If $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is an injective (surjective, regular) normed operator ideal, then $[\mathcal{I}\circ\hat{\mathcal{B}},\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is an injective (surjective, regular) normed normalized Bloch ideal.

Proof.

(1) Let $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ be a normed operator ideal. Then $[\mathcal{I}\circ\hat{\mathcal{B}},\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is a seminormed Bloch ideal by Proposition 2.2. Furthermore, if $f \in \mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$ and $\|f\|_{\mathcal{I}\circ\mathcal{B}} = 0$, the inequality $\rho_{\mathcal{B}}(f) \leq \|f\|_{\mathcal{I}\circ\mathcal{B}}$ implies that $f = 0$, and thus $\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}$ is a norm on $\mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$.

Since $(\mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X),\|\cdot\|_{\mathcal{I}\circ\mathcal{B}})$ is isometrically isomorphic to $(\mathcal{I}(\mathcal{G}(\mathbb{D}),X),\|\cdot\|_{\mathcal{I}})$ by Theorem 2.2, then $[\mathcal{I}\circ\hat{\mathcal{B}},\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is a Banach space whenever $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is so.

(2) Suppose that the normed operator ideal $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is injective (surjective, regular). We have:

(I) Assume that $\iota\circ f \in \mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},Y)$, where Y is a complex Banach space and $\iota\colon X \to Y$ is a metric injection. Since $\iota\circ S_f = S_{\iota\circ f} \in \mathcal{I}(\mathcal{G}(\mathbb{D}),Y)$ by Theorems 2.1 and 2.2, and the operator ideal $\mathcal{I}$ is injective, we deduce that $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}),X)$ with $||S_f||_{\mathcal{I}} = ||\iota\circ S_f||_{\mathcal{I}}$, thus $f \in \mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$ by Theorem 2.2 with

$$\|f\|_{\mathcal{I}\circ\mathcal{B}} = \|S_f\|_{\mathcal{I}} = \|\iota\circ S_f\|_{\mathcal{I}} = \|S_{\iota\circ f}\|_{\mathcal{I}} = \|\iota\circ f\|_{\mathcal{I}\circ\mathcal{B}},$$

and this proves that $[\mathcal{I} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is injective.

(S) Suppose that $f \circ \pi \in \mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$, where $\pi \in \hat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$ such that $\hat{\pi}$ is a metric surjection. Since $S_f \circ \hat{\pi} = S_{f\circ\pi} \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$ by Theorems 2.1 and 2.2, and the operator ideal $\mathcal{I}$ is surjective, we have that $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$ with $||S_f||_{\mathcal{I}} = ||S_f \circ \hat{\pi}||_{\mathcal{I}}$, hence $f \in \mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$ by Theorem 2.2 with

$$\|f\|_{\mathcal{I}\circ\mathcal{B}} = \|S_f\|_{\mathcal{I}} = \|S_f \circ \hat{\pi}\|_{\mathcal{I}} = \|S_{f\circ\pi}\|_{\mathcal{I}} = \|f \circ \pi\|_{\mathcal{I}\circ\mathcal{B}},$$

and thus $[\mathcal{I} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is surjective.

(R) It follows with a proof similar to that of (I).

□

3. New examples of ideals of $\mathcal{I}$-factorizable Bloch mappings

In this section, we will present some notable subclasses of Bloch mappings: separable (Rosenthal, Asplund) Bloch mappings and p-integral (strictly p-integral, p-nuclear) Bloch mappings for any $p \in [1, \infty)$. We will show their most important properties and demonstrate that they correspond to $\mathcal{I}$-factorizable Bloch mappings when $\mathcal{I}$ is the corresponding operator ideal.

Given two normed normalized Bloch ideals $[\mathcal{I}^{\hat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}}]$ and $[\mathcal{J}, \|\cdot\|_{\mathcal{J}^{\hat{\mathcal{B}}}}]$, we will write

$$[\mathcal{I}^{\hat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}}] \leq [\mathcal{J}, \|\cdot\|_{\mathcal{J}^{\hat{\mathcal{B}}}}]$$

to indicate that for any Banach space X, we have $\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X) \subseteq \mathcal{J}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ and $\|f\|_{\mathcal{J}^{\hat{\mathcal{B}}}} \leq \|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ for all $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$.

3.1. Mappings whose Bloch range is separable, Rosenthal or Asplund.

Given a Banach space X, let us recall that a set $A \subseteq X$ is called:

- Rosenthal if every sequence in A admits a weak Cauchy subsequence,
- Asplund if A is bounded and for any countable set $D \subseteq A$, the seminormed space $(X^*, \|\cdot\|_D)$ is separable, where $||x^*||_D = \sup_{x\in D} |x^*(x)|$.

Let us recall that an operator $T \in \mathcal{L}(X, Y)$ is said to be compact (resp., weakly compact, separable, Rosenthal, Asplund) if $T(B_X)$ is a relatively compact (resp., relatively weakly compact, separable, Rosenthal, Asplund) subset of Y.

For $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$, we will denote by $\mathcal{I}(X, Y)$ the linear space of all finite-rank (approximable, compact, weakly compact, separable, Rosenthal, Asplund) bounded linear operators from X to Y, respectively. We refer to the monograph [15] for a complete study of the ideal structure of such operators. The following inclusions are known:

$$\mathcal{F}(X,Y) \subseteq \overline{\mathcal{F}}(X,Y) \subseteq \mathcal{K}(X,Y) \subseteq \mathcal{W}(X,Y) \subseteq \mathcal{R}(X,Y) \cap \mathcal{AS}(X,Y),$$
$$\mathcal{K}(X,Y) \subseteq \mathcal{S}(X,Y).$$

Our aim is to study some Bloch variants of these classes of operators introduced with the aid of the following set. Given a complex Banach space X and a mapping $f \in \mathcal{H}(\mathbb{D}, X)$, the Bloch range of f is defined as

$$\mathrm{rang}_{\mathcal{B}}(f) := \{(1 - |z|^2)f'(z) \colon z \in \mathbb{D}\} \subseteq X.$$

Note that $f \in \mathcal{H}(\mathbb{D}, X)$ is Bloch if and only if $\mathrm{rang}_{\mathcal{B}}(f)$ is a norm-bounded subset of X. Some of the following concepts were introduced in [13, Definitions 5.1 and 5.2].

