Eastern Anatolian Journal of Science
Volume X, Issue II, 2024, 37-46
Eastern Anatolian Journal of Science

On the Pseudo Starlike and Pseudo Convex Bi-univalent Function Classes of Complex Order

Nizami MUSTAFA [1], Veysel NEZİR [2] and Arzu KANKILIÇ [3*]

[1] *Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey, nizammstafa@gmail.com*

[2] *Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey, veyselnezir@yahoo.com*

[*3] *Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey, arzucalimgil@gmail.com*

Abstract

In this paper, we defined a new subclass of starlike and convex bi-univalent functions and examine some geometric properties this function class. For this definition class, we gave some coefficient estimates and solve Fekete-Sezöge problem.

Keywords: Starlike function, convex function, pseudo starlike function, pseudo convex function

1. Introduction

In this section, we give some basic information which we will use in our study.

Let $H(U)$ be the class of analytic functions on the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ in the complex plane $\mathbb{C}$. By A, we will denote the class of the functions $f\in H(U)$ given by the following series expansions

$$f(z)=z+a_2z^2+a_3z^3+\cdots = z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}. \tag{1.1}$$

The subclass of univalent functions of A is denoted by S in the literature. This class was first introduced by Köebe (Köebe 1909) and has become the core ingredient of advanced research in this field. Many mathematicians were interested coefficient estimates for this class. Within a short period, Bieberbach published a (Bieberbach 1916) paper in which the famous coefficient hypothesis was proposed. This hypothesis states that if $f\in S$ and has the series form (1.1), then $|a_n|\le n$ for each $n\ge 2$. In 1985, it was de-Branges (de-Branges 1985), who settled this long-lasting conjecture. There were a lot of papers devoted to this conjecture and its related coefficient problems (Brannan, Kirwan 1969, Janowski 1970, Sokol, Stankiewcz 1996, Sharmaet all 1996, Frasin, Aouf 2011, Sharma et all 2016, Arif et all 2019, Cho et all 2019, Kumar, Arora 2020, Mendiratta et all 2015, Bano, Raza 2020, Alotaibi et all 2020, Ullah et all 2021, Mustafa, Nezir, Kankılıç 2023a)

As is known that the function $f(z)$ is called a bi-univalent function, if itself and inverse is univalent in U and $f(U)$, respectively. The class of bi-univalent functions is denoted by Σ.

We will denote $g(w)=f^{-1}(w),\ w\in f(U)$. In this case, if $f\in\Sigma$

$$\begin{aligned} g(w) &= w+A_2w^2+A_3w^3+A_4w^4+\dots \\ &= w+\sum_{n=2}^{\infty}A_nz^n,\ w\in f(U), \end{aligned} \tag{1.2}$$

where

$A_2=-a_2,\ A_3=2a_2^2-a_3,\ A_4=-a_2^3+5a_2a_3-a_4,\dots$.

It is well known that the bi-starlike and bi-convex function classes defined on the open unit disk U are defined analytically as follows

$$S_{\Sigma}^{*}=\left\{\begin{array}{l} f\in S:\ \operatorname{Re}\left(\dfrac{zf'(z)}{f(z)}\right)>0,\ z\in U \\ \text{and } \operatorname{Re}\left(\dfrac{wg'(w)}{g(w)}\right)>0,\ w\in f(U)\end{array}\right\},$$

Received:19.07.2024
Revised:10.12.2024
Accepted:18.12.2024
**Corresponding author:* Arzu Kankılıç, *PhD*
Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey,
E-mail: arzucalimgil@gmail.com
Cite this article as: A. Kankılıç et al., On the Pseudo Starlike Convex Bi-univalent Function Classes of Complex Order, *Eastern Anatolian Journal of Science, Vol.10 Issue 2, 37-46,2024.*

$$C_\Sigma = \left\{ \begin{array}{l} f \in S: \ \mathrm{Re}\left(\dfrac{(zf'(z))'}{f'(z)} \right) > 0, \ z \in U \\ \text{and } \mathrm{Re}\left(\dfrac{(zg'(w))'}{g'(w)} \right) > 0, \ w \in f(U) \end{array} \right\}.$$

