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Jensen's type inequalities for two times differentiable functions with applications

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Abstract. In the main body, first of all this work recommends an inequality of Jensen's type involving Green functions for a class of two times differentiable functions. This result enables further to obtain some related interpolating inequalities for a function f such that $|f''|^q$ is either concave or convex for $q \geq 1$. Then manipulation of certain existing results in the corresponding interpolating inequalities gives bounds for the differences of the Jensen-Steffensen and Jensen's inequalities. In the similar fashion, they provide some new variants for the reverse form of aforementioned inequalities. Further, the obtained results about Jensen's inequality yield different novel adaptations of Hölder's inequality, fresh insights into the discrepancy of the well known Hermite-Hadamard inequality, and inequalities for geometric mean, quasi-arithmetic mean, and power mean. As a resultant, this work also suggests graphical interpretation of some results to verify the authenticity and sharpness of the obtained results about Jensen's inequality. Finally, this research work put forward some applications involving Zipf-Mandelbrot entropy and various types of Csiszár divergence from information theory.

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1. Introduction and Preliminaries

Mathematical inequalities form a foundation of investigation in modern applied analysis, offering a platform for exploring a wide array of phenomena and scenarios. The mutual relationship between mathematical inequalities and convex functions has caught a profound pursuit among researchers, culminating in the formulation of various inequalities revealed through the concepts of convex analysis. [1, 2, 7, 9–15, 17, 21, 22, 24, 29, 30]. Jensen's inequality stands out as the most dynamic and potent inequality within this set of mathematical principles. The classical discrete form of Jensen's inequality is follows as:

Theorem 1. *Let $f : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a convex function, $s_i \in [\sigma_1, \sigma_2]$, $u_i \geq 0$ for $i = 1, 2, \ldots, n$ with $U_n = \sum_{i=1}^{n} u_i > 0$, then*

$$f\left(\frac{1}{U_n}\sum_{i=1}^{n} u_i s_i\right) \leq \frac{1}{U_n}\sum_{i=1}^{n} u_i f(s_i). \tag{1}$$

If the function f becomes concave, the direction of inequality in (1) reverses. Also, the integral form of Jensen's inequality in Riemann sense can be found in [19]. Esteemed as a crucial gateway to the

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classical inequalities including Minkowski's, Levinson's, Hölder's, Young's, the arithmatic-geometric, Ky-Fan's, and the Hermite-Hadamard inequalities, this inequality serves as a cornerstone for their derivation. A valuable and substantial body of work has been produced regarding converse results, extensions, improvements, refinements, and generalizations of Jensen's inequality and their applications in various aspects [3, 6, 8, 20, 23, 25, 26].

Steffensen has relaxed the condition of non-negativity of u_i by virtue of restricting s_i for $i = 1, 2, \ldots, n$ [28]:

Theorem 2. *Let $f : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a convex function, $s_i \in [\sigma_1, \sigma_2]$, $u_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. If $s_1 \le s_2 \le \cdots \le s_n$ or $s_1 \ge s_2 \ge \cdots \ge s_n$ and*

$$0 \le \sum_{c=1}^{i} u_c \le \sum_{c=1}^{n} u_c, \quad i = 1, 2, \ldots, n, \quad \sum_{c=1}^{n} u_c > 0,$$

then (1) holds.

Theorem 2 can be found in its integral form in [16].

A reverse of Jensen's inequality is given as follows, which may also be found in [27, p. 83]:

Theorem 3. *Let $f : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a convex function, $s_i \in [\sigma_1, \sigma_2]$, $u_1 > 0$, $u_i \le 0$ for $i = 2, 3, \ldots, n$ with $U_n = \sum_{i=1}^{n} u_i > 0$. Also, let $\frac{1}{U_n} \sum_{i=1}^{n} u_i s_i \in [\sigma_1, \sigma_2]$, then*

$$f\left(\frac{1}{U_n} \sum_{i=1}^{n} u_i s_i\right) \ge \frac{1}{U_n} \sum_{i=1}^{n} u_i f(s_i). \tag{2}$$

A reverse of the Jensen-Steffensen inequality is follows as [27, p. 83]:

Theorem 4. *Let $f : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a convex function, $s_i \in [\sigma_1, \sigma_2], u_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$. Let $U_i = \sum_{j=1}^{i} u_j$ for $i = 1, 2, \ldots, n$ with $U_n > 0$ and $\frac{1}{U_n} \sum_{i=1}^{n} u_i s_i \in [\sigma_1, \sigma_2]$. Also, let $(s_1, s_2, \ldots, s_n)$ be a monotonic n-tuple, and $p \in \{1, 2, ...n\}$ be such that*

$$U_i \le 0 \quad \text{for} \quad i < p \qquad \text{and} \qquad U_n - U_{i-1} \le 0 \quad \text{for} \quad i > p,$$

then (2) holds.

We use the following continuous and convex Green functions G_k for $k = 1, 2, 3, 4, 5$, which are defined on the rectangle $[\sigma_1, \sigma_2] \times [\sigma_1, \sigma_2]$ [5]:

$$G_1(z, x) = \begin{cases} \sigma_1 - x, & \sigma_1 \le x \le z, \\ \sigma_1 - z, & z \le x \le \sigma_2. \end{cases} \tag{3}$$

$$G_2(z, x) = \begin{cases} z - \sigma_2, & \sigma_1 \le x \le z, \\ x - \sigma_2, & z \le x \le \sigma_2. \end{cases} \tag{4}$$

$$G_3(z, x) = \begin{cases} z - \sigma_1, & \sigma_1 \le x \le z, \\ x - \sigma_1, & z \le x \le \sigma_2. \end{cases} \tag{5}$$

$$G_4(z, x) = \begin{cases} \sigma_2 - x, & \sigma_1 \le x \le z, \\ \sigma_2 - z, & z \le x \le \sigma_2. \end{cases} \tag{6}$$

$$G_5(z, x) = \begin{cases} \frac{(z-\sigma_2)(x-\sigma_1)}{\sigma_2-\sigma_1}, & \sigma_1 \le x \le z, \\ \frac{(x-\sigma_2)(z-\sigma_1)}{\sigma_2-\sigma_1}, & z \le x \le \sigma_2. \end{cases} \tag{7}$$

Lemma 1 ([5]). *The following integral identities hold for a function $f \in C^2[\sigma_1, \sigma_2]$ and the Green functions G_k for $k = 1, 2, 3, 4, 5$ given in (3)–(7) respectively.*

$$f(z) = f(\sigma_1) + (z - \sigma_1) f'(\sigma_2) + \int_{\sigma_1}^{\sigma_2} G_1(z, x) f''(x) dx, \tag{8}$$

$$f(z) = f(\sigma_2) + (z - \sigma_2) f'(\sigma_1) + \int_{\sigma_1}^{\sigma_2} G_2(z, x) f''(x) dx, \tag{9}$$

$$f(z) = f(\sigma_2) + (z - \sigma_1) f'(\sigma_1) - (\sigma_2 - \sigma_1) f'(\sigma_2) + \int_{\sigma_1}^{\sigma_2} G_3(z, x) f''(x) dx, \tag{10}$$

$$f(z) = f(\sigma_1) + (\sigma_2 - \sigma_1)f'(\sigma_1) - (\sigma_2 - z)f'(\sigma_2) + \int_{\sigma_1}^{\sigma_2} G_4(z,x)f''(x)dx, \tag{11}$$

$$f(z) = \frac{\sigma_2 - z}{\sigma_2 - \sigma_1}f(\sigma_1) + \frac{z - \sigma_1}{\sigma_2 - \sigma_1}f(\sigma_2) + \int_{\sigma_1}^{\sigma_2} G_5(z,x)f''(x)dx. \tag{12}$$

Lemma 2 ([5]). *Let the function $f : [\sigma_1, \sigma_2] \to \mathbb{R}$ be convex, then for a nonnegative weight function $w(x)$ with $\int_{\sigma_1}^{\sigma_2} w(x)dx > 0$, the following inequality holds*

$$f\left(\frac{1}{\int_{\sigma_1}^{\sigma_2} w(x)dx}\int_{\sigma_1}^{\sigma_2} xw(x)dx\right) \leq \frac{1}{\int_{\sigma_1}^{\sigma_2} w(x)dx}\int_{\sigma_1}^{\sigma_2} f(x)w(x)dx. \tag{13}$$

The inequality takes on a reversed form when the function f exhibits concavity.

Lemma 3. *Consider the scenario that for a nonnegative real valued weight function $w : [\sigma_1, \sigma_2] \to \mathbb{R}$, $\int_{\sigma_1}^{\sigma_2} w(x)dx$ is positive. Further with the condition that $|g|^q (q > 1)$ is concave for the function $g \in L^q([\sigma_1, \sigma_2])$. Henceforth, the subsequent inequality can be derived*

$$\int_{\sigma_1}^{\sigma_2} w(x)|g(x)|^q dx \leq \int_{\sigma_1}^{\sigma_2} w(x)dx. \left| g\left(\frac{\int_{\sigma_1}^{\sigma_2} xw(x)dx}{\int_{\sigma_1}^{\sigma_2} w(x)dx}\right)\right|^q. \tag{14}$$

Proof. Upon replacing $|g|^q$ as a concave function in (13), we get

$$\left| g\left(\frac{1}{\int_{\sigma_1}^{\sigma_2} w(x)dx}\int_{\sigma_1}^{\sigma_2} xw(x)dx\right)\right|^q \geq \frac{1}{\int_{\sigma_1}^{\sigma_2} w(x)dx}\int_{\sigma_1}^{\sigma_2} w(x)|g(x)|^q dx.$$

Now multiplying $\int_{\sigma_1}^{\sigma_2} w(x)dx$ to the above inequality, we get (14). □

A variant of the Lah-Ribarič's inequality is follows as [27]:

Lemma 4. *Let w be a nonnegative real valued weight function defined on $[\sigma_1, \sigma_2]$ and $g \in L^q([\sigma_1, \sigma_2])$ be a function with the condition that $|g|^q$ for $q > 1$ is convex, under these conditions the following result holds*

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2} w(x)|g(x)|^q dx\\ &\leq \frac{1}{\sigma_2 - \sigma_1}\left((\sigma_2|g(\sigma_1)|^q - \sigma_1|g(\sigma_2)|^q)\int_{\sigma_1}^{\sigma_2} w(x)dx + (|g(\sigma_2)|^q - |g(\sigma_1)|^q)\int_{\sigma_1}^{\sigma_2} xw(x)dx\right).\end{aligned} \tag{15}$$

2. Main Results

The aforementioned Green functions enable us to establish the first main result as follows:

Theorem 5. *Let $f \in C^2([\sigma_1, \sigma_2])$ be a function and $s_i \in [\sigma_1, \sigma_2]$, $u_i \in \mathbb{R}$ for $i = 1, 2, \dots, n$ with $U_n := \sum_{i=1}^n u_i \neq 0$ and $\frac{1}{U_n}\sum_{i=1}^n u_i s_i \in [\sigma_1, \sigma_2]$. Also, let G_k for $k = 1, 2, 3, 4, 5$ be as defined in (3)-(7) and $q > 1$.*

1. *If for $x \in [\sigma_1, \sigma_2]$ and each $k \in \{1, 2, 3, 4, 5\}$ the following inequality holds*

$$\frac{1}{U_n}\sum_{i=1}^n u_i G_k(s_i, x) - G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_i s_i, x\right) \geq 0, \tag{16}$$

then

$$\begin{aligned}&\left|\frac{1}{U_n}\sum_{i=1}^n u_i f(s_i) - f\left(\frac{1}{U_n}\sum_{i=1}^n u_i s_i\right)\right|\\ &\leq \frac{1}{2^{1-\frac{1}{q}}}\left(\frac{1}{U_n}\sum_{i=1}^n u_i s_i^2 - \left(\frac{1}{U_n}\sum_{i=1}^n u_i s_i\right)^2\right)^{1-\frac{1}{q}}\\ &\times\left(\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^n u_i G_k(s_i, x) - G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_i s_i, x\right)\right)|f''(x)|^q dx\right)^{\frac{1}{q}}.\end{aligned} \tag{17}$$

2. *In case the inequality in (16) is reversed, then*

$$\left|\frac{1}{U_n}\sum_{i=1}^{n}u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\right)\right|$$
$$\leq\frac{1}{2^{1-\frac{1}{q}}}\left(\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\right)^2-\frac{1}{U_n}\sum_{i=1}^{n}u_is_i^2\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_{\sigma_1}^{\sigma_2}\left(G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)-\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)\right)|f''(x)|^qdx\right)^{\frac{1}{q}}. \tag{18}$$

Proof. **1**. Using (8)-(12) in $\frac{1}{U_n}\sum_{i=1}^{n}u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\right)$, we obtain

$$\frac{1}{U_n}\sum_{i=1}^{n}u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\right)$$
$$=\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\right)f''(x)dx. \tag{19}$$

After applying the triangle inequality to the absolute value of (19), we arrive at

$$\left|\frac{1}{U_n}\sum_{i=1}^{n}u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\right)\right|$$
$$=\left|\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\right)f''(x)dx\right|$$
$$\leq\int_{\sigma_1}^{\sigma_2}\left|\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\right||f''(x)|dx$$
$$=\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\right)|f''(x)|dx. \tag{20}$$

By employing the power mean inequality in (20), we arrive at

$$\left|\frac{1}{U_n}\sum_{i=1}^{n}u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\right)\right|$$
$$\leq\left(\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\right)dx\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\right)|f''(x)|^qdx\right)^{\frac{1}{q}}. \tag{21}$$

We have $f''(y)=1$ for $f(y)=\frac{y^2}{2}$. Now using (19) for such functions, we get

$$\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\right)dx$$
$$=\frac{1}{2}\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i^2-\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\right)^2\right). \tag{22}$$

Using (22) in (21), we get (17).
2. Adopting the procedure of part 1, we get (18). □

Theorem 6. *Let $|f''|^q$ is convex for $q>1$, $f\in C^2[\sigma_1,\sigma_2]$. Also, $s_i\in[\sigma_1,\sigma_2]$, $u_i\in\mathbb{R}$ for $i=1,2,\ldots,n$ with $U_n:=\sum_{i=1}^n u_i\neq 0$ and $\frac{1}{U_n}\sum_{i=1}^n u_is_i\in[\sigma_1,\sigma_2]$. Further, assume that G_k for $k=1,2,3,4,5$ are as defined in (3)-(7).*

1. *If for $x\in[\sigma_1,\sigma_2]$ and each $k\in\{1,2,3,4,5\}$ the following inequality holds*

$$\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)\geq 0, \tag{23}$$

then

$$\begin{aligned}&\left|\frac{1}{U_n}\sum_{i=1}^n u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)\right|\\&\leq\frac{1}{2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}}\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i^2-\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2\right)^{1-\frac{1}{q}}\\&\times\left[\frac{\sigma_2|f''(\sigma_1)|^q-\sigma_1|f''(\sigma_2)|^q}{2}\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i^2-\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2\right)\right.\\&\left.+\frac{|f''(\sigma_2)|^q-|f''(\sigma_1)|^q}{6}\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i^3-\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^3\right)\right]^{\frac{1}{q}}.\end{aligned} \tag{24}$$

2. *In case the inequality in (23) is reversed, then*

$$\begin{aligned}&\left|\frac{1}{U_n}\sum_{i=1}^n u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)\right|\\&\leq\frac{1}{2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}}\left(\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2-\frac{1}{U_n}\sum_{i=1}^n u_is_i^2\right)^{1-\frac{1}{q}}\\&\times\left[\frac{\sigma_2|f''(\sigma_1)|^q-\sigma_1|f''(\sigma_2)|^q}{2}\left(\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2-\frac{1}{U_n}\sum_{i=1}^n u_is_i^2\right)\right.\\&\left.+\frac{|f''(\sigma_2)|^q-|f''(\sigma_1)|^q}{6}\left(\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^3-\frac{1}{U_n}\sum_{i=1}^n u_is_i^3\right)\right]^{\frac{1}{q}}.\end{aligned} \tag{25}$$

Proof. **1**. Using Lemma 4 for $w(x)=\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)$ and $|g|^q=|f''|^q$ in inequality (17), we get the following inequality

$$\begin{aligned}&\left|\frac{1}{U_n}\sum_{i=1}^n u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)\right|\\&\leq\frac{1}{2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}}\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i^2-\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2\right)^{1-\frac{1}{q}}\\&\times\left[(\sigma_2|f''(\sigma_1)|^q-\sigma_1|f''(\sigma_2)|^q)\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)\right)dx\right.\\&\left.+(|f''(\sigma_2)|^q-|f''(\sigma_1)|^q)\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)\right)xdx\right]^{\frac{1}{q}}.\end{aligned} \tag{26}$$

Let $f(y)=\frac{y^3}{6}$, then $f''(y)=y$. Using (19) for these functions we obtain

$$\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\right)xdx=\frac{1}{6}\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i^3-\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\right)^3\right). \quad (27)$$

Now using (22) and (27) in (26), we get (24).

2. Using Lemma 4 for $w(x)=G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)-\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)$ and $|g|^q=|f''|^q$ in inequality (18) and adopting the procedure of part 1, we obtain (25). □

The Jensen gap is estimated from the right side in the following theorem as an application of Theorem 6:

Theorem 7. *Suppose that the function $|f''|^q$ for $q>1$ is convex for $f\in C^2[\sigma_1,\sigma_2]$ and $s_i\in[\sigma_1,\sigma_2]$, $u_i\geq 0$ for $i=1,2,\ldots,n$ with the condition that $\sum_{i=1}^{n}u_i=U_n>0$, then inequality (24) holds.*

Proof. Since G_k are convex functions for all k and $u_i\geq 0$ for all i with $U_n>0$. Therefore using the Jensen inequality, result (23) can be obtained. Thus inequality (24) can be obtained easily by using Theorem 6 for such facts. □

A new result for the Jensen-Steffensen inequality is established as follows, which is an application of Theorem 6:

Corollary 1. *Let $|f''|^q$ is convex for $q>1$, $f\in C^2[\sigma_1,\sigma_2]$. Also for $i=1,2,\ldots,n$, assume that $u_i\in\mathbb{R}$ and $s_i\in[\sigma_1,\sigma_2]$. Now, if*

$$\sum_{c=1}^{n}u_c\geq\sum_{c=1}^{i}u_c\geq 0,\ \ i=1,2,\ldots,n,\ \ \ \sum_{c=1}^{n}u_c>0,$$

and $s_1\geq s_2\geq\cdots\geq s_n$ or $s_1\leq s_2\leq\cdots\leq s_n$, then (24) holds.

Proof. Since G_k are convex functions for all k and Jensen-Steffensen conditions also hold, therefore the result (23) can be acquired as direct consequence of the Jensen-Steffensen inequality. Thus we get (24) by using Theorem 6. □

A bound to the discrepancy of Jensen's inequality is given in the following corollary by proposing the assumptions of Theorem 3:

Corollary 2. *Let $|f''|^q$ is convex for $q>1$, $f\in C^2[\sigma_1,\sigma_2]$. Also, $s_i\in[\sigma_1,\sigma_2]$, $u_1>0$ and $u_i\leq 0$ for $i=2,3,\ldots,n$ with $U_n=\sum_{i=1}^{n}u_i>0$. Further assume that $\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\in[\sigma_1,\sigma_2]$, then the proposed inequality in (25) holds.*

Proof. It may be noted that G_k are convex functions for all $k\in\{1,2,3,4,5\}$, therefore Theorem 3 enables to prove that $\bar{G}(x):=\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\leq 0$. Thus, by using Theorem 6 one may get (25) easily. □

The Jensen-Steffensen gap is estimated from the right hand side in the following corollary by proposing the assumptions of Theorem 4:

Corollary 3. *Let $|f''|^q$ is convex for $q>1$ and $f\in C^2[\sigma_1,\sigma_2]$. Also, assume that $s_i\in[\sigma_1,\sigma_2],u_i\in\mathbb{R}$ for $i=1,2,\ldots,n$. Further, suppose that $U_i=\sum_{j=1}^{i}u_j$ for $i=1,2,\ldots,n$ with $U_n>0$ and $\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\in[\sigma_1,\sigma_2]$. If $p\in\{1,2,...,n\}$ and $(s_1,s_2,\ldots,s_n)$ is a monotonic n-tuple with*

$$U_i\leq 0\quad \text{for}\quad p>i\qquad \text{and}\qquad 0\geq U_n-U_{i-1}\quad \text{for}\quad p<i,$$

then the proposed inequality in (25) holds.