Definition 3.2. *Let X be a complex Banach space. A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be compact (resp., weakly compact, separable, Rosenthal, Asplund) if $\mathrm{rang}_{\mathcal{B}}(f)$ is a relatively compact (resp., relatively weakly compact, separable, Rosenthal, Asplund) subset of X.*

A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to have finite dimensional Bloch rank if $\mathrm{lin}(\mathrm{rang}_{\mathcal{B}}(f))$ is a finite dimensional subspace of X, and f is said to be approximable Bloch if it is the limit in the Bloch seminorm $\rho_{\mathcal{B}}$ of a sequence of finite-rank Bloch mappings of $\mathcal{B}(\mathbb{D}, X)$.

For $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$, we denote by $\mathcal{B}_{\mathcal{I}}(\mathbb{D}, X)$ the linear space of all finite-rank (resp., approximable, compact, weakly compact, separable, Rosenthal, Asplund) Bloch mappings from $\mathbb{D}$ into X. We write $\hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$ to represent the subspace consisting of all functions $f \in \mathcal{B}_{\mathcal{I}}(\mathbb{D}, X)$ so that $f(0) = 0$.

The following two results were established in [13] for the cases $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}$. We now complete it here for $\mathcal{I} = \mathcal{S}, \mathcal{R}, \mathcal{AS}$ with similar proofs.

Proposition 3.3. *Let X be a complex Banach space. For $\mathcal{I} = S, \mathcal{R}, \mathcal{AS}$, the space $(\mathcal{B}_{\mathcal{I}}(\mathbb{D}, X), \rho_{\mathcal{B}})$ is Möbius-invariant.*

Proof. Given $f \in \mathcal{H}(\mathbb{D}, X)$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, for all $z \in \mathbb{D}$ we have

$$(1 - |z|^2)(f \circ \phi)'(z) = (1 - |z|^2) f'(\phi(z))\phi'(z) = (1 - |\phi(z)|^2) f'(\phi(z)) \frac{\phi'(z)}{|\phi'(z)|}.$$

Hence $\mathrm{rang}_{\mathcal{B}}(f \circ \phi) \subseteq \mathbb{T}\,\mathrm{rang}_{\mathcal{B}}(f)$, and thus if $\mathrm{rang}_{\mathcal{B}}(f)$ has the $\mathcal{I}$-property (the terminology should be self-explanatory), it is readily seen that $f \circ \phi$ has the $\mathcal{I}$-property with $p_{\mathcal{B}}(f \circ \phi) = p_{\mathcal{B}}(f)$. □

We next analyse the relationship of a mapping f in $\hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$ with its linearization S_f in $\mathcal{L}(\mathcal{G}(\mathbb{D}), X)$.

Theorem 3.3. *Let X be a complex Banach space and $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. For the operator ideal $\mathcal{I} = \mathcal{S}, \mathcal{R}, \mathcal{AS}$, the following conditions are equivalent:*

(1) f belongs to $\hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$,
(2) S_f belongs to $\mathcal{I}(\mathcal{G}(\mathbb{D}), X)$.

Furthermore, $f \mapsto S_f$ is an isometric isomorphism from $(\hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X), \rho_{\mathcal{B}})$ onto $(\mathcal{I}(\mathcal{G}(\mathbb{D}), X), \|\cdot\|)$.

Proof. First, using Theorem 2.1, we obtain the relations:

$$\begin{aligned}\mathrm{rang}_{\mathcal{B}}(f) = S_f(\mathcal{M}_{\mathcal{B}}(\mathbb{D})) \subseteq S_f(B_{\mathcal{G}(\mathbb{D})}) &= S_f(\overline{\mathrm{aco}}(\mathcal{M}_{\mathcal{B}}(\mathbb{D}))) \\ &\subseteq \overline{\mathrm{aco}}(S_f(\mathcal{M}_{\mathcal{B}}(\mathbb{D}))) = \overline{\mathrm{aco}}(\mathrm{rang}_{\mathcal{B}}(f)).\end{aligned}$$

(1) ⇒ (2): Assume $\mathcal{I} = \mathcal{S}, \mathcal{R}, \mathcal{AS}$. If $f \in \hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$, then $\mathrm{rang}_{\mathcal{B}}(f)$ has the $\mathcal{I}$-property. It is known that $\overline{\mathrm{aco}}(\mathrm{rang}_{\mathcal{B}}(f))$ has the $\mathcal{I}$-property and since the $\mathcal{I}$-property is hereditary, the second inclusion above tells us that $S_f(B_{\mathcal{G}(\mathbb{D})})$ has the $\mathcal{I}$-property. This means that $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$.

(2) ⇒ (1): If $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$, then $S_f(B_{\mathcal{G}(\mathbb{D})})$ has the $\mathcal{I}$-property, hence $\mathrm{rang}_{\mathcal{B}}(f)$ has the $\mathcal{I}$-property by the first inclusion above, and this means that $f \in \hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$.

The last assertion of the statement follows using Theorem 2.1 and what was proved above. □

It is known that the Banach operator ideals $\mathcal{S}, \mathcal{R}, \mathcal{AS}$ are injective and surjective (see [15] or a list of such examples in [12]). In view of Theorem 3.3, the combination of Theorem 2.2 and Corollary 2.1 yield the following.