2. Materials and Methods

It is well-known that an analytical function ω satisfying the conditions $\omega(0)=0$ and $|\omega(z)|<1$ is called Schwartz function. Let's $f, g \in H(U)$, then it is said that f is subordinate to g and denoted by $f \prec g$, if there exists a Schwartz function ω, such that $f(z) = g(\omega(z))$.
Ma and Minda using subordination terminology presented unified version of the classes $S^*(\varphi)$ and $C(\varphi)$ as follows

$$(S^* \vee C)(\varphi) = \left\{ \begin{array}{l} f \in S: \ (1-\beta)\dfrac{zf'(z)}{f(z)} + \beta \dfrac{(zf'(z))'}{f'(z)} \prec \varphi(z), \\ z \in U \end{array} \right\},$$

$\beta \in [0,1]$,
where $\varphi(z)$ is a univalent function with $\varphi(0)=1$, $\varphi'(0)>0$ and the region $\varphi(U)$ is star-shaped about the point $\varphi(0)=1$ and symmetric with respect to real axis. Such a function has a series expansion of the following form

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$$
$$= 1 + \sum_{n=1}^{\infty} b_n z^n, \ b_1 > 0.$$

In the past few years, numerous subclasses of the collection S have been introduced as special choices of the classes $S^*(\varphi)$ and $C(\varphi)$ (Brannan, Kirwan 1969, Sokol, Stankiewcz 1996, Mendiratta et all 2015, Sharma et all 2016, Arif et all 2019, Shi et all 2019, Cho et all 2019, Kumar, Arora 2020, Alotaibi et all 2020, Ullah et all 2021, Mustafa, Nezir, Kankılıç 2023a, Frasin, Aouf 2011, Mustafa, Nezir, Kankılıç 2023b, Mustaf, Nezir 2023, Mustafa , Demir 2023a, Mustafa, Demir, 2023b, Mustafa, Nezir, Kankılıç 2023c, Mustafa, Nezir, Kankılıç 2023d, Mustafa, Demir 2023d).

Now, let's define some new subclass of bi-univalent functions in the open unit disk U.

Definition 2.1. For $\beta \in [0,1]$, $\lambda > \frac{1}{2}$ and $\tau \in \mathbb{C} - \{0\}$ the function $f \in \Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$, if the following conditions are satisfied

$$(1-\beta)\left\{1 + \frac{1}{\tau}\left[\frac{z(f'(z))^\lambda}{f(z)} - 1\right]\right\}$$
$$+\beta\left\{1 + \frac{1}{\tau}\left[\frac{\left[(zf'(z))'\right]^\lambda}{f'(z)} - 1\right]\right\}$$
$$\prec 1 + \sinh z, \ z \in U,$$
$$(1-\beta)\left\{1 + \frac{1}{\tau}\left[\frac{w(g'(w))^\lambda}{g(w)} - 1\right]\right\}$$
$$+\beta\left\{1 + \frac{1}{\tau}\left[\frac{\left[(wg'(w))'\right]^\lambda}{g'(w)} - 1\right]\right\}$$
$$\prec 1 + \sinh w, \ w \in f(U).$$

From the Definition 2.1, in the special values of the parameters, we obtain the following function classes.

Definition 2.2. For $\lambda > \frac{1}{2}$ and $\tau \in \mathbb{C} - \{0\}$ the function $f \in \Sigma$ is said to be in the class $S^*_{\Sigma,\sinh}(\lambda,\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right]\right\}\prec 1+\sinh z,\ z\in U,$$

$$\left\{1+\frac{1}{\tau}\left[\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}-1\right]\right\}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.2.1. For $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $S^{*}_{\Sigma,\sinh}(\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}\prec 1+\sinh z,\ z\in U,$$

$$\left\{1+\frac{1}{\tau}\left[\frac{wg'(w)}{g(w)}-1\right]\right\}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.2.2. For $\lambda>\frac{1}{2}$ the function $f\in\Sigma$ is said to be in the class $S^{*}_{\Sigma,\sinh}(\lambda)$, if the following conditions are satisfied

$$\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\prec 1+\sinh z,\ z\in U,$$

$$\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.2.3. For the function $f\in\Sigma$ is said to be in the class $S^{*}_{\Sigma,\sinh}$, if the following conditions are satisfied

$$\frac{zf'(z)}{f(z)}\prec 1+\sinh z,\ z\in U,$$

$$\frac{wg'(w)}{g(w)}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.3. For $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\lambda,\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}-1\right]\right\}\prec 1+\sinh z,\ z\in U,$$

$$\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}-1\right]\right\}\prec 1+\sinh w,$$

$w\in f(U)$.