Proof. By following the steps of the proof of Corollary 2, We achieve the desired outcome through the utilization of Theorem 4 instead of Theorem 3 therein. □

Presented below are two novel variations of the Hölder inequality, derived directly from Theorem 7:

Corollary 4. *Let* $p_2 > 1,\ p_1 \notin \left(2, 2+\frac{1}{q}\right)$ *for* $q > 1$ *and* $\frac{1}{p_1}+\frac{1}{p_2} = 1$. *Further if the interval* $[\sigma_1, \sigma_2]$ *and the tuples* $(a_1, a_2, \dots, a_n)$ *and* $(b_1, b_2, \dots, b_n)$ *are positive with* $\frac{\sum_{i=1}^n a_i b_i}{\sum_{i=1}^n b_i^{p_2}},\ a_i b_i^{-\frac{p_2}{p_1}} \in [\sigma_1, \sigma_2]$ *for* $i = 1, 2, \dots, n$, *then*

$$\left(\sum_{i=1}^n a_i^{p_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n b_i^{p_2}\right)^{\frac{1}{p_2}} - \sum_{i=1}^n a_i b_i$$
$$\leq \frac{(p_1(p_1-1))^{\frac{1}{p_1}}}{2^{\frac{q-1}{p_1 q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}} \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^2 b_i^{1-\frac{p_2}{p_1}} - \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i b_i\right)^2\right)^{\frac{q-1}{p_1 q}}$$
$$\times \left[\frac{\sigma_2 \sigma_1^{q(p_1-2)} - \sigma_1 \sigma_2^{q(p_1-2)}}{2} \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^2 b_i^{1-\frac{p_2}{p_2}} - \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i b_i\right)^2\right)\right.$$
$$\left. + \frac{\sigma_2^{q(p_1-2)} - \sigma_1^{q(p_1-2)}}{6} \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^3 b_i^{1-2\frac{p_2}{p_1}} - \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i b_i\right)^3\right)\right]^{\frac{1}{p_1 q}} \sum_{i=1}^n b_i^{p_2}. \tag{28}$$

Proof. Using (24) for $f(y) = y^{p_1},\ y \in [\sigma_1, \sigma_2]$ and $u_i = b_i^{p_2},\ s_i = a_i b_i^{-\frac{p_2}{p_1}}$, we derive

$$\left(\left(\sum_{i=1}^n a_i^{p_1}\right)\left(\sum_{i=1}^n b_i^{p_2}\right)^{p_1-1} - \left(\sum_{i=1}^n a_i b_i\right)^{p_1}\right)^{\frac{1}{p_1}}$$
$$\leq \frac{(p_1(p_1-1))^{\frac{1}{p_1}}}{2^{\frac{q-1}{p_1 q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}} \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^2 b_i^{1-\frac{p_2}{p_1}} - \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i b_i\right)^2\right)^{\frac{q-1}{p_1 q}}$$
$$\times \left[\frac{\sigma_2 \sigma_1^{q(p_1-2)} - \sigma_1 \sigma_2^{q(p_1-2)}}{2} \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^2 b_i^{1-\frac{p_2}{p_2}} - \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i b_i\right)^2\right)\right.$$
$$\left. + \frac{\sigma_2^{q(p_1-2)} - \sigma_1^{q(p_1-2)}}{6} \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^3 b_i^{1-2\frac{p_2}{p_1}} - \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i b_i\right)^3\right)\right]^{\frac{1}{p_1 q}} \sum_{i=1}^n b_i^{p_2}. \tag{29}$$

By utilizing the inequality $\alpha^e - \beta^e \leq (\alpha-\beta)^e,\ 0 \leq \beta \leq \alpha,\ e \in [0,1]$ for $\beta = \left(\sum_{i=1}^n a_i b_i\right)^{p_1}$, $\alpha = \left(\sum_{i=1}^n a_i^{p_1}\right)\left(\sum_{i=1}^n b_i^{p_2}\right)^{p_1-1}$ and $e = \frac{1}{p_1}$, we obtain

$$\left(\sum_{i=1}^n a_i^{p_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n b_i^{p_2}\right)^{\frac{1}{p_2}} - \sum_{i=1}^n a_i b_i \leq \left(\left(\sum_{i=1}^n a_i^{p_1}\right)\left(\sum_{i=1}^n b_i^{p_2}\right)^{p_1-1} - \left(\sum_{i=1}^n a_i b_i\right)^{p_1}\right)^{\frac{1}{p_1}}. \tag{30}$$

Now using (30) in (29), we get (28). □

Corollary 5. *Let* $1 > p_1 > 0,\ p_2 = \frac{p_1}{p_1-1}$ *with* $p_1 \notin \left(\frac{q}{2q+1}, \frac{1}{2}\right)$ *for* $q > 1$. *Also, let* $(a_1, a_2, \dots, a_n)$ *and* $(b_1, b_2, \dots, b_n)$ *be two positive n-tuples with* $\frac{\sum_{i=1}^n a_i^{p_1}}{\sum_{i=1}^n b_i^{p_2}},\ a_i^{p_1} b_i^{-p_2} \in [\sigma_1, \sigma_2] \subseteq \mathbb{R}^+$ *for* $i = 1, 2, \dots, n$, *then*

$$\sum_{i=1}^n a_i b_i - \left(\sum_{i=1}^n a_i^{p_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n b_i^{p_2}\right)^{\frac{1}{p_2}}$$
$$\leq \frac{(1-p_1)}{p_1^2 2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}} \left(\sum_{i=1}^n a_i^{2p_1} b_i^{-p_2} - \frac{\left(\sum_{i=1}^n a_i^{p_1}\right)^2}{\left(\sum_{i=1}^n b_i^{p_2}\right)}\right)$$

$$\times\left[\frac{\sigma_2\sigma_1^{q\left(\frac{1}{p_1}-2\right)}-\sigma_1\sigma_2^{q\left(\frac{1}{p_1}-2\right)}}{2}\left(\sum_{i=1}^{n}a_i^{2p_1}b_i^{-p_2}-\frac{\left(\sum_{i=1}^{n}a_i^{p_1}\right)^2}{\left(\sum_{i=1}^{n}b_i^{p_2}\right)}\right)\right.$$
$$\left.+\frac{\sigma_2^{q\left(\frac{1}{p_1}-2\right)}-\sigma_1^{q\left(\frac{1}{p_1}-2\right)}}{6}\left(\sum_{i=1}^{n}a_i^{3p_1}b_i^{-2p_2}-\frac{\left(\sum_{i=1}^{n}a_i^{p_1}\right)^3}{\left(\sum_{i=1}^{n}b_i^{p_2}\right)^2}\right)\right]^{\frac{1}{q}}. \tag{31}$$

Proof. When considering the specified values of p_1 and q, the functions $f(y) = y^{\frac{1}{p_1}}$ and $|f''|^q$ for $y \in [\sigma_1, \sigma_2]$, are convex. Therefore by using (24) for $f(y) = y^{\frac{1}{p_1}}$, $u_i = b_i^{p_2}$ and $s_i = a_i^{p_1}b_i^{-p_2}$, we get (31). □

Definition 1. *Taking* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ *as positive tuples with* $U_n = \sum_{i=1}^{n} u_i$, *and* $\lambda \in \mathbb{R}$, *then the power mean of order* λ *is declared by*

$$\mathcal{M}_\lambda(\mathbf{u},\mathbf{s}) = \begin{cases} \left(\frac{1}{U_n}\sum_{i=1}^{n} u_i s_i^{\lambda}\right)^{\frac{1}{\lambda}}, & \lambda \neq 0, \\ \left(\prod_{k=1}^{m} s_i^{u_i}\right)^{\frac{1}{U_n}}, & \lambda = 0. \end{cases}$$

Theorem 7 gives us an inequality for the power mean as follows:

Corollary 6. *Let* $0 < \sigma_1 < \sigma_2$ *and* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ *be two positive n-tuples with* $U_n = \sum_{i=1}^{n} u_i$. *Also, suppose that* $\upsilon, \lambda \in \mathbb{R} - \{0\}$ *such that* $\upsilon < \lambda$ *and* $q > 1$.
1. If $\lambda, \upsilon > 0$, *then*

$$\mathcal{M}_\lambda^\upsilon(\mathbf{u},\mathbf{s}) - \mathcal{M}_\upsilon^\upsilon(\mathbf{u},\mathbf{s}) \leq \frac{\upsilon(\lambda-\upsilon)}{\lambda^2 2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}}\left(\mathcal{M}_{2\lambda}^{2\lambda}(\mathbf{u},\mathbf{s}) - \mathcal{M}_\lambda^{2\lambda}(\mathbf{u},\mathbf{s})\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{\sigma_2\sigma_1^{q\left(\frac{\upsilon}{\lambda}-2\right)}-\sigma_1\sigma_2^{q\left(\frac{\upsilon}{\lambda}-2\right)}}{2}\left(\mathcal{M}_{2\lambda}^{2\lambda}(\mathbf{u},\mathbf{s}) - \mathcal{M}_\lambda^{2\lambda}(\mathbf{u},\mathbf{s})\right)\right.$$
$$\left.+\frac{\sigma_2^{q\left(\frac{\upsilon}{\lambda}-2\right)}-\sigma_1^{q\left(\frac{\upsilon}{\lambda}-2\right)}}{6}\left(\mathcal{M}_{3\lambda}^{3\lambda}(\mathbf{u},\mathbf{s}) - \mathcal{M}_\lambda^{3\lambda}(\mathbf{u},\mathbf{s})\right)\right]^{\frac{1}{q}}. \tag{32}$$

2. If $\lambda, \upsilon < 0$ *with* $\frac{\upsilon}{\lambda} \notin \left(2, 2+\frac{1}{q}\right)$, *then*

$$\mathcal{M}_\upsilon^\upsilon(\mathbf{u},\mathbf{s}) - \mathcal{M}_\lambda^\upsilon(\mathbf{u},\mathbf{s}) \leq \frac{\upsilon(\upsilon-\lambda)}{\lambda^2 2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}}\left(\mathcal{M}_{2\lambda}^{2\lambda}(\mathbf{u},\mathbf{s}) - \mathcal{M}_\lambda^{2\lambda}(\mathbf{u},\mathbf{s})\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{\sigma_2\sigma_1^{q\left(\frac{\upsilon}{\lambda}-2\right)}-\sigma_1\sigma_2^{q\left(\frac{\upsilon}{\lambda}-2\right)}}{2}\left(\mathcal{M}_{2\lambda}^{2\lambda}(\mathbf{u},\mathbf{s}) - \mathcal{M}_\lambda^{2\lambda}(\mathbf{u},\mathbf{s})\right)\right.$$
$$\left.+\frac{\sigma_2^{q\left(\frac{\upsilon}{\lambda}-2\right)}-\sigma_1^{q\left(\frac{\upsilon}{\lambda}-2\right)}}{6}\left(\mathcal{M}_{3\lambda}^{3\lambda}(\mathbf{u},\mathbf{s}) - \mathcal{M}_\lambda^{3\lambda}(\mathbf{u},\mathbf{s})\right)\right]^{\frac{1}{q}}. \tag{33}$$