Proposition 3.4. *For $\mathcal{I} = S, \mathcal{R}, \mathcal{AS}$, we have $[\hat{\mathcal{B}}_{\mathcal{I}}, \rho_{\mathcal{B}}] = [\mathcal{I} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ and, in particular, $[\hat{\mathcal{B}}_{\mathcal{I}}, \rho_{\mathcal{B}}]$ is a surjective and injective Banach normalized Bloch ideal.* □

3.2. **p-Integral Bloch mappings.** Following [15, Section 19.2], given two Banach spaces X, Y and $p \in [1, \infty)$, an operator $T \in \mathcal{L}(X, Y)$ is said to be p-integral if there exist a probability measure μ and two operators $R \in \mathcal{L}(L_p(\mu), Y^{**})$ and $S \in \mathcal{L}(X, L_\infty(\mu))$ such that

$$\kappa_Y \circ T = R \circ I^{\mu}_{\infty,p} \circ S,$$

being $I^{\mu}_{\infty,p} \colon L_\infty(\mu) \to L_p(\mu)$ the formal identity. The set of all p-integral operators from X into Y is denoted by $\mathcal{I}_p(X, Y)$, and the p-integral norm of $T \in \mathcal{I}_p(X, Y)$ is

$$\iota_p(T) = \inf\{\|R\| \, \|S\|\},$$

where the infimum is taken over all such factorizations of $\kappa_Y \circ T$ as above. It is well known that $[\mathcal{I}_p, \iota_p]$ is a Banach operator ideal.

We now introduce a Bloch variant of this concept.

Definition 3.3. *Let X be a complex Banach space and $p \in [1, \infty)$. A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be p-integral Bloch if there exists a probability measure μ, an operator $T \in \mathcal{L}(L_p(\mu), X^{**})$ and a Bloch mapping $g \in \mathcal{B}(\mathbb{D}, L_\infty(\mu))$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[r, "\kappa_X"] & X^{**} \\
L_\infty(\mu) \arrow[rr, "I^{\mu}_{\infty,p}"] & & L_p(\mu) \arrow[u, "T"']
\end{tikzcd}$$

The triple $(T, I^{\mu}_{\infty,p}, g)$ is termed a p-integral Bloch representation of f. We define

$$\iota^{\mathcal{B}}_p(f) = \inf\{\|T\| \rho_{\mathcal{B}}(g)\},$$

where the infimum is taken over all p-integral Bloch representations of f. The set of all p-integral Bloch mappings from $\mathbb{D}$ into X is denoted by $\mathcal{I}^{\mathcal{B}}_p(\mathbb{D}, X)$. If in the factorization of $\kappa_X \circ f$, g verifies also that $g(0) = 0$, then the set of such mappings f is denoted by $\mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$.

A proof similar to that of Proposition 2.1 shows the following fact.

Proposition 3.5. *Let X be a complex Banach space and $p \in [1, \infty)$. Then $(\mathcal{I}^{\mathcal{B}}_p(\mathbb{D}, X), \iota^{\mathcal{B}}_p)$ is Möbius-invariant.* □

Proof. Let $f \in \mathcal{I}^{\mathcal{B}}_p(\mathbb{D}, X)$ and $(T, I^{\mu}_{\infty,p}, g)$ be a p-integral Bloch representation of f. Hence $\kappa_X \circ f = T \circ I^{\mu}_{\infty,p} \circ g$.

(1) Then $\kappa_X \circ f' = T \circ I^{\mu}_{\infty,p} \circ g'$. Therefore, for all $z \in \mathbb{D}$, we have

$$\begin{aligned}
(1 - |z|^2) \, \|f'(z)\| = (1 - |z|^2) \, \|\kappa_X(f'(z))\| &= (1 - |z|^2) \, \left\|T(I^{\mu}_{\infty,p}(g'(z)))\right\| \\
&\leq (1 - |z|^2) \, \|T\| \, \|g'(z)\| \leq \|T\| \, \rho_{\mathcal{B}}(g),
\end{aligned}$$

and so $f \in \mathcal{B}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(f) \leq \|T\| \, \rho_{\mathcal{B}}(g)$. Taking the infimum over all such $(T, I^{\mu}_{\infty,p}, g)$, we have $\rho_{\mathcal{B}}(f) \leq \iota^{\mathcal{B}}_p(f)$.

(2) Let $\phi \in \mathrm{Aut}(\mathbb{D})$. Then $\kappa_X \circ f \circ \phi = T \circ I^{\mu}_{\infty,p} \circ g \circ \phi$, where $g \circ \phi \in \mathcal{B}(\mathbb{D}, Y)$ with $\rho_{\mathcal{B}}(g \circ \phi) = \rho_{\mathcal{B}}(g)$. Hence $f \circ \phi \in \mathcal{I}^{\mathcal{B}}_p(\mathbb{D}, X)$ and $\iota^{\mathcal{B}}_p(f \circ \phi) \leq \|T\| \, \rho_{\mathcal{B}}(g)$. Taking the infimum over all p-integral Bloch representations of f, we obtain $\iota^{\mathcal{B}}_p(f \circ \phi) \leq \iota^{\mathcal{B}}_p(f)$. The converse inequality follows from what we have proved.

□

We now study the linearization of p-integral normalized Bloch mappings.