Definition 2.3.1. For $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)}-1\right]\right\}\prec 1+\sinh z,\ z\in U,$$

$$\left\{1+\frac{1}{\tau}\left[\frac{\left(wg'(w)\right)'}{g'(w)}-1\right]\right\}\prec 1+\sinh w, w\in f(U).$$

Definition 2.3.2. For $\lambda>\frac{1}{2}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\lambda)$, if the following conditions are satisfied

$$\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}\prec 1+\sinh z,\ z\in U,$$

$$\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.3.3. For the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}$, if the following conditions are satisfied

$$\frac{\left(zf'(z)\right)'}{f'(z)}\prec 1+\sinh z,\ z\in U,$$

$$\frac{\left(wg'(w)\right)'}{g'(w)}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.4. For $\beta \in [0,1]$ and $\tau \in \mathbb{C}-\{0\}$ the function $f \in \Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta,1,\tau)$, if the following conditions are satisfied

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}$$
$$+\beta\left\{1+\frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)}-1\right]\right\}$$
$$\prec 1+\sinh z,\ z\in U,$$
$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{wg'(w)}{g(w)}-1\right]\right\}$$
$$+\beta\left\{1+\frac{1}{\tau}\left[\frac{(wg'(w))'}{g'(w)}-1\right]\right\}$$
$$\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.4.1. For $\beta \in [0,1]$ the function $f \in \Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta)$, if the following conditions are satisfied

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{(zf'(z))'}{f'(z)}\prec 1+\sinh z,\ z\in U$$
,
$$(1-\beta)\frac{w(g'(w))}{g(w)}+\beta\frac{(wg'(w))'}{g'(w)}\prec 1+\sinh w,$$
$$w\in f(U).$$

Definition 2.5. For $\beta \in [0,1]$ and $\lambda > \frac{1}{2}$ the function $f \in \Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta,\lambda)$, if the following conditions are satisfied

$$(1-\beta)\frac{z(f'(z))^{\lambda}}{f(z)}+\beta\frac{\left[(zf'(z))'\right]^{\lambda}}{f'(z)}$$
$$\prec 1+\sinh z,\ z\in U,$$
$$(1-\beta)\frac{w(g'(w))^{\lambda}}{g(w)}+\beta\frac{\left[(wg'(w))'\right]^{\lambda}}{g'(w)}.$$
$$\prec 1+\sinh w,\ w\in f(U).$$

Remark 2.1. Let's point out that the classes $S^{*}_{\Sigma,\sinh}(\lambda,\tau)$ and $C_{\Sigma,\cosh}(\lambda,\tau)$ was investigated by Kankılıç and Mustafa (Kankılıç, Mustafa 2023a, Kankılıç, Mustafa 2023b).

Let P be the class of analytic functions in U satisfied the conditions $p(0)=1$ and $\mathrm{Re}(p(z))>0$, $z\in U$, which from the subordination principle easily can written

$$\mathrm{P}=\left\{p\in A:\ p(z)\prec\frac{1+z}{1-z}, z\in U\right\},$$

where $p(z)$ has the series expansion of the form

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$$
$$=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U\ . \tag{2.1}$$

The class P defined above is known as the class Caratheodory functions (Caratheodory 1907).

Now, let us present some necessary lemmas known in the literature for the proof of our main results.

Lemma 2.1 (Duren 1983). Let the function $p(z)$ belong in the class P. Then,

$|p_n|\le 2$ for each $n\in\mathbb{N}$ and $|p_n-\lambda p_k p_{n-k}|\le 2$ for $n,k\in\mathbb{N},\ n>k$ and $\lambda\in[0,1]$.

The equalities hold for

$$p(z)=\frac{1+z}{1-z}.$$

Lemma 2.2 (Duren 1983) Let the an analytic function $p(z)$ be of the form (2.1, then

$$2p_2=p_1^2+(4-p_1^2)x,$$
$$4p_3$$
$$=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2$$
$$+2(4-p_1^2)(1-|x|^2)y$$

for $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$.