3. If $\lambda > 0$ *and* $\upsilon < 0$, *then inequality in (33) holds .*

Proof. **1**. Taking $f(y) = y^{\frac{\upsilon}{\lambda}}$ for $y \in [\sigma_1, \sigma_2]$. For the proposed values of λ, υ, the function f becomes concave while the function $|f''|^q$ becomes convex. Thus using (24) for $f(y) = y^{\frac{\upsilon}{\lambda}}$ and letting $s_i \to s_i^\lambda$, we get (32).
2. For the proposed values of λ and υ the functions $f(y) = y^{\frac{\upsilon}{\lambda}}$ and $|f''|^q$ become convex, thus by adhering to the methodology outlined in part 1, one can acquire the result proposed in (33).
3. The functions $f(y) = y^{\frac{\upsilon}{\lambda}}$ and $|f''|^q$ for $y \in [\sigma_1, \sigma_2]$ become convex in this case as well. Thus adhering to the methodology outlined in part 2, one can obtain the inequality proposed in (33). □

FIGURE 1. $[\sigma_1, \sigma_2] = [1, 250]$, $(u_1, u_2) = (0.5, 0.5)$, $(s_1, s_2) = (2, 3)$, q=1

Discussion 1. *Figure 1 demonstrates that how sharp is the discrepancy of left and right hand sides of inequality (32) for the corresponding parameters, provided specifically. Consequently, it explores the sharpness of the Jensen gap because this inequality is a direct consequence of the main result around Jensen inequality, provided in inequality (24).*

Two different types of mean are connected as follows, as direct applications of Theorem 7:

Corollary 7. *Let $1 \leq q$, $0 < \sigma_1 < \sigma_2$, and $(u_1, u_2, \ldots, u_n) = \mathbf{u}$, $(s_1, s_2, \ldots, s_n) = \mathbf{s}$ be some positive tuples with $U_n = \sum_{i=1}^n u_i$, then*

1. $\dfrac{\mathcal{M}_1(\mathbf{u},\mathbf{s})}{\mathcal{M}_0(\mathbf{u},\mathbf{s})}$

$$\leq \exp\left[\frac{1}{2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}}\left(\mathcal{M}_2^2(\mathbf{u},\mathbf{s})-\mathcal{M}_1^2(\mathbf{u},\mathbf{s})\right)^{1-\frac{1}{q}}\left(\frac{\sigma_2^{2q+1}-\sigma_1^{2q+1}}{2(\sigma_1\sigma_2)^{2q}}\left(\mathcal{M}_2^2(\mathbf{u},\mathbf{s})-\mathcal{M}_1^2(\mathbf{u},\mathbf{s})\right)\right.\right.$$
$$\left.\left.-\frac{\sigma_2^{2q}-\sigma_1^{2q}}{6(\sigma_1\sigma_2)^{2q}}\left(\mathcal{M}_3^3(\mathbf{u},\mathbf{s})-\mathcal{M}_1^3(\mathbf{u},\mathbf{s})\right)\right)^{\frac{1}{q}}\right]. \tag{34}$$

2. $\mathcal{M}_1(\mathbf{u},\mathbf{s}) - \mathcal{M}_0(\mathbf{u},\mathbf{s})$

$$\leq \frac{1}{2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}}\left(\frac{1}{U_n}\sum_{i=1}^n u_i\ln^2 s_i - \ln^2\mathcal{M}_0(\mathbf{u},\mathbf{s})\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{\sigma_2 e^{q\sigma_1}-\sigma_1 e^{q\sigma_2}}{2}\left(\frac{1}{U_n}\sum_{i=1}^n u_i\ln^2 s_i - \ln^2\mathcal{M}_0(\mathbf{u},\mathbf{s})\right)\right.$$
$$\left.+\frac{e^{q\sigma_2}-e^{q\sigma_1}}{6}\left(\frac{1}{U_n}\sum_{i=1}^n u_i\ln^3 s_i - \ln^3\mathcal{M}_0(\mathbf{u},\mathbf{s})\right)\right]^{\frac{1}{q}}. \tag{35}$$

Proof. **1**. Let $f(y) = -\ln y$ for $y \in [\sigma_1, \sigma_2]$, then the function $|f''|^q$ becomes convex. Therefore using (24) here, we get (34).
2. Let $f(y) = e^y$ for $y \in [\sigma_1, \sigma_2]$, then $|f''|^q$ becomes a convex function. Therefore using (24) here and letting $s_i = \ln s_i$, we get (35). ☐

Definition 2. *Assume that φ is a strictly monotone and continuous function. Let $(u_1, u_2, \ldots, u_n) = \mathbf{u}$ and $(s_1, s_2, \ldots, s_n) = \mathbf{s}$ be positive tuples with $\sum_{i=1}^{n} u_i = U_n$. The quasi arithmetic mean for these assumptions is defined by*

$$\mathcal{M}_\varphi(\mathbf{u}, \mathbf{s}) = \varphi^{-1}\left(\frac{1}{U_n}\sum_{i=1}^{n} u_i\varphi(s_i)\right).$$

An estimate for quasi arithmetic mean can be acquired by using Theorem 7 as follows:

Corollary 8. *Let the function $|(\beta \circ \varphi^{-1})''|^q$ be convex on $[\sigma_1, \sigma_2]$, $\sigma_1 > 0$ for $q > 1$ and $\beta \circ \varphi^{-1}$ as an arbitrary function provided that φ is a strictly monotone and continuous function. Also, assume that $(u_1, u_2, \ldots, u_n) = \mathbf{u}$, $(s_1, s_2, \ldots, s_n) = \mathbf{s}$ be some positive tuples with $U_n = \sum_{i=1}^{n} u_i$. Then*

$$\begin{aligned}
&\left|\frac{1}{U_n}\sum_{i=1}^{n} u_i\beta(s_i) - \beta\left(\mathcal{M}_\varphi(\mathbf{u}, \mathbf{s})\right)\right| \\
&\leq \frac{1}{2^{1-\frac{1}{q}}(\sigma_2 - \sigma_1)^{\frac{1}{q}}}\left(\frac{1}{U_n}\sum_{i=1}^{n} u_i\varphi^2(s_i) - \varphi^2\left(\mathcal{M}_\varphi(\mathbf{u}, \mathbf{s})\right)\right)^{1-\frac{1}{q}} \\
&\times\left[\frac{\sigma_2|(\beta\circ\varphi^{-1})''(\sigma_1)|^q - \sigma_1|(\beta\circ\varphi^{-1})''(\sigma_2)|^q}{2}\left(\frac{1}{U_n}\sum_{i=1}^{n} u_i\varphi^2(s_i) - \varphi^2\left(\mathcal{M}_\varphi(\mathbf{u}, \mathbf{s})\right)\right)\right. \\
&\left.+\frac{|(\beta\circ\varphi^{-1})''(\sigma_2)|^q - |(\beta\circ\varphi^{-1})''(\sigma_1)|^q}{6}\left(\frac{1}{U_n}\sum_{i=1}^{n} u_i\varphi^3(s_i) - \varphi^3\left(\mathcal{M}_\varphi(\mathbf{u}, \mathbf{s})\right)\right)\right]^{\frac{1}{q}}. \qquad (36)
\end{aligned}$$

Proof. (36) follows directly from (24) by letting $f \to \beta \circ \varphi^{-1}$ and $s_i \to \varphi(s_i)$. ☐

A generalization of Theorem 6 to its Reimann integral form can be followed as:

Theorem 8. *Let $|f''|^q$ is convex for $q > 1$, $f \in C^2[\sigma_1, \sigma_2]$. Also, let the functions $J_1, J_2 : [a_1, a_2] \to \mathbb{R}$ be integrable such that $J_1(y) \in [\sigma_1, \sigma_2]$ for all $y \in [a_1, a_2]$ with $H := \int_{a_1}^{a_2} J_2(y)dy \neq 0$ and $\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy \in [\sigma_1, \sigma_2]$. Suppose that $G_k (k = 1, 2, 3, 4, 5)$ are defined as in (3)-(7).*
1. *Provided that for each and every $k \in \{1, 2, 3, 4, 5\}$ and $x \in [\sigma_1, \sigma_2]$, one has*

$$\frac{1}{H}\int_{a_1}^{a_2} J_2(y)G_k(J_1(y), x)dy - G_k\left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy, x\right) \geq 0, \qquad (37)$$

then

$$\begin{aligned}
&\left|\frac{1}{H}\int_{a_1}^{a_2}(f\circ J_1)(y)J_2(y)dy - f\left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)\right| \\
&\leq \frac{1}{2^{1-\frac{1}{q}}(\sigma_2 - \sigma_1)^{\frac{1}{q}}}\left(\frac{1}{H}\int_{a_1}^{a_2} J_1^2(y)J_2(y)dy - \left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)^2\right)^{1-\frac{1}{q}} \\
&\times\left[\frac{\sigma_2|f''(\sigma_1)|^q - \sigma_1|f''(\sigma_2)|^q}{2}\left(\frac{1}{H}\int_{a_1}^{a_2} J_1^2(y)J_2(y)dy - \left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)^2\right)\right. \\
&\left.+\frac{|f''(\sigma_2)|^q - |f''(\sigma_1)|^q}{6}\left(\frac{1}{H}\int_{a_1}^{a_2} J_1^3(y)J_2(y)dy - \left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)^3\right)\right]^{\frac{1}{q}}. \qquad (38)
\end{aligned}$$

2. *If the inverted inequality in (37) is satisfied, then*

$$\left|\frac{1}{H}\int_{a_1}^{a_2}(f\circ J_1)(y)J_2(y)dy - f\left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)\right|$$