Theorem 3.4. *Let X be a complex Banach space, $p \in [1, \infty)$ and $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. Then $f \in \mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ if and only if $S_f \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), X)$. In this case, $\iota^{\mathcal{B}}_p(f) = \iota_p(S_f)$. Moreover, $(\mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X), \iota^{\mathcal{B}}_p)$ and $(\mathcal{I}_p(\mathcal{G}(\mathbb{D}), X), \iota_p)$ are isometrically isomorphic through the map $f \mapsto S_f$.*

Proof. Assume that $f \in \mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ and let $\varepsilon > 0$. Then there exist a probability measure μ, a mapping $g \in \hat{\mathcal{B}}(\mathbb{D}, L_\infty(\mu))$ and an operator $T \in \mathcal{L}(L_p(\mu), X^{**})$ such that

$$\kappa_X \circ f = T \circ I^{\mu}_{\infty,p} \circ g \colon \mathbb{D} \xrightarrow{g} L_\infty(\mu) \xrightarrow{I^{\mu}_{\infty,p}} L_p(\mu) \xrightarrow{T} X^{**},$$

with $\|T\|\rho_{\mathcal{B}}(g) \leq \iota^{\mathcal{B}}_p(f) + \varepsilon$. By Theorem 2.1, from the equality

$$\kappa_X \circ S_f \circ \Gamma = \kappa_X \circ f' = T \circ I^{\mu}_{\infty,p} \circ g' = T \circ I^{\mu}_{\infty,p} \circ S_g \circ \Gamma,$$

we infer that $\kappa_X \circ S_f = T \circ I^{\mu}_{\infty,p} \circ S_g$, where $S_g \in \mathcal{L}(\mathcal{G}(\mathbb{D}), L_\infty(\mu))$. Hence $S_f \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), X)$ with

$$\iota_p(S_f) \leq \|T\|\|S_g\| = \|T\|\rho_{\mathcal{B}}(g) \leq \iota^{\mathcal{B}}_p(f) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\iota_p(S_f) \leq \iota^{\mathcal{B}}_p(f)$.

Conversely, suppose that $S_f \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), X)$. Then, for each $\varepsilon > 0$ there exist a probability measure μ and two operators $T \in \mathcal{L}(L_p(\mu), X^{**})$ and $S \in \mathcal{L}(\mathcal{G}(\mathbb{D}), L_\infty(\mu))$ such that

$$\kappa_X \circ S_f = T \circ I^{\mu}_{\infty,p} \circ S : \mathcal{G}(\mathbb{D}) \xrightarrow{S} L_\infty(\mu) \xrightarrow{I^{\mu}_{\infty,p}} L_p(\mu) \xrightarrow{T} X^{**},$$

with $\|T\|\|S\| \leq \iota_p(S_f) + \varepsilon$. As $S \circ \Gamma \in \mathcal{H}(\mathbb{D}, L_\infty(\mu))$, [13, Lemma 2.9] provides a mapping $g \in \mathcal{H}(\mathbb{D}, L_\infty(\mu))$ with $g(0) = 0$ such that $g' = S \circ \Gamma$, and hence

$$(1 - |z|^2)\|g'(z)\| = (1 - |z|^2)\|(S \circ \Gamma)(z)\| \leq \|S\| \qquad (z \in \mathbb{D}).$$

Thus, $g \in \hat{\mathcal{B}}(\mathbb{D}, L_\infty(\mu))$ with $\rho_{\mathcal{B}}(g) \leq \|S\|$. Moreover

$$(\kappa_X \circ f)' = \kappa_X \circ f' = \kappa_X \circ S_f \circ \Gamma = T \circ I^{\mu}_{\infty,p} \circ S \circ \Gamma = T \circ I^{\mu}_{\infty,p} \circ g' = (T \circ I^{\mu}_{\infty,p} \circ g)',$$

which implies that $\kappa_X \circ f = T \circ I^{\mu}_{\infty,p} \circ g$, and then $f \in \mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ with

$$\iota^{\mathcal{B}}_p(f) \leq \|T\|\rho_{\mathcal{B}}(g) \leq \|T\| \, \|S\| \leq \iota_p(S_f) + \varepsilon.$$

Just letting $\varepsilon \to 0$, the proof can be concluded. □

Since $[\mathcal{I}_p, \iota_p]$ is a Banach operator ideal, Theorems 2.2 and 3.4 and Corollary 2.1 yield the following result.

Corollary 3.2. *Let $p \in [1, \infty)$. Then $[\mathcal{I}^{\hat{\mathcal{B}}}_p, \iota^{\mathcal{B}}_p] = [\mathcal{I}_p \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I}_p \circ \hat{\mathcal{B}}}]$. In particular, $[\mathcal{I}^{\hat{\mathcal{B}}}_p, \iota^{\mathcal{B}}_p]$ is a Banach ideal of normalized Bloch mappings.* □

Applying Theorem 3.4 and [15, Proposition 19.2.10], we deduce immediately the following.

Corollary 3.3. *If $1 \leq p \leq q < \infty$, then $[\mathcal{I}^{\hat{\mathcal{B}}}_p, \iota^{\mathcal{B}}_p] \leq [\mathcal{I}^{\hat{\mathcal{B}}}_q, \iota^{\mathcal{B}}_q]$.* □

Let us recall from [9] that a mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be p-summing Bloch for $p \in [1,\infty)$ if there is a constant $C \geq 0$ such that for any $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $z_1, \ldots, z_n \in \mathbb{D}$, we have

$$\left(\sum_{i=1}^{n} |\lambda_i|^p \|f'(z_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{g \in B_{\hat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} |\lambda_i|^p |g'(z_i)|^p\right)^{\frac{1}{p}}.$$

The least of all the constants C for which such an inequality holds, denoted $\pi_p^{\mathcal{B}}(f)$, defines a norm on the linear space of all p-summing Bloch map $f\colon \mathbb{D} \to X$ so that $f(0) = 0$, denoted $\Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$.