In this paper, we give some coefficient estimates and examine Fekete-Szegö problem for the class $\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$. Additionally, the results obtained for specific values of the parameters in our study are compared with the results obtained in the literature.

3. *Results*

In this section, we examine the coefficient estimates problem for the function class $\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$.

Theorem 3.1. If $f\in\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$, then are provided the following inequalities

$$|a_2|\leq\frac{|\tau|}{(2\lambda-1)(1+\beta)} \text{ and}$$

$$|a_3| \leq|\tau|\begin{cases}\dfrac{1}{(3\lambda-1)(1+2\beta)}, & |\tau|\leq\dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)},\\ \dfrac{|\tau|}{(2\lambda-1)^2(1+\beta)^2}, & |\tau|\geq\dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}.\end{cases} \tag{3.1}$$

Proof. Let $f\in\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$, $\beta\in[0,1]$, $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$. Then, are Schwartz functions $\omega:U\to U,\varpi:U_{r_0}\to U_{r_0}$, such that

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}-1\right]\right\},z\in U$$
$$=1+\sinh\omega(z)$$

and

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}-1\right]\right\} \tag{3.2}$$
$$=1+\sinh\varpi(w),w\in f(U).$$

Let's the functions $p,q\in P$ defined as follows:

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+p_3z^3+...$$
$$=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U,$$

$$q(w)=\frac{1+\varpi(w)}{1-\varpi(w)}=1+q_1w+q_2w^2+q_3w^3+...$$
$$=1+\sum_{n=1}^{\infty}q_nw^n,\ w\in f(U). \tag{3.3}$$

From these equalities, we can write

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_1}{2}z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+\frac{1}{2}\left(p_3-p_1p_2-\frac{p_1^3}{4}\right)z^3...,\ z\in U,$$

$$\varpi(w)=\frac{q(w)-1}{q(w)+1}=\frac{q_1}{2}w+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)w^2+\frac{1}{2}\left(q_3-q_1q_2-\frac{q_1^3}{4}\right)w^3...,w\in f(U). \tag{3.4}$$

Then, from the (3.1) and (3.4) can written the following equalities

$$(1-\beta)\times \left\{1+\frac{1}{\tau}\left[\begin{array}{l} a_2(2\lambda-1)z \\ +\left((3\lambda-1)a_3+\left(2\lambda^2-4\lambda+1\right)a_2^2\right)z^2+\ldots \end{array}\right]\right\}$$

$$+\beta\left\{\begin{array}{l}1+ \\ \frac{1}{\tau}\left[\begin{array}{l} 2a_2(2\lambda-1)z+ \\ \left(3(3\lambda-1)a_3+\left(8\lambda^2-16\lambda+4\right)a_2^2\right)z^2+\ldots \end{array}\right]\end{array}\right\}$$

$$=1+\frac{p_1}{2}z+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\ldots,\ z\in U,$$

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\begin{array}{l} A_2(2\lambda-1)w \\ +\left(\begin{array}{l}(3\lambda-1)A_3 \\ +\left(2\lambda^2-4\lambda+1\right)A_2^2\end{array}\right)w^2+\ldots \end{array}\right]\right\}$$

$$+\beta\left\{1+\frac{1}{\tau}\left[\begin{array}{l} 2A_2(2\lambda-1)w \\ +\left(\begin{array}{l}3(3\lambda-1)A_3 \\ +\left(8\lambda^2-16\lambda+4\right)A_2^2\end{array}\right)w^2+\ldots \end{array}\right]\right\}$$

$$=1+\frac{q_1}{2}w+\left(\frac{q_2}{2}-\frac{q_1^2}{4}\right)w^2+\ldots,$$

$$w\in f(U). \tag{3.5}$$

Comparing the coefficients of the same degree terms on the right and left sides of the equalities (3.5), we obtain the following equalities for the coefficients a_2 and a_3 of the function f