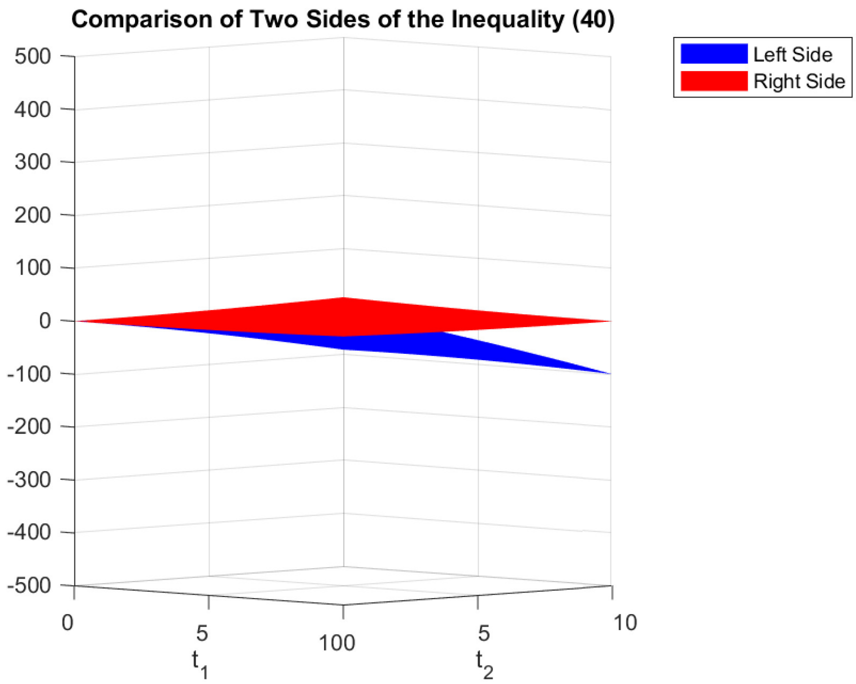

FIGURE 2. $q=1,\ f(y)=y^2$

$$\leq \frac{1}{2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}}\left(\left(\frac{1}{H}\int_{a_1}^{a_2}J_1(y)J_2(y)dy\right)^2-\frac{1}{H}\int_{a_1}^{a_2}J_1^2(y)J_2(y)dy\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{\sigma_2|f''(\sigma_1)|^q-\sigma_1|f''(\sigma_2)|^q}{2}\left(\left(\frac{1}{H}\int_{a_1}^{a_2}J_1(y)J_2(y)dy\right)^2-\frac{1}{H}\int_{a_1}^{a_2}J_1^2(y)J_2(y)dy\right)\right.$$
$$\left.+\frac{|f''(\sigma_2)|^q-|f''(\sigma_1)|^q}{6}\left(\left(\frac{1}{H}\int_{a_1}^{a_2}J_1(y)J_2(y)dy\right)^3-\frac{1}{H}\int_{a_1}^{a_2}J_1^3(y)J_2(y)dy\right)\right]^{\frac{1}{q}}. \tag{39}$$

A generalization of Theorem 7 to its Reimann integral version can be established as follows:

Theorem 9. *Let $|f''|^q$ be a convex function for $q>1$ and $f\in C^2[\sigma_1,\sigma_2]$. Also, assuming two real valued integrable functions J_2, J_1 defined on $[a_1,a_2]$ with the constraints that $J_1(y)\in[\sigma_1,\sigma_2]$ and $J_2(y)\geq 0$ for all $y\in[a_1,a_2]$ and $0<H:=\int_{a_1}^{a_2}J_2(y)dy$, then the inequality given by (38) holds.*

Remark 1. *The Corollaries: 1, 2 and 3 can be presented in their integral form as direct applications of Theorem 8.*

Remark 2. *Exploiting the implications of Theorem 9, we derive the integral expressions of the Corollaries: 4–8.*

Applying Theorem 9, we can establish an inequality concerning the Hermite-Hadamard gap as follows:

Corollary 9. *Let $|f''|^q$ be a convex function for $q>1$, $f\in C^2[t_1,t_2]$. Then*

$$\left|\frac{1}{t_2-t_1}\int_{t_1}^{t_2}f(y)dy-f\left(\frac{t_1+t_2}{2}\right)\right|\leq\frac{(t_2-t_1)^2}{24}\left(\frac{|f''(t_1)|^q+|f''(t_2)|^q}{2}\right)^{\frac{1}{q}}. \tag{40}$$

Proof. Using (38) for $[\sigma_1,\sigma_2]=[t_1,t_2]$, $J_2(y)=1$ and $J_1(y)=y$ for all $y\in[t_1,t_2]$, we get (40). □

The second main result is follows as:

Theorem 10. *Let $|f''|^q$ is concave for $q>1$ and $f\in C^2[\sigma_1,\sigma_2]$. Let $s_i\in[\sigma_1,\sigma_2]$, $u_i\in\mathbb{R}$ for $i=1,2,\dots,n$ with $\frac{1}{U_n}\sum_{i=1}^n u_is_i\in[\sigma_1,\sigma_2]$ and $U_n:=\sum_{i=1}^n u_i\neq 0$. Also, let G_k for $k=1,2,3,4,5$ be as defined in (3)-(7). There are two possibilities.*
1. *If for $x\in[\sigma_1,\sigma_2]$ and each $k\in\{1,2,3,4,5\}$, (41) holds*

$$\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)\geq 0, \tag{41}$$

then

$$\begin{aligned}&\left|\frac{1}{U_n}\sum_{i=1}^n u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)\right|\\
&\qquad\leq\frac{1}{2}\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i^2-\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2\right)\\
&\qquad\times\left|f''\left(\frac{\frac{1}{U_n}\sum_{i=1}^n u_is_i^3-\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^3}{3\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i^2-\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2\right)}\right)\right|.\end{aligned} \tag{42}$$

2. *If the inverted inequality in (41) is satisfied, then*

$$\begin{aligned}&\left|\frac{1}{U_n}\sum_{i=1}^n u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)\right|\\
&\qquad\leq\frac{1}{2}\left(\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2-\frac{1}{U_n}\sum_{i=1}^n u_is_i^2\right)\\
&\qquad\times\left|f''\left(\frac{\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^3-\frac{1}{U_n}\sum_{i=1}^n u_is_i^3}{3\left(\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2-\frac{1}{U_n}\sum_{i=1}^n u_is_i^2\right)}\right)\right|.\end{aligned} \tag{43}$$

Proof. **1**. Using Lemma 3 for $w(x)=\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)$ and $|g|^q=|f''|^q$ in (17), we get

$$\begin{aligned}&\left|\frac{1}{U_n}\sum_{i=1}^n u_if(s_i)-f\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)\right|\\
&\qquad\leq\left(\frac{1}{2}\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i^2-\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i\right)^2\right)\right)^{1-\frac{1}{q}}\\
&\qquad\times\left[\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)\right)dx\right.\\
&\times\left.\left|f''\left(\frac{\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)\right)xdx}{\int_{\sigma_1}^{\sigma_2}\left(\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)\right)dx}\right)\right|^q\right]^{\frac{1}{q}}.\end{aligned} \tag{44}$$

Using (22) and (27) in (44) and simplifying, we get the result (42).
2. Making use of Lemma 3 for $w(x)=G_k\left(\frac{1}{U_n}\sum_{i=1}^n u_is_i,x\right)-\frac{1}{U_n}\sum_{i=1}^n u_iG_k(s_i,x)$ and $|g|^q=|f''|^q$ in (18), next implementing the process described in part 1, we attain (43). □

Remark 3. *In light of the assumption as follows*

$$\frac{1}{U_n}\sum_{i=1}^{n}u_iG_k(s_i,x)-G_k\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i,x\right)\geq 0,\ \textit{for each}\ \ k\in\{1,2,3,4,5\}.$$

Then using (22), it can be verified that

$$\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i^2-\left(\frac{1}{U_n}\sum_{i=1}^{n}u_is_i\right)^2\right)\geq 0.$$

Remark 4. *The inequalities in Theorem 6 and Theorem 10 are in fact some interpolations by using Theorem 5.*

In fact a bound for the Jensen gap can be extracted from Theorem 10 and is follows as:

Theorem 11. *Let the function $|f''|^q$ for $q>1$ and $f\in C^2[\sigma_1,\sigma_2]$ be concave and $s_i\in[\sigma_1,\sigma_2],\ u_i\geq 0$ for $i=1,2,\ldots,n$ with restriction that $\sum_{i=1}^{n}u_i=U_n>0$, then inequality (42) holds.*

Proof. Since it is well known that for each k, G_k are convex functions and $u_i\geq 0$ for all i with its positive sum U_n. By invoking Jensen's inequality now, the inequality in (41) can be deduced. Therefore result (42) follows directly from Theorem 10. □

The subsequent corollary introduces an enhancement to the Jensen-Steffensen inequality, serving as an application of Theorem 10:

Corollary 10. *Let $|f''|^q$ for $q>1$ and $f\in C^2[\sigma_1,\sigma_2]$ be a concave function. Also, assume that $s_i\in[\sigma_1,\sigma_2],\ u_i\in\mathbb{R}$ for $i=1,2,\ldots,n$. Further, if*

$$\sum_{c=1}^{n}u_c\geq\sum_{c=1}^{i}u_c\geq 0,\ \ i=1,2,\ldots,n,\quad \sum_{c=1}^{n}u_c>0,$$

and $s_1\geq s_2\geq\cdots\geq s_n$ or $s_1\leq s_2\leq\cdots\leq s_n$, then (42) holds.

Proof. We get the required result easily by following the procedure of proving Corollary 1. □

Corollary 11. *Let the function $|f''|^q$ is concave and all other assumptions are same as in the statement of Corollary 2, then inequality in (43) holds.*

Proof. One can obtain the required result as we have obtained in Corollary 2. □

Corollary 12. *Let the function $|f''|^q$ be concave and all other assumptions are same as provided in the statement of Corollary 3, then inequality in (43) holds.*

Proof. One can obtain the required result as we have obtained in Corollary 3. □

The forthcoming two corollaries unveil novel variations of the Hölder inequality, each serving as a direct application of Theorem 11:

Corollary 13. *Let $p_2>1,\ p_1\in\left(2,2+\frac{1}{q}\right)$ for $q>1$ be such that $\frac{1}{p_1}+\frac{1}{p_2}=1$. Also, let the interval $[\sigma_1,\sigma_2]$ and the tuples $(a_1,a_2,\ldots,a_n),(b_1,b_2,\ldots,b_n)$ be positive such that $\frac{\sum_{i=1}^{n}a_ib_i}{\sum_{i=1}^{n}b_i^{p_2}},\ a_ib_i^{-\frac{p_2}{p_1}}\in[\sigma_1,\sigma_2]$ for $i=1,2,\ldots,n$, then*

$$\left(\sum_{i=1}^{n}a_i^{p_1}\right)^{\frac{1}{p_1}}\left(\sum_{i=1}^{n}b_i^{p_2}\right)^{\frac{1}{p_2}}-\sum_{i=1}^{n}a_ib_i$$
$$\leq\left[\frac{p_1(p_1-1)}{2}\left(\frac{\Delta_2}{3}\right)^{p_1-2}\Delta_1^{3-p_1}\right]^{\frac{1}{p_1}}\sum_{i=1}^{n}b_i^{p_2},\tag{45}$$

where

$$\Delta_1=\frac{1}{\sum_{i=1}^{n}b_i^{p_2}}\sum_{i=1}^{n}a_i^2b_i^{1-\frac{p_2}{p_1}}-\left(\frac{1}{\sum_{i=1}^{n}b_i^{p_2}}\sum_{i=1}^{n}a_ib_i\right)^2$$

and

$$\Delta_2 = \frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^3 b_i^{1-2\frac{p_2}{p_1}} - \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i b_i \right)^3 .$$