We can connect p-integral Bloch mappings with this class of Bloch maps. Compare the following result with [10, Proposition 5.5] and [15, Proposition 19.2.12].

Corollary 3.4. *Let $p \in [1,\infty)$. Then:*

(1) $[\mathcal{I}_p^{\hat{\mathcal{B}}}, \iota_p^{\mathcal{B}}] \leq [\Pi_p^{\hat{\mathcal{B}}}, \pi_p^{\mathcal{B}}]$.

(2) $[\mathcal{I}_p^{\hat{\mathcal{B}}}, \iota_p^{\mathcal{B}}] \leq [\hat{\mathcal{B}}_{\mathcal{W}}, \rho_{\mathcal{B}}]$.

Proof. Let $f \in \mathcal{I}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$. Then $S_f \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), X)$ with $\iota_p(S_f) = \iota_p^{\mathcal{B}}(f)$ by Theorem 3.4:

(1) Then $S_f \in \Pi_p(\mathcal{G}(\mathbb{D}), X)$ with $\pi_p(S_f) \leq \iota_p(S_f)$ by [10, Proposition 5.5]. It is immediate to prove that $f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ and $\pi_p^{\mathcal{B}}(f) \leq \pi_p(S_f)$, and this completes the proof.

(2) Now, $S_f \in \mathcal{W}(\mathcal{G}(\mathbb{D}), X)$ with $\|S_f\| \leq \iota_p(S_f)$ by [15, Proposition 19.2.12]. Hence $f \in \hat{\mathcal{B}}_{\mathcal{W}}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(f) = \|S_f\|$ by [13, Theorem 5.6], and we have finished.

□

3.3. **Strictly p-integral Bloch mappings.** In the linear context, an important subclass of p-integral operators appears in [14] when the passage to the bidual in their definition is superfluous. Given two Banach spaces X, Y and $p \in [1,\infty)$, an operator $T \in \mathcal{L}(X, Y)$ is said to be strictly p-integral (or Pietsch p-integral) if there exist a probability measure μ and two operators $R \in \mathcal{L}(L_p(\mu), Y)$ and $S \in \mathcal{L}(X, L_\infty(\mu))$ such that $T = R \circ I_{\infty,p}^{\mu} \circ S$. We set

$$\upsilon_p(T) = \inf\{\|R\|\,\|S\|\},$$

where the infimum is taken over all such factorizations of T as above. The set of all strictly p-integral operators from X into Y is denoted by $\mathcal{PI}_p(X, Y)$, and it is known that $[\mathcal{PI}_p, \upsilon_p]$ is a Banach operator ideal.

The corresponding Bloch version of this concept introduces a new class of Bloch mappings.

Definition 3.4. *Given $p \in [1,\infty)$, a mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be strictly p-integral Bloch or Pietsch p-integral Bloch if there exist a probability measure μ, an operator $T \in \mathcal{L}(L_p(\mu), X)$ and a Bloch mapping $g \in \mathcal{B}(\mathbb{D}, L_\infty(\mu))$ giving rise the commutative diagram*

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "f"] \arrow[d, "g"'] & X \\
L_\infty(\mu) \arrow[r, "I_{\infty,p}^{\mu}"] & L_p(\mu) \arrow[u, "T"']
\end{tikzcd}$$

The triple $(T, I^{\mu}_{\infty,p}, g)$ is called a strictly p-integral Bloch representation of f. We denote by $\mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ the set of all strictly p-integral Bloch maps from $\mathbb{D}$ to X. If in the factorization of f, we also demand that $g(0) = 0$, then the set of such maps f is represented by $\mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$. We set

$$\upsilon^{\mathcal{B}}_p(f) = \inf\{\|T\|\rho_{\mathcal{B}}(g)\},$$

where the infimum is taken over all strictly p-integral Bloch representations of f.

From Definitions 3.3 and 3.4, it is clear that $\mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X) \subseteq \mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ and $\iota^{\mathcal{B}}_p(f) \leq \upsilon^{\mathcal{B}}_p(f)$ for all $f \in \mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$.

With proofs similar to those of Subsection 3.2, we establish the following results.

Proposition 3.6. *Let X be a complex Banach space and $p \in [1, \infty)$. Then $(\mathcal{PI}^{\mathcal{B}}_p(\mathbb{D}, X), \upsilon^{\mathcal{B}}_p)$ is Möbius-invariant.* $\square$

Theorem 3.5. *Let X be a complex Banach space, $p \in [1, \infty)$ and $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. Then $f \in \mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ if and only if $S_f \in \mathcal{PI}_p(\mathcal{G}(\mathbb{D}), X)$, in whose case $\upsilon^{\mathcal{B}}_p(f) = \upsilon_p(S_f)$. As a consequence,*

(1) $f \mapsto S_f$ is an isometric isomorphism from $(\mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X), \upsilon^{\mathcal{B}}_p)$ onto $(\mathcal{PI}_p(\mathcal{G}(\mathbb{D}), X), \upsilon_p)$,

(2) $[\mathcal{PI}^{\hat{\mathcal{B}}}_p, \upsilon^{\mathcal{B}}_p] = [\mathcal{PI} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{PI}\circ\hat{\mathcal{B}}}]$, and thus $[\mathcal{PI}^{\hat{\mathcal{B}}}_p, \upsilon^{\mathcal{B}}_p]$ is a Banach normalized Bloch ideal. $\square$

3.4. **p-Nuclear Bloch mappings.** Let us recall from [15, Definition 18.1.1] that an operator $T \in \mathcal{L}(X, Y)$ is said to be p-nuclear for $p \in [1, \infty)$ if $T = R \circ M_\lambda \circ S$, where $S \in \mathcal{L}(X, \ell_\infty)$, $R \in \mathcal{L}(\ell_p, Y)$ and M_λ denotes the diagonal operator from ℓ_∞ to ℓ_p defined by $M_\lambda((x_n)) = (\lambda_n x_n)$ for all $(x_n) \in \ell_\infty$, being $\lambda = (\lambda_n) \in \ell_p$. We denote $\mathcal{N}_p(X, Y)$ to the set of all p-nuclear operators from X into Y. It is known that $\mathcal{N}_p$ is a Banach operator ideal under the norm

$$\nu_p(T) = \inf\{\|R\| \, \|\lambda\|_p \, \|S\|\},$$

where the infimum is taken over all representations of T as above.