$$\frac{1}{\tau}a_2(2\lambda-1)(1+\beta)=\frac{p_1}{2}, \tag{3.6}$$

$$\frac{1}{\tau}\left[\begin{array}{l}(3\lambda-1)(1+2\beta)a_3 \\ +\left(2\lambda^2-4\lambda+1\right)(1+3\beta)a_2^2\end{array}\right]=\frac{p_2}{2}-\frac{p_1^2}{4}, \tag{3.7}$$

$$-\frac{1}{\tau}a_2(2\lambda-1)(1+\beta)=\frac{q_1}{2}, \tag{3.8}$$

$$\frac{1}{\tau}\left[\begin{array}{l}(3\lambda-1)(1+2\beta)\left(2a_2^2-a_3\right) \\ +\left(2\lambda^2-4\lambda+1\right)(1+3\beta)a_2^2\end{array}\right]=\frac{q_2}{2}-\frac{q_1^2}{4}. \tag{3.9}$$

From the equalities (3.6) and (3.8), we can write

$$\frac{\tau p_1}{2(2\lambda-1)(1+\beta)}=a_2=-\frac{\tau q_1}{2(2\lambda-1)(1+\beta)}; \tag{3.10}$$

that is,

$$p_1=-q_1. \tag{3.11}$$

Thus, according to the Lemma 1.1, we obtain the first result of the theorem.

Using the equality (3.11), from the equalities (3.7) and (3.9) we obtain the following equality for a_3

$$a_3=\frac{\tau^2}{4(2\lambda-1)^2(1+\beta)^2}p_1^2+\frac{\tau(p_2-q_2)}{4(3\lambda-1)(1+2\beta)}. \tag{3.12}$$

From the Lemma 1.2, we can write

$$p_2-q_2=\frac{\left(4-p_1^2\right)}{2}(x-y)$$

for $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$. Substitute this expression for the p_2-q_2 difference in (3.12), we get

$$a_3= =\frac{\tau^2}{4(2\lambda-1)^2(1+\beta)^2}p_1^2+\frac{\tau\left(4-p_1^2\right)(x-y)}{8(3\lambda-1)(1+2\beta)} \tag{3.13}$$

Applying triangle inequality to the last equality, we obtain

$$|a_3| \le \frac{|\tau|^2}{4(2\lambda-1)^2(1+\beta)^2}t^2 + \frac{|\tau|}{4(3\lambda-1)(1+2\beta)}.\frac{(4-t^2)}{2}(\xi+\eta), \quad (3.14)$$
$$(\xi,\eta)\in[0,1]^2,$$

where $\xi=|x|$, $\eta=|y|$ and $t=|p_1|$.

From the inequality (3.14), we can write

$$|a_3| \le \frac{|\tau|}{4}\left[a(|\tau|,\lambda,\beta)t^2 + \frac{4}{(3\lambda-1)(1+2\beta)}\right], \quad t\in[0,2], \quad (3.15)$$

where

$$a(|\tau|,\lambda,\beta) = \left(\frac{|\tau|}{(2\lambda-1)^2(1+\beta)^2} - \frac{1}{(3\lambda-1)(1+2\beta)}\right).$$

Then, maximizing the function

$$\chi(t)=|\tau|\left(a(\lambda,\beta)t^2+\frac{1}{(3\lambda-1)(1+2\beta)}\right),$$

we obtain the second result of theorem.

With this the proof of theorem is completed.

Taking $\beta=0$ and $\beta=1$ in the Theorem 3.1, we obtain the following results, respectively.

Corollary 3.1. If $f\in S^*_{\Sigma,\sinh}(\lambda,\tau)$, then

$$|a_2| \le \frac{|\tau|}{2\lambda-1} \text{ and}$$

$$|a_3| \le |\tau| \begin{cases} \frac{1}{3\lambda-1}, & |\tau| \le \frac{(2\lambda-1)^2}{3\lambda-1}, \\ \frac{|\tau|}{(2\lambda-1)^2}, & |\tau| \ge \frac{(2\lambda-1)^2}{3\lambda-1}. \end{cases}$$