Proof. Let $f(y) = y^{p_1}$, $y \in [\sigma_1, \sigma_2]$, then $f''(y) = p_1(p_1-1)y^{p_1-2} > 0$, shows that f is convex. Also, $|f''(y)|^q = p_1^q(p_1-1)^q y^{q(p_1-2)}$, this function is concave for the proposed values of q and p_1. Thus using (42) for $f(y) = y^{p_1}$, $u_i = b_i^{p_2}$ and $r_i = a_i b_i^{-\frac{p_2}{p_1}}$, we derive

$$\left(\left(\sum_{i=1}^n a_i^{p_1} \right) \left(\sum_{i=1}^n b_i^{p_2} \right)^{p_1-1} - \left(\sum_{i=1}^n a_i b_i \right)^{p_1} \right)^{\frac{1}{p_1}}$$

$$\leq \left[\frac{p_1(p_1-1)}{2} \left(\frac{\Delta_2}{3} \right)^{p_1-2} \Delta_1^{3-p_1} \right]^{\frac{1}{p_1}} \sum_{i=1}^n b_i^{p_2}. \tag{46}$$

By utilizing the inequality $\alpha^e - \beta^e \leq (\alpha - \beta)^e$, $0 \leq \beta \leq \alpha$, $e \in [0,1]$ for $\alpha = \left(\sum_{i=1}^n a_i^{p_1}\right)\left(\sum_{i=1}^n b_i^{p_2}\right)^{p_1-1}$, $\beta = \left(\sum_{i=1}^n a_i b_i\right)^{p_1}$ and $e = \frac{1}{p_1}$, we obtain

$$\left(\sum_{i=1}^n a_i^{p_1} \right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n b_i^{p_2} \right)^{\frac{1}{p_2}} - \sum_{i=1}^n a_i b_i$$

$$\leq \left(\left(\sum_{i=1}^n a_i^{p_1} \right) \left(\sum_{i=1}^n b_i^{p_2} \right)^{p_1-1} - \left(\sum_{i=1}^n a_i b_i \right)^{p_1} \right)^{\frac{1}{p_1}} . \tag{47}$$

Now using (47) in (46), we get (45). □

Corollary 14. *Let* $0 < \sigma_1 < \sigma_2$ *while* $p_1 \in \left(\frac{q}{2q+1}, \frac{1}{2}\right)$ *for* $q > 1$, $p_2 = \frac{p_1}{p_1-1}$ *and* $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$ *be positive tuples such that* $\frac{\sum_{i=1}^n a_i^{p_1}}{\sum_{i=1}^n b_i^{p_2}}$, $a_i^{p_1} b_i^{-p_2} \in [\sigma_1, \sigma_2]$ *for* $i = 1, 2, \ldots, n$, *then*

$$\sum_{i=1}^n a_i b_i - \left(\sum_{i=1}^n a_i^{p_1} \right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n b_i^{p_2} \right)^{\frac{1}{p_2}}$$

$$\leq \left[\frac{1}{2p_1} \left(\frac{1}{p_1} - 1 \right) \Theta_1^{3-\frac{1}{p_1}} \left(\frac{\Theta_2}{3} \right)^{\frac{1}{p_1}-2} \right] \sum_{i=1}^n b_i^{p_2}, \tag{48}$$

where

$$\Theta_1 = \frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^{2p_1} b_i^{-p_2} - \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^{p_1} \right)^2$$

and

$$\Theta_2 = \frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^{3p_1} b_i^{-2p_2} - \left(\frac{1}{\sum_{i=1}^n b_i^{p_2}} \sum_{i=1}^n a_i^{p_1} \right)^3 .$$

Proof. Assuming a convex function $f(y) = y^{\frac{1}{p_1}}$, $y \in [\sigma_1, \sigma_2]$. For the specified values of p_1 and q, it is observed that the function $|f''|^q$ exhibits concavity. Therefore substituting $u_i = b_i^{p_2}$ and $r_i = a_i^{p_1} b_i^{-p_2}$ in (42), we get (48). □

An inequality can be established for the power mean as a direct consequence of Theorem 11 as follows:

Corollary 15. *Suppose that the interval* $[\sigma_1, \sigma_2]$ *and the tuples* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ *are positive with* $\sum_{i=1}^n u_i = U_n$. *Additionally, let* υ *and* λ *represent negative real numbers, satisfying* $\upsilon < \lambda$ *with* $\frac{\upsilon}{\lambda} \in \left(2, 2 + \frac{1}{q}\right)$ *for* $q > 1$, *then the subsequent inequality is valid*

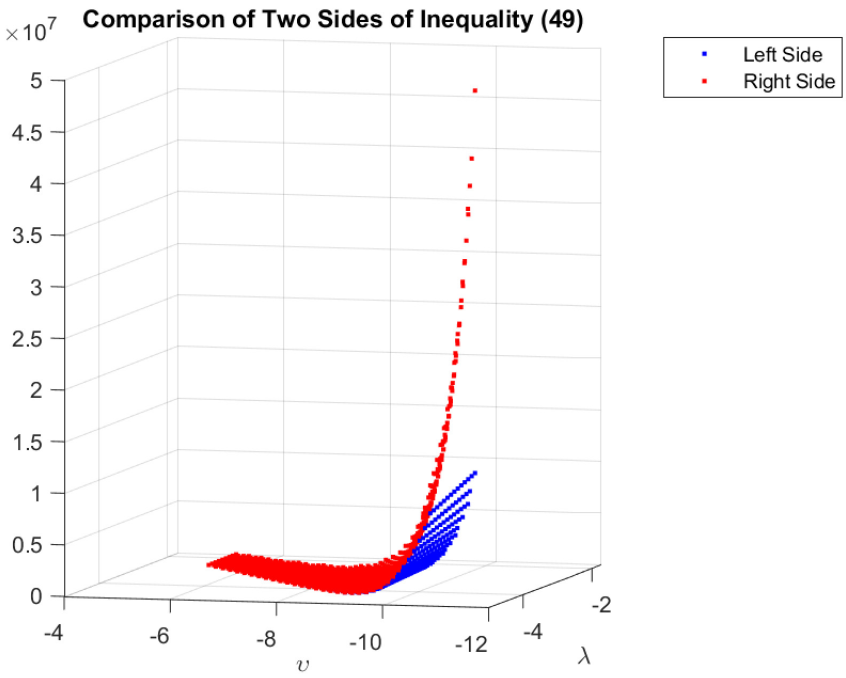

FIGURE 3. $q=1,\ [\sigma_1,\sigma_2]=[1,3200],\ (u_1,u_2)=(0.5,0.5),\ (s_1,s_2)=(0.2,0.3)$

$$\mathcal{M}_\upsilon^\upsilon(\mathbf{u},\mathbf{s})-\mathcal{M}_\lambda^\upsilon(\mathbf{u},\mathbf{s}) \leq \frac{\upsilon(\upsilon-\lambda)}{2\lambda^2}\left(\mathcal{M}_{2\lambda}^{2\lambda}(\mathbf{u},\mathbf{s})-\mathcal{M}_\lambda^{2\lambda}(\mathbf{u},\mathbf{s})\right)^{3-\frac{\upsilon}{\lambda}}\left(\frac{\mathcal{M}_{3\lambda}^{3\lambda}(\mathbf{u},\mathbf{s})-\mathcal{M}_\lambda^{3\lambda}(\mathbf{u},\mathbf{s})}{3}\right)^{\frac{\upsilon}{\lambda}-2}. \tag{49}$$

Proof. For the assumed values of λ,υ and q, the function $f(y)=y^{\frac{\upsilon}{\lambda}}$ is convex while on the other hand the function $|f''|^q$ is valid for concavity on its respective domain in $[\sigma_1,\sigma_2]$, therefore using (42) for $f(y)=y^{\frac{\upsilon}{\lambda}}$ and $s_i\to s_i^\lambda$, we obtain (49). □

Quasi arithmetic mean can be connected in an inequality as an application of Theorem 11 as follows:

Corollary 16. *Let $|(\beta\circ\varphi^{-1})''|^q$ be a concave function for $q>1$ and $\beta\circ\varphi^{-1}$ being an arbitrary function, provided that φ is a continuous and strictly monotone function. Also, assume that the intervals $(u_1,u_2,\ldots,u_n)=\mathbf{u}$ and $(s_1,s_2,\ldots,s_n)=\mathbf{s}$ are positive with $\sum_{i=1}^n u_i=U_n$. Then*

$$\left|\frac{1}{U_n}\sum_{i=1}^n u_i\beta(s_i)-\beta\left(\mathcal{M}_\varphi(\mathbf{u},\mathbf{s})\right)\right| \leq \frac{1}{2}\left(\frac{1}{U_n}\sum_{i=1}^n u_i\varphi^2(s_i)-\varphi^2\left(\mathcal{M}_\varphi(\mathbf{u},\mathbf{s})\right)\right) \times\left|(\beta\circ\varphi^{-1})''\left(\frac{\frac{1}{U_n}\sum_{i=1}^n u_i\varphi^3(s_i)-\varphi^3\left(\mathcal{M}_\varphi(\mathbf{u},\mathbf{s})\right)}{3\left(\frac{1}{U_n}\sum_{i=1}^n u_i\varphi^2(s_i)-\varphi^2\left(\mathcal{M}_\varphi(\mathbf{u},\mathbf{s})\right)\right)}\right)\right|. \tag{50}$$

Proof. (50) follows from (42) by assuming $f\to\beta\circ\varphi^{-1}$ and $s_i\to\varphi(s_i)$. □

A generalization of Theorem 10 to its integral form can be followed as follows:

Theorem 12. *Let $|f''|^q$ for $q>1$ and $f\in C^2[\sigma_1,\sigma_2]$ be a concave function. Also, let the functions $J_1,J_2:[a_1,a_2]\to\mathbb{R}$ be integrable such that $J_1(y)\in[\sigma_1,\sigma_2]$ for all $y\in[a_1,a_2]$ with $H:=\int_{a_1}^{a_2}J_2(y)dy\neq 0$*

and $\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy \in [\sigma_1, \sigma_2]$.