Next, we introduce the analogue to the concept of p-nuclear operator in the Bloch setting.

Definition 3.5. *Let $p \in [1, \infty)$, let X be a complex Banach space and $f \in \mathcal{H}(\mathbb{D}, X)$. We say that f is p-nuclear Bloch if the following diagram commutes*

$$\begin{tikzcd} \mathbb{D} \arrow[r, "f"] \arrow[d, "g"'] & X \\ \ell_\infty \arrow[r, "M_\lambda"] & \ell_p \arrow[u, "T"'] \end{tikzcd}$$

where $g \in \mathcal{B}(\mathbb{D}, \ell_\infty), T \in \mathcal{L}(\ell_p, X)$ and $\lambda = (\lambda_n) \in \ell_p$. We say that the triple (T, M_λ, g) is a p-nuclear Bloch representation of f. The set of all p-nuclear Bloch mappings from $\mathbb{D}$ to X is denoted by $\mathcal{N}^{\mathcal{B}}_p(\mathbb{D}, X)$. If in the factorization of f, we also require that $g(0) = 0$, then $\mathcal{N}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ stands the set of all such mappings f. Define

$$\nu^{\mathcal{B}}_p(f) = \inf\{\|T\| \, \|\lambda\|_p \, \rho_{\mathcal{B}}(g)\},$$

where the infimum is taken over all the above factorizations.

As in the preceding sections, we can prove the following.

Proposition 3.7. *Let X be a complex Banach space and $p \in [1,\infty)$. Then $(\mathcal{N}_p^{\mathcal{B}}(\mathbb{D},X),\nu_p^{\mathcal{B}})$ is Möbius-invariant.* □

We first study the relationship between a p-nuclear Bloch mapping and its linearization.

Theorem 3.6. *Let $p \in [1,\infty)$, let X be a complex Banach space and $f \in \hat{\mathcal{B}}(\mathbb{D},X)$. Then $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D},X)$ if and only if $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}),X)$. In this case, $\nu_p^{\mathcal{B}}(f) = \nu_p(S_f)$. Furthermore, $(\mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D},X),\nu_p^{\mathcal{B}})$ and $(\mathcal{N}_p(\mathcal{G}(\mathbb{D}),X),\nu_p)$ are isometrically isomorphic via the mapping $f \mapsto S_f$.*

Proof. Let $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D},X)$. We can write $f = T \circ M_\lambda \circ g$, where $T \in \mathcal{L}(\ell_p,X)$, $\lambda \in \ell_p$ and $g \in \hat{\mathcal{B}}(\mathbb{D},\ell_\infty)$. An application of Theorem 2.1 shows that $S_f = T \circ M_\lambda \circ S_g$ where $S_g \in \mathcal{L}(\mathcal{G}(\mathbb{D}),\ell_\infty)$. Hence $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}),X)$ with

$$\nu_p(S_f) \leq \|T\| \|M_\lambda\|_p \, \|S_g\| = \|T\| \|M_\lambda\|_p \, \rho_{\mathcal{B}}(g).$$

Taking the infimum over all p-nuclear Bloch representations of f, we conclude that $\nu_p(S_f) \leq \nu_p^{\mathcal{B}}(f)$.

Conversely, suppose that $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}),X)$ and let $\varepsilon > 0$. Hence we can assure the existence of $\lambda \in \ell_p$, $R \in \mathcal{L}(\mathcal{G}(\mathbb{D}),\ell_\infty)$ and $S \in \mathcal{L}(\ell_p,X)$ such that $S_f = S \circ M_\lambda \circ R$ and $\|S\| \|\lambda\|_p \, \|R\| \leq (1+\varepsilon)\nu_p(S_f)$. Thus $f' = S_f \circ \Gamma = S \circ M_\lambda \circ R \circ \Gamma$.

Since $R \circ \Gamma \in \mathcal{H}(\mathbb{D},\ell_\infty)$, by [13, Lemma 2.9], we can find $h \in \mathcal{H}(\mathbb{D},\ell_\infty)$ with $h(0) = 0$ such that $h' = R \circ \Gamma$, and in fact $h \in \hat{\mathcal{B}}(\mathbb{D},\ell_\infty)$ with $\rho_{\mathcal{B}}(h) \leq \|R\|$. Hence we have

$$f' = S \circ M_\lambda \circ R \circ \Gamma = S \circ M_\lambda \circ h' = (S \circ M_\lambda \circ h)'.$$

It follows that $f = S \circ M_\lambda \circ h$, and therefore $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D},X)$ with

$$\nu_p^{\mathcal{B}}(f) \leq \|S\| \|\lambda\|_p \, \rho_{\mathcal{B}}(h) \leq \|S\| \|\lambda\|_p \, \|R\| \leq (1+\varepsilon)\nu_p(S_f).$$

Just letting $\varepsilon \to 0$, we obtain $\nu_p^{\mathcal{B}}(f) \leq \nu_p(S_f)$. □

In view of Theorem 3.6, both Theorems 2.2 and Corollary 2.1 yield the following.