Corollary 3.2. If $f\in C_{\Sigma,\sinh}(\lambda,\tau)$, then

$$|a_2| \le \frac{|\tau|}{2(2\lambda-1)} \text{ and}$$

$$|a_3| \le |\tau| \begin{cases} \frac{1}{3(3\lambda-1)}, & |\tau| \le \frac{4(2\lambda-1)^2}{3(3\lambda-1)}, \\ \frac{|\tau|}{4(2\lambda-1)^2}, & |\tau| \ge \frac{4(2\lambda-1)^2}{3(3\lambda-1)}. \end{cases}$$

Note: 3.1. In the special values of the parameters λ and τ from Corollary 3.1 and Corollary 3.2, we obtain the results for the classes $S^*_{\Sigma,\sinh}(\tau)$, $S^*_{\Sigma,\sinh}(\lambda)$, $S^*_{\Sigma,\sinh}$ and $C_{\Sigma,\sinh}(\tau)$, $C_{\Sigma,\sinh}(\lambda)$, $C_{\Sigma,\sinh}$, respectively.

4. The Fekete-Szegö problem

In this section, we focused on the solution of the Fekete-Szegö problem for the class $\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$.

Theorem 4.1. Let $f\in\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$ and $\mu\in\mathbb{C}$. Then,

$$|a_3-\mu a_2^2| \le |\tau| \begin{cases} \frac{1}{(3\lambda-1)(1+2\beta)} & \text{if } |1-\mu||\tau| \le l(\lambda,\beta), \\ \frac{|\tau||1-\mu|}{(2\lambda-1)^2(1+\beta)^2} & \text{if } |1-\mu||\tau| \ge l(\lambda,\beta), \end{cases} \quad (4.1)$$

where

$$l(\lambda,\beta)=\frac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}.$$

Obtained here result is sharp.

Proof. Let $f\in\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$, $\beta\in[0,1]$, $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$. From the equalities (3.10), (3.12) and (3.13), we can write the folloing equality for the expression $a_3-\mu a_2^2$

$$a_3-\mu a_2^2=(1-\mu)\frac{\tau^2p_1^2}{4(2\lambda-1)^2(1+\beta)^2}+\frac{\tau(4-p_1^2)}{8(3\lambda-1)(1+2\beta)}(x-y) \qquad (4.2)$$

for $x,y\in\mathbb{C}$ with $|x|\le1$ and $|y|\le1$.

Applying triangle inequality to the equality (4.2), we obtain

$$|a_3-\mu a_2^2|\le|1-\mu|\frac{|\tau|^2t^2}{4(2\lambda-1)^2(1+\beta)^2}+\frac{|\tau|(4-t^2)}{8(3\lambda-1)(1+2\beta)}(\xi+\eta),$$

$$(\xi,\eta)\in[0,1]^2,$$

where $\xi=|x|$, $\eta=|y|$ and $t=|p_1|$.

From the last inequality, we can write

$$|a_3-\mu a_2^2|\le\frac{|\tau|}{4(2\lambda-1)^2(1+\beta)^2}\left[\begin{matrix}|1-\mu||\tau|-\\ l(\lambda,\beta)\end{matrix}\right]t^2+\frac{|\tau|}{(3\lambda-1)(1+2\beta)},$$

$$t\in[0,2], \qquad (4.3)$$

where

$$l(\lambda,\beta)=\frac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}.$$

Maximizing the expression on the right hand side of the inequality (4.3) according to the parameter t, we get

$$|a_3-\mu a_2^2|\le|\tau|\begin{cases}\dfrac{1}{(3\lambda-1)(1+2\beta)} & \text{if } |1-\mu||\tau|\le l(\lambda,\beta),\\[2ex] \dfrac{|\tau||1-\mu|}{(2\lambda-1)^2(1+\beta)^2} & \text{if } |1-\mu||\tau|\ge l(\lambda,\beta).\end{cases}$$

The result of the theorem is sharp in the case $|1-\mu||\tau|\le l(\lambda,\beta)$ for the function

$$f_1(z)=z+\frac{\sqrt{|\tau|}}{\sqrt{|1-\mu|(3\lambda-1)(1+2\beta)}}z^2+\frac{|\tau|}{|1-\mu|(3\lambda-1)(1+2\beta)}z^3,\ z\in U$$

and in the case $|1-\mu||\tau|\ge l(\lambda,\beta)$ for the function

$$f_2(z)=z+\frac{|\tau|}{(2\lambda-1)(1+\beta)}z^2+\frac{|\tau|^2}{(2\lambda-1)^2(1+\beta)^2}z^3,\ z\in U.$$

Thus, the proof of theorem is completed.