1. *Assume that (37) holds, then*

$$\begin{aligned}&\left|\frac{1}{H}\int_{a_1}^{a_2}(f\circ J_1)(y)J_2(y)dy - f\left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)\right| \\ &\leq \frac{1}{2}\left(\frac{1}{H}\int_{a_1}^{a_2} J_1^2(y)J_2(y)dy - \left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)^2\right) \\ &\times\left|f''\left(\frac{\frac{1}{H}\int_{a_1}^{a_2} J_1^3(y)J_2(y)dy - \left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)^3}{3\left(\frac{1}{H}\int_{a_1}^{a_2} J_1^2(y)J_2(y)dy - \left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)^2\right)}\right)\right|. \end{aligned} \tag{51}$$

2. *If the reverse inequality holds in (37), then*

$$\begin{aligned}&\left|\frac{1}{H}\int_{a_1}^{a_2}(f\circ J_1)(y)J_2(y)dy - f\left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)\right| \\ &\leq \frac{1}{2}\left(\left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)^2 - \frac{1}{H}\int_{a_1}^{a_2} J_1^2(y)J_2(y)dy\right) \\ &\times\left|f''\left(\frac{\left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)^3 - \frac{1}{H}\int_{a_1}^{a_2} J_1^3(y)J_2(y)dy}{3\left(\left(\frac{1}{H}\int_{a_1}^{a_2} J_1(y)J_2(y)dy\right)^2 - \frac{1}{H}\int_{a_1}^{a_2} J_1^2(y)J_2(y)dy\right)}\right)\right|. \end{aligned} \tag{52}$$

Remark 5. *Integral versions of Corollary 10, Corollary 11 and of Corollary 12 can be presented as applications of Theorem 12.*

From Theorem 12, we derive a bound for the Jensen gap, representing the integral form of Theorem 11:

Theorem 13. *Let* $|f''|^q$ *for* $q > 1$ *and* $f \in C^2[\sigma_1, \sigma_2]$ *be a concave function. Let the function* $J_1 : [a_1, a_2] \to \mathbb{R}$ *be integrable with the restriction that* $J_1(y) \in [\sigma_1, \sigma_2]$ *for all* $y \in [a_1, a_2]$ *and the function* $J_2 : [a_1, a_2] \to \mathbb{R}$ *be nonnegative with the condition that* $\int_{a_1}^{a_2} J_2(y)dy = H > 0$, *then (51) holds.*

Remark 6. *The integral form of Corollaries 13, 14, 15 and 16 can be established as direct applications of Theorem 13.*

A variant of the Hermite-Hadamard gap is extracted from Theorem 13 as follows:

Corollary 17. *Let* $f \in C^2[t_1, t_2]$ *be a function such that* $|f''|^q$ *for* $q > 1$ *is concave, then*

$$\left|\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(y)dy - f\left(\frac{t_1 + t_2}{2}\right)\right| \leq \frac{(t_2 - t_1)^2}{24}\left|f''\left(\frac{t_1 + t_2}{2}\right)\right|. \tag{53}$$

Proof. Using (51) for $[\sigma_1, \sigma_2] = [t_1, t_2]$ and $J_2(y) = 1,\ J_1(y) = y$ for all $y \in [t_1, t_2]$, we get (53). □

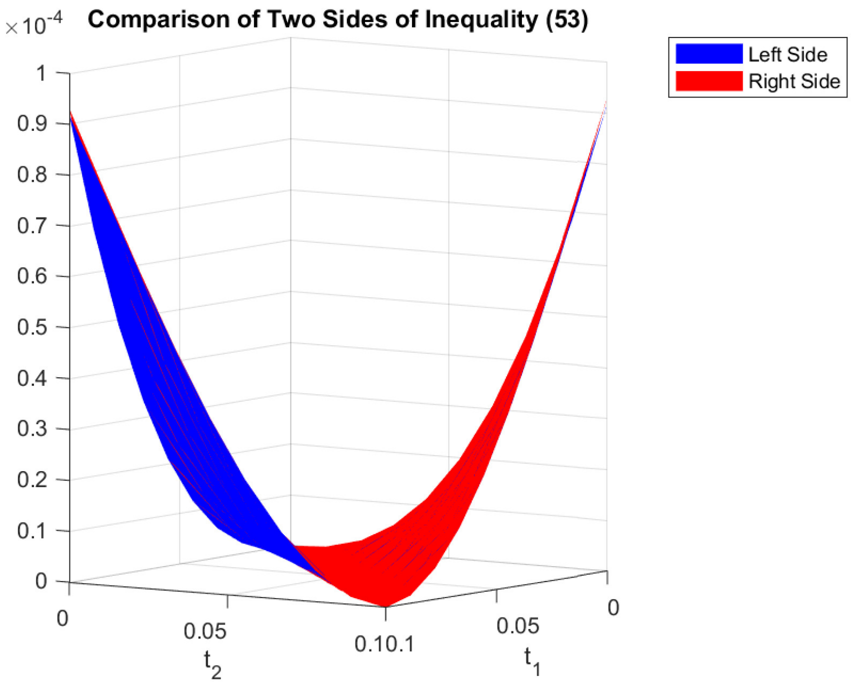

FIGURE 4. $q = 1,\ f(y) = \frac{4}{15}y^{\frac{5}{2}}$

Remark 7. *By substituting $q = 1$ in Theorem 7 and its related results, we extract the results presented in [18]. Similarly taking Theorem 9 and its related results for $q = 1$, we obtain the results presented in [4].*

3. Applications in Information Theory

The corollaries that follow illustrate various applications to different divergences stemming from Theorem 7:

Definition 3. ***(Csiszár divergence)*** *Let $[\sigma_1, \sigma_2] \subset \mathbb{R}$, and $f : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a function, then for $(r_1, r_2, \ldots, r_n) = \mathbf{r} \in \mathbb{R}^n$ and $(w_1, w_2, \ldots, w_n) = \mathbf{w} \in \mathbb{R}^n_+$ with the condition $\frac{r_i}{w_i} \in [\sigma_1, \sigma_2]$ $(i = 1, 2, \ldots, n)$, the mathematical formula of Csiszár divergence is following*

$$\bar{D}_c(\mathbf{r}, \mathbf{w}) = \sum_{i=1}^{n} w_i f\left(\frac{r_i}{w_i}\right).$$

Theorem 14. *Consider that the function $|f''|^q$ is convex for $q > 1$ and $f \in C^2[\sigma_1, \sigma_2]$. Further if, $(r_1, r_2, \ldots, r_n) = \mathbf{r} \in \mathbb{R}^n$, $(w_1, w_2, \ldots, w_n) = \mathbf{w} \in \mathbb{R}^n_+$ such that $\frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n w_i}, \frac{r_i}{w_i} \in [\sigma_1, \sigma_2]$ for $i = 1, 2, \ldots, n$, then*

$$\begin{aligned}
&\left|\frac{1}{\sum_{i=1}^n w_i}\bar{D}_c(\mathbf{r}, \mathbf{w}) - f\left(\frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n w_i}\right)\right| \\
&\quad\le \frac{1}{2^{1-\frac{1}{q}}(\sigma_2 - \sigma_1)^{\frac{1}{q}}}\left(\frac{1}{\sum_{i=1}^n w_i}\sum_{i=1}^{n}\frac{r_i^2}{w_i} - \left(\frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n w_i}\right)^2\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\frac{\sigma_2|f''(\sigma_1)|^q - \sigma_1|f''(\sigma_2)|^q}{2}\left(\frac{1}{\sum_{i=1}^n w_i}\sum_{i=1}^{n}\frac{r_i^2}{w_i} - \left(\frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n w_i}\right)^2\right)\right. \\
&\quad\left.+\frac{|f''(\sigma_2)|^q - |f''(\sigma_1)|^q}{6}\left(\frac{1}{\sum_{i=1}^n w_i}\sum_{i=1}^{n}\frac{r_i^3}{w_i^2} - \left(\frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n w_i}\right)^3\right)\right]^{\frac{1}{q}}. \qquad (54)
\end{aligned}$$

Proof. Choosing $s_i = \frac{r_i}{w_i}$, $u_i = \frac{w_i}{\sum_{i=1}^n w_i}$ in (24), we get the result (54). □

Definition 4. ***(Rényi-divergence)*** *Assume that $(r_1, r_2, \ldots, r_n) = \mathbf{r}$ and $(w_1, w_2, \ldots, w_n) = \mathbf{w}$ are positive probability distributions and $\mu \neq 1$ is a nonnegative real number, then the following mathematical formula holds*

$$D_{re}(\mathbf{r}, \mathbf{w}) = \frac{1}{\mu - 1} \log \left(\sum_{i=1}^{n} r_i^{\mu} w_i^{1-\mu} \right).$$

Corollary 18. *Let $1 < \mu$ and $(r_1, r_2, \ldots, r_n) = \mathbf{r}$, $(w_1, w_2, \ldots, w_n) = \mathbf{w}$ are positive probability distributions, and provided that $\sum_{i=1}^{n} w_i \left(\frac{r_i}{w_i}\right)^{\mu}, \left(\frac{r_i}{w_i}\right)^{\mu-1} \in [\sigma_1, \sigma_2] \subseteq \mathbb{R}^+$ for $i = 1, 2, \ldots, n$. Then for $q > 1$, the following inequality holds*

$$\begin{aligned}
&D_{re}(\mathbf{r}, \mathbf{w}) - \frac{1}{\mu - 1} \sum_{i=1}^{n} r_i \log \left(\frac{r_i}{w_i}\right)^{\mu-1} \\
&\leq \frac{1}{2^{1-\frac{1}{q}} (\sigma_2 - \sigma_1)^{\frac{1}{q}} (\mu - 1) \sigma_1^2 \sigma_2^2} \left(\sum_{i=1}^{n} r_i \left(\frac{r_i}{w_i}\right)^{2(\mu-1)} - \left(\sum_{i=1}^{n} r_i^{\mu} w_i^{1-\mu} \right)^2 \right)^{1-\frac{1}{q}} \\
&\times \left[\frac{\sigma_2^{2q+1} - \sigma_1^{2q+1}}{2} \left(\sum_{i=1}^{n} r_i \left(\frac{r_i}{w_i}\right)^{2(\mu-1)} - \left(\sum_{i=1}^{n} r_i^{\mu} w_i^{1-\mu} \right)^2 \right) \right. \\
&\left. - \frac{\sigma_2^{2q} - \sigma_1^{2q}}{6} \left(\sum_{i=1}^{n} r_i \left(\frac{r_i}{w_i}\right)^{3(\mu-1)} - \left(\sum_{i=1}^{n} r_i^{\mu} w_i^{1-\mu} \right)^3 \right) \right]^{\frac{1}{q}}. \qquad (55)
\end{aligned}$$