Corollary 3.5. *Let $p \in [1,\infty)$. Then $[\mathcal{N}_p^{\hat{\mathcal{B}}},\nu_p^{\mathcal{B}}] = [\mathcal{N}_p \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{N}_p \circ \hat{\mathcal{B}}}]$. In particular, $[\mathcal{N}_p^{\hat{\mathcal{B}}},\nu_p^{\mathcal{B}}]$ is a Banach ideal of normalized Bloch mappings.* □

Applying Theorem 3.6, [10, Corollary 5.24] and [13, Theorem 5.9] gives some relations.

Corollary 3.6. *(1) $[\mathcal{N}_p^{\hat{\mathcal{B}}},\nu_p^{\mathcal{B}}] \leq [\mathcal{N}_q^{\hat{\mathcal{B}}},\nu_q^{\mathcal{B}}]$ whenever $1 \leq p \leq q < \infty$,*

(2) $[\mathcal{N}_p^{\hat{\mathcal{B}}},\nu_p^{\mathcal{B}}] \leq [\mathcal{PI}_p^{\hat{\mathcal{B}}},\upsilon_p^{\mathcal{B}}]$ and $[\mathcal{N}_p^{\hat{\mathcal{B}}},\nu_p^{\mathcal{B}}] \leq [\hat{\mathcal{B}}_{\overline{\mathcal{F}}},\rho_{\mathcal{B}}]$ whenever $1 \leq p < \infty$.

□

With Theorems 5.27 and 5.28 of [10] in mind, it is natural to expect some composition results.

Corollary 3.7. *Let $p \in [1,\infty)$, let X be a complex Banach space and $f \in \hat{\mathcal{B}}(\mathbb{D},X)$. The following are equivalent:*

(1) $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D},X)$.

(2) There exist a complex Banach space Y, $T \in \mathcal{K}(Y,X)$ and $g \in \mathcal{I}_p^{\hat{\mathcal{B}}}(\mathbb{D},Y)$ such that $f = T \circ g$.

(3) There is a complex Banach space Y, $T \in \mathcal{PI}_p(Y,X)$ and $g \in \hat{\mathcal{B}}_{\mathcal{K}}(\mathbb{D},Y)$ such that $f = T \circ g$.

In this case, taking the infimum extended over all such factorizations yields

$$\nu_p^{\mathcal{B}}(f) = \inf\left\{\|T\| \, \iota_p^{\mathcal{B}}(g)\right\} = \inf\left\{\upsilon_p(T)\rho_{\mathcal{B}}(g)\right\}.$$

Proof.

(1) ⇒ (2): If $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$, then $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}), X)$ with $\nu_p^{\mathcal{B}}(f) = \nu_p(S_f)$ by Theorem 3.6. Given $\varepsilon > 0$, by [10, Theorem 5.27], we can take a complex Banach space Y, $T \in \mathcal{K}(Y, X)$ and $S \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), Y)$ such that $S_f = T \circ S$ and $\|T\|\, \iota_p(S) \leq \nu_p(S_f) + \varepsilon$. By Theorem 3.4, there is a $g \in \mathcal{I}_p^{\hat{\mathcal{B}}}(\mathbb{D}, Y)$ such that $S_g = S$ and $\iota_p^{\mathcal{B}}(g) = \iota_p(S)$. Hence $S_f = T \circ S_g$, therefore $f' = S_f \circ \Gamma = T \circ S_g \circ \Gamma = T \circ g' = (T \circ g)'$ and this implies that $f = T \circ g$. Further, $\|T\|\, \iota_p^{\mathcal{B}}(g) = \|T\|\, \iota_p(S) \leq \nu_p(S_f) + \varepsilon = \nu_p^{\mathcal{B}}(f) + \varepsilon$, and the arbitrariness of ε gives $\|T\|\, \iota_p^{\mathcal{B}}(g) \leq \nu_p^{\mathcal{B}}(f)$.

(2) ⇒ (1): Assume that $f = T \circ g$ with $T \in \mathcal{K}(Y, X)$ and $g \in \mathcal{I}_p^{\hat{\mathcal{B}}}(\mathbb{D}, Y)$ for some complex Banach space Y. Hence $S_f = T \circ S_g$ with $S_g \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), Y)$ and $\iota_p(S_g) = \iota_p^{\mathcal{B}}(g)$ by Theorem 3.4. Then $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}), X)$ with $\nu_p(S_f) \leq \|T\|\, \iota_p(S_g)$ by [10, Theorem 5.27]. We conclude that $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ with $\nu_p^{\mathcal{B}}(f) = \nu_p(S_f)$ by Theorem 3.6. Moreover, $\nu_p^{\mathcal{B}}(f) \leq \|T\|\, \iota_p^{\mathcal{B}}(g)$ and since we are working with an arbitrary factorization $T \circ g$ for f, we get that $\nu_p^{\mathcal{B}}(f) \leq \inf\left\{\|T\|\, \iota_p^{\mathcal{B}}(g)\right\}$.

(1) ⇔ (3): This can be proved as the preceding implications by using now [10, Theorem 5.28] and [13, Theorem 5.4] instead of [10, Theorem 5.27] and Theorem 3.4, respectively.

□

4. Bloch dual ideal of an operator ideal

Following [15, Section 4.4], given a normed operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$, recall that for any normed spaces X and Y, the components

$$\mathcal{I}^{\text{dual}}(X, Y) := \{T \in \mathcal{L}(X, Y) \colon T^* \in \mathcal{I}(Y^*, X^*)\},$$

endowed with the norm

$$\|T\|_{\mathcal{I}^{\text{dual}}} = \|T^*\|_{\mathcal{I}} \qquad (T \in \mathcal{I}^{\text{dual}}(X, Y)),$$

define a normed operator ideal $[\mathcal{I}^{\text{dual}}, \|\cdot\|_{\mathcal{I}^d}]$ called dual ideal of $\mathcal{I}$. Moreover, $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is called symmetric if $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}] \leq [\mathcal{I}^{\text{dual}}, \|\cdot\|_{\mathcal{I}^{\text{dual}}}]$. In the case $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}] = [\mathcal{I}^{\text{dual}}, \|\cdot\|_{\mathcal{I}^{\text{dual}}}]$, the operator ideal is said to be completely symmetric.