If we take $\beta=0$ and $\beta=1$ in Theorem 4.1, we obtain the following results, respectively.

Corollary 4.1. If $f\in S^*_{\Sigma,\sinh}(\lambda,\tau)$, then

$$\left|a_3-\mu a_2^2\right|\le|\tau|\begin{cases}\dfrac{1}{3\lambda-1} & \text{if } |1-\mu||\tau|\le\dfrac{2\lambda-1^{\;2}}{3\lambda-1},\\[2ex] \dfrac{|\tau||1-\mu|}{(2\lambda-1)^2} & \text{if } |1-\mu||\tau|\ge\dfrac{(2\lambda-1)^2}{3\lambda-1}.\end{cases}$$

Corollary 4.2. If $f\in C_{\Sigma,\sinh}(\lambda,\tau)$, then

$$\left|a_3-\mu a_2^2\right|\le|\tau|\begin{cases}\dfrac{1}{3(3\lambda-1)} & \text{if } |1-\mu||\tau|\le\dfrac{4(2\lambda-1)^2}{3(3\lambda-1)},\\[2ex] \dfrac{|\tau||1-\mu|}{4(2\lambda-1)^2} & \text{if } |1-\mu||\tau|\ge\dfrac{4(2\lambda-1)^2}{3(3\lambda-1)}.\end{cases}$$

Note: 4.1. In the special values of the parameters λ and τ from Corollary 4.1 and Corollary 4.2, we obtain the results for the classes $S^*_{\Sigma,\sinh}(\tau)$, $S^*_{\Sigma,\sinh}(\lambda)$, $S^*_{\Sigma,\sinh}$ and $C_{\Sigma,\sinh}(\tau)$, $C_{\Sigma,\sinh}(\lambda)$, $C_{\Sigma,\sinh}$, respectively.

5. *Discussion*

In this study, we defined a new subclass of starlike and convex bi-univalent functions, which we will call pseudo-starlike and pseudo-convex function class. For this definition class, we examine some geometric properties, like as coefficient and Fekete-Szegö problem.

References

ALOTAIBI, A., ARIF, M., ALGHAMDI, M. A., HUSSAIN, S. (2020) Starlikness associated with cosine hyperbolic function. *Mathematics*. 8(7), p. 1118.

ARIF, M., AHMAD, K., LIU, J.-L., SOKOL, J. (2019) A new class of analytic functions associated with Salagean operator. *Hindawi Journal of Function Spaces*. Article ID 6157394, 8 pages https://doi.org/10.1155/2019/6157394.

BANO, K., RAZA, M. (2020) Starlike functions associated with cosine function. *Bull. Iran. Math. Soc*. 47, pp. 1513-1532.

BRANNAN, D. A., KIRWAN, W. E. (1969) On some classes of bounded univalent functions. J. Lond. Math. Soc. 2, pp. 431-443.

BIEBERBACH, L. (1916) Über die Koeffizienten derjenigen Potenzreihen welche eine schlichte Abbidung des Einbeiskreises vermitteln. *Sitzungsberichte Preuss. Akad. Der Wiss*. 138, pp. 940-955.

CARATHEODORY, C. (1907) Über den Variabilitatsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annehmen. *Math. Ann*. 64, pp.95-115.

CHO, N. E., KUMAR, V., KUMAR, S. S., RAVICHANDRAN, V. (2019) Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc*. 45, pp. 213-232.

DE BRANGES, L. (1985) A proof of the Bieberbach conjecture. Acta Math. 154, pp. 137-152.

DUREN, P. L. (1983) Univalent Functions. In Grundlehren der Mathematischen Wissenschaften, *New York, Berlin, Heidelberg and Tokyo, Springer-Verlag, Volume* 259.

FRASIN, B. AND AOUF, M.K. (2011) New Subclasses of Bi-univalent Functions. *Applied Mathematics Letters.* 24(9), pp. 1569-1573.

JANOWSKI, W. (1970) Extremal problems for a family of functions with positive real part and for some related families. *Ann. Pol. Math*. 23, pp. 159-177.

SOKOL, J. A certain class of starlike functions. *Comput. Math. Appl*. 2011, 62, pp. 611-619.

KANKILIÇ. A., MUSTAFA, N. (2024a) On the Pseudo Starlike Bi-univalent Function Class of Complex Order. *Journal of Scientific and Engineering Research.* 11(5), pp. 26-34.

KANKILIÇ. A., MUSTAFA, N. (2024b) On the Pseudo Convex Bi-univalent Function Class of Complex Order. *Journal of Scientific and Engineering Research. 11(7).* pp. 259-267.

KÖEBE, P. (1909) Über die Uniformisierrung der algebraishen Kurven, durch automorpher Funktionen mit imaginarer Substitutions gruppe. *Nachr. Akad. Wiss. Göttingen Math. -Phys*. pp. 68-76.

KUMAR, S. S., ARORA, K. (2020) Starlike functions associated with a petal shaped domain. *arXiv:* 2010.10072.

MENDIRATTA, R., NAGPAL, S., RAVICHANDRAN, V. (2015) On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Soc.* 38, pp. 365-386.

MILLER, S. S. (1975) Differential inequalities and Caratheodory functions. *Bull. Am. Math. Soc*. 81, pp. 79-81.

MUSTAFA, N., NEZIR, V., KANKILIÇ. A. (2023a) Coefficient estimates for certain subclass of analytic and univalent functions associated with sine hyperbolic function. *13th International Istanbul Scientific Research Congress on Life, Engineering and*

Applied Sciences on April 29-30, pp.234-241, Istanbul, Turkey.

MUSTAFA, N., NEZIR, V., KANKILIÇ. A. (2023b) The Fekete-Szegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function. *13th International Istanbul Scientific Research Congress on Life, Engineering and Applied Sciences* on April 29-30, pp. 242-249, Istanbul, Turkey.

MUSTAFA, N., NEZIR, V. (2023) Coefficient estimates and Fekete-Szegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function. *13th International Istanbul Scientific Research Congress on Life, Engineering and Applied Sciences* on May 1-2, pp. 475-481, Istanbul, Türkiye.

MUSTAFA, N., DEMIR, H. A. (2023a) Coefficient estimates for certain subclass of analytic and univalent functions with associated with sine and cosine functions. *4th International Black Sea Congress on Modern Scientific Research* on June 6-8, pp. 2555-2563, Rize Turkey.

MUSTAFA, N., DEMIR, H. A. (2023b) Fekete-Szegö problem for certain subclass of analytic and univalent functions associated with sine and cosine functions. *4th International Black Sea Congress on Modern Scientific Research* on June 6-8, pp. 2564-2572, Rize Turkey.

MUSTAFA, N., NEZIR, V., KANKILIÇ. A. (2023c) Coefficient estimates for certain subclass of analytic and univalent functions associated with sine hyperbolic function with complex order. *Journal of Scientific and Engineering Research.* 10(6), pp.18-25.

MUSTAFA, N., NEZIR, V., KANKILIÇ. A. (2023d) The Fekete-Szegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function with complex order. *Eastern Anatolian Journal of Science.* 9(1), pp. 1-6.

MUSTAFA, N., DEMIR, H. A. (2023) Coefficient estimates for certain subclass of analytic and univalent functions with associated with sine and cosine functions with complex order. *Journal of Scientific and Engineering Research.* 10(6), pp.131-140.

SHARMA, K., JAIN, N. K., RAVICHANDRAN, V. (2016) Starlike function associated with a cardioid. *Afr. Math.* 27, pp. 923-939.

SHI, L., SRIVASTAVA, H. M. ARIF, M., HUSSAIN, S., KHAN, H. (2019) An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function. *Symmetry*. 11, p.598.

SOKOL, J., STANKIEWCZ, J. (1996) Radius of convexity of some subclasses of strongly starlike functions. *Zesz. Nauk. Politech. Rzesz. Math.* vol.*19*, pp.101–105.

ULLAH, K., ZAINAB, S., ARIF, M., DARUS, M., SHUTAYI, M. (2021) Radius Problems for Starlike Functions Associated with the Tan Hyperbolic Function. *Hindawi Journal of Function Spaces.*2021, 9967640.