With the aid of the notion of transpose of a Bloch mapping, we now introduce the next concept.

Definition 4.6. *Let $\mathcal{I}$ be an operator ideal. For any complex Banach space X, we define*

$$\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X) = \{f \in \hat{\mathcal{B}}(\mathbb{D}, X) \colon f^t \in \mathcal{I}(X^*, \hat{\mathcal{B}}(\mathbb{D}))\}.$$

If $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is a normed operator ideal, we set

$$\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}} = \|f^t\|_{\mathcal{I}} \qquad (f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)).$$

We now show that $[\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}}]$ is really an ideal of normalized Bloch mappings.

Theorem 4.7. *Let X be a complex Banach space and $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. If $\mathcal{I}$ is an operator ideal, then $f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)$ if and only if $f \in \mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$. Moreover, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal, then $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}} = \|f\|_{\mathcal{I}^{dual} \circ \hat{\mathcal{B}}}$ for all $f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)$.*

Proof. Let us assume that $f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)$. Then $f^t \in \mathcal{I}(X^*, \hat{\mathcal{B}}(\mathbb{D}))$. By Theorem 2.1, there exists $S_f \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$ such that $S_f \circ \Gamma = f'$ and also $(S_f)^* = \Lambda \circ f^t$. Hence $(S_f)^* \in \mathcal{I}(X^*, \mathcal{G}(\mathbb{D})^*)$ and therefore $S_f \in \mathcal{I}^{\text{dual}}(\mathcal{G}(\mathbb{D}), X)$. Thus, by Theorem 2.2, we have $f \in \mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$ with $\|f\|_{\mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}} = \|S_f\|_{\mathcal{I}^{\text{dual}}}$. Further,

$$\|f\|_{\mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}} = \|S_f\|_{\mathcal{I}^{\text{dual}}} = \|(S_f)^*\|_{\mathcal{I}} = \|\Lambda \circ f^t\|_{\mathcal{I}} \leq \|\Lambda\| \|f^t\|_{\mathcal{I}} = \|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}}.$$

Conversely, let $f \in \mathcal{I}^{\mathrm{dual}} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$. Then there are a complex Banach space Y, a mapping $g \in \hat{\mathcal{B}}(\mathbb{D}, Y)$ and an operator $T \in \mathcal{I}^{\mathrm{dual}}(Y, X)$ such that $f = T \circ g$. Given $x^* \in X^*$, we have

$$f^t(x^*) = (T \circ g)^t(x^*) = x^* \circ (T \circ g) = (x^* \circ T) \circ g = T^*(x^*) \circ g = g^t(T^*(x^*)) = (g^t \circ T^*)(x^*),$$

and thus $f^t = g^t \circ T^*$. Since $T^* \in \mathcal{I}(X^*, Y^*)$ and $g^t \in \mathcal{L}(Y^*, \hat{\mathcal{B}}(\mathbb{D}))$, we obtain that $f^t \in \mathcal{I}(X^*, \hat{\mathcal{B}}(\mathbb{D}))$. Hence $f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)$ and moreover, we have

$$\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}} = \left\|f^t\right\|_{\mathcal{I}} = \left\|g^t \circ T^*\right\|_{\mathcal{I}} \leq \left\|g^t\right\| \left\|T^*\right\|_{\mathcal{I}} = \rho_{\mathcal{B}}(g) \left\|T\right\|_{\mathcal{I}^{\mathrm{dual}}},$$

and taking the infimum over all representations $T \circ g$ of f, we conclude that $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}} \leq \|f\|_{\mathcal{I}^{\mathrm{dual}} \circ \hat{\mathcal{B}}}$. □

An immediate consequence of Theorem 4.7 is the following.

Corollary 4.8. $[\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}}] = [\mathcal{I} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I} \circ \mathcal{B}}]$ *whenever* $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ *is a completely symmetric normed operator ideal.* □

Since the ideal $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}$ is completely symmetric by [15, Proposition 4.4.7], Corollary 4.8 combined with Theorem 2.2 and [13, Theorems 5.4, 5.6, 5.7 and 5.9] yield the following identifications.

Corollary 4.9. $[\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}}] = [\hat{\mathcal{B}}_{\mathcal{I}}, \rho_{\mathcal{B}}]$ *for* $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}$. □

Since the normed operator ideal $[\mathcal{I}_1, \iota_1]$ is completely symmetric by [10, Theorem 5.15], Corollaries 4.8 and 3.2 give the following result.

Corollary 4.10. $[(\mathcal{I}_1)^{\hat{\mathcal{B}}\text{-dual}}, \|\cdot\|_{(\mathcal{I}_1)^{\hat{\mathcal{B}}\text{-dual}}}] = [\mathcal{I}_1^{\hat{\mathcal{B}}}, \iota_1^{\mathcal{B}}]$. □

5. Conclusions

This study has presented a unified method for generating ideals of Möbius-invariant Bloch mappings by composition of a member of a distinguished Banach operator ideal and a Bloch mapping on the complex unit open disc. Our approach is based on the application of a known technique of linearization of Bloch mappings. The aforementioned method permits an extensive exploration into new classes of Bloch mappings in connection with known Banach operator ideals. This highlights the close interconnection between the linear setting and the holomorphic setting. Notably, the study has drawn meaningful results which contributes to understanding of richness of Bloch mappings.

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