Proof. Since the function $f(y) = -\frac{1}{\mu-1} \log y$, $y \in [\sigma_1, \sigma_2]$ satisfies the conditions of Theorem 6, therefore using (24) for this function and substituting u_i by r_i and s_i by $\left(\frac{r_i}{w_i}\right)^{\mu-1}$, we obtain (55). ☐

Definition 5. ***(Shannon entropy)*** *Provided that $(w_1, w_2, \ldots, w_n) = \mathbf{w}$ is a positive probability distribution, the Shannon entropy for it, is given by*

$$E_s(\mathbf{w}) = -\sum_{i=1}^{n} w_i \log w_i.$$

Corollary 19. *Let $q > 1$ and $[\sigma_1, \sigma_2] \subseteq \mathbb{R}^+$ while $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a positive probability distribution such that $\frac{1}{w_i} \in [\sigma_1, \sigma_2]$ for $i = 1, 2, \ldots, n$, then*

$$\begin{aligned}
\log n - E_s(\mathbf{w}) &\leq \frac{1}{2^{1-\frac{1}{q}} (\sigma_2 - \sigma_1)^{\frac{1}{q}} \sigma_1^2 \sigma_2^2} \left(\sum_{i=1}^{n} \frac{1}{w_i} - n^2 \right)^{1-\frac{1}{q}} \\
&\times \left[\frac{\sigma_2^{2q+1} - \sigma_1^{2q+1}}{2} \left(\sum_{i=1}^{n} \frac{1}{w_i} - n^2 \right) - \frac{\sigma_2^{2q} - \sigma_1^{2q}}{6} \left(\sum_{i=1}^{n} \frac{1}{w_i^2} - n^3 \right) \right]^{\frac{1}{q}}. \qquad (56)
\end{aligned}$$

Proof. Since the function $f(y) = -\log y$, $y \in [\sigma_1, \sigma_2]$ satisfies the assumptions of Theorem 14 therefore by using (54) for this function and substituting $(r_1, r_2, \ldots, r_n)$ by $(1, 1, \ldots, 1)$, we get (56). ☐

Definition 6. ***(Kullback-Leibler divergence)*** *It is presented by the following expression*

$$D_{kl}(\mathbf{r}, \mathbf{w}) = \sum_{i=1}^{n} r_i \log \frac{r_i}{w_i},$$

provided that $(r_1, r_2, \ldots, r_n) = \mathbf{r}$ and $(w_1, w_2, \ldots, w_n) = \mathbf{w}$ are positive probability distributions.

Corollary 20. *Let $q > 1$ and $[\sigma_1, \sigma_2] \subseteq \mathbb{R}^+$ while $(r_1, r_2, \ldots, r_n) = \mathbf{r}$, $(w_1, w_2, \ldots, w_n) = \mathbf{w}$ be positive probability distributions such that $\frac{r_i}{w_i} \in [\sigma_1, \sigma_2]$ for $i = 1, 2, \ldots, n$, then*

$$D_{kl}(\mathbf{r}, \mathbf{w}) \leq \frac{1}{2^{1-\frac{1}{q}} (\sigma_2 - \sigma_1)^{\frac{1}{q}} \sigma_1 \sigma_2} \left(\sum_{i=1}^{n} \frac{r_i^2}{w_i} - 1 \right)^{1-\frac{1}{q}}$$

$$\times \left[\frac{\sigma_2^{q+1} - \sigma_1^{q+1}}{2} \left(\sum_{i=1}^{n} \frac{r_i^2}{w_i} - 1 \right) - \frac{\sigma_2^q - \sigma_1^q}{6} \left(\sum_{i=1}^{n} \frac{r_i^3}{w_i^2} - 1 \right) \right]^{\frac{1}{q}}. \tag{57}$$

Proof. Since the function $f(y) = y \log y,\ y \in [\sigma_1, \sigma_2]$ satisfies the assumptions of Theorem 14, therefore we get (57) by using (54) for $f(y) = y \log y$. □

Definition 7. ***(χ^2-divergence)*** *This is presented by the following expression*

$$D_{\chi^2}(\mathbf{r}, \mathbf{w}) = \sum_{i=1}^{n} \frac{(r_i - w_i)^2}{w_i},$$

for $(r_1, r_2, \ldots, r_n) = \mathbf{r},\ (w_1, w_2, \ldots, w_n) = \mathbf{w}$ *as two positive probability distributions.*

Corollary 21. *If* $q > 1$ *and* $[\sigma_1, \sigma_2] \subseteq \mathbb{R}^+$ *while* $(r_1, r_2, \ldots, r_n) = \mathbf{r},\ (w_1, w_2, \ldots, w_n) = \mathbf{w}$ *are two positive probability distributions such that* $\frac{r_i}{w_i} \in [\sigma_1, \sigma_2]$ *for* $i = 1, 2, \ldots, n$, *then*

$$D_{\chi^2}(\mathbf{r}, \mathbf{w}) \le \frac{1}{2^{1-\frac{1}{q}}} \left(\sum_{i=1}^{n} \frac{r_i^2}{w_i} - 1 \right). \tag{58}$$

Proof. Since the function $f(y) = (y-1)^2$ for $y \in [\sigma_1, \sigma_2]$ satisfies the assumptions of Theorem 14, therefore inequality (58) follows by using (54) for such function. □

Definition 8. ***(Bhattacharyya-coefficient)*** *This is given by the following mathematical expression*

$$C_b(\mathbf{r}, \mathbf{w}) = \sum_{i=1}^{n} \sqrt{r_i w_i},$$

provided that $(w_1, w_2, \ldots, w_n) = \mathbf{w}$ *and* $(r_1, r_2, \ldots, r_n) = \mathbf{r}$ *are two positive probability distributions.*

Corollary 22. *For* $q > 1$ *and* $[\sigma_1, \sigma_2] \subseteq \mathbb{R}^+$, *the following inequality holds*

$$1 - C_b(\mathbf{r}, \mathbf{w}) \le \frac{1}{2^{3-\frac{1}{q}} (\sigma_2 - \sigma_1)^{\frac{1}{q}} \sigma_1^{\frac{3}{2}} \sigma_2^{\frac{3}{2}}} \left(\sum_{i=1}^{n} \frac{r_i^2}{w_i} - 1 \right)^{1-\frac{1}{q}}$$

$$\times \left[\frac{\sigma_2^{\frac{3q}{2}+1} - \sigma_1^{\frac{3q}{2}+1}}{2} \left(\sum_{i=1}^{n} \frac{r_i^2}{w_i} - 1 \right) - \frac{\sigma_2^{\frac{3q}{2}} - \sigma_1^{\frac{3q}{2}}}{6} \left(\sum_{i=1}^{n} \frac{r_i^3}{w_i^2} - 1 \right) \right]^{\frac{1}{q}}, \tag{59}$$

where $(r_1, r_2, \ldots, r_n) = \mathbf{r},\ (w_1, w_2, \ldots, w_n) = \mathbf{w}$ *are positive probability distributions such that* $\frac{r_i}{w_i} \in [\sigma_1, \sigma_2]$ *for* $i = 1, 2, \ldots, n$.

Proof. Since the function $f(y) = -\sqrt{y},\ y \in [\sigma_1, \sigma_2]$ satisfies the conditions imposed on it in Theorem 14, therefore we get (59) by following (54) for $f(y) = -\sqrt{y}$. □

Remark 8. *The integral counterparts of the corollaries presented in this section can be gladly derived by applying Theorem 9.*

3.1. **Estimation of the Zipf-Mandelbrot Entropy.** The actual mathematical expression of the probability mass function for the Zipf-Mandelbrot law is follows as:

$$f_{(i,n,\theta,s)} = \frac{1/(i+\theta)^s}{M_{n,\theta,s}},$$

for $i = 1, 2, \ldots, n,\ n \in \{1, 2, 3, \ldots\},\ s > 0,\ \theta \ge 0$ and $M_{n,\theta,s} = \sum_{i=1}^{n} \frac{1}{(i+\theta)^s}$ is a generalized harmonic number. As documented in the research by [5], the mathematical structure of the Zipf-Mandelbrot entropy is elucidated:

$$Z(M, \theta, s) = \frac{s}{M_{n,\theta,s}} \sum_{i=1}^{n} \frac{\log(i+\theta)}{(i+\theta)^s} + \log M_{n,\theta,s}.$$

Corollary 23. *Let $0 < \sigma_1 < \sigma_2$, $q > 1$, $s > 0$, $\theta \geq 0$ and $w_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n w_i = 1$, then*

$$-\frac{1}{M_{n,\theta,s}} \sum_{i=1}^{n} \frac{\log w_i}{(i+\theta)^s} - Z(M, \theta, s)$$

$$\leq \frac{1}{2^{1-\frac{1}{q}} (\sigma_2 - \sigma_1)^{\frac{1}{q}} \sigma_1 \sigma_2} \left(\sum_{i=1}^{n} \frac{1}{w_i (i+\theta)^{2s} M_{n,\theta,s}^2} - 1 \right)^{1-\frac{1}{q}}$$

$$\times \left[\frac{\sigma_2^{q+1} - \sigma_1^{q+1}}{2} \left(\sum_{i=1}^{n} \frac{1}{w_i (i+\theta)^{2s} M_{n,\theta,s}^2} - 1 \right) - \frac{\sigma_2^q - \sigma_1^q}{6} \left(\sum_{i=1}^{n} \frac{1}{w_i^2 (i+\theta)^{3s} M_{n,\theta,s}^3} - 1 \right) \right]^{\frac{1}{q}}. \tag{60}$$

Proof. For $r_i = \frac{1}{(i+\theta)^s M_{n,\theta,s}}$, $i = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^{n} r_i \log \frac{r_i}{w_i} = \sum_{i=1}^{n} \frac{1}{(i+\theta)^s M_{n,\theta,s}} \left(-s \log(i+\theta) - \log M_{n,\theta,s} - \log w_i \right)$$

$$= -Z(M, \theta, s) - \frac{1}{M_{n,\theta,s}} \sum_{i=1}^{n} \frac{\log w_i}{(i+\theta)^s}, \tag{61}$$

and

$$\frac{1}{2^{1-\frac{1}{q}} (\sigma_2 - \sigma_1)^{\frac{1}{q}} \sigma_1 \sigma_2} \left(\sum_{i=1}^{n} \frac{r_i^2}{w_i} - 1 \right)^{1-\frac{1}{q}} \left[\frac{\sigma_2^{q+1} - \sigma_1^{q+1}}{2} \left(\sum_{i=1}^{n} \frac{r_i^2}{w_i} - 1 \right) - \frac{\sigma_2^q - \sigma_1^q}{6} \left(\sum_{i=1}^{n} \frac{r_i^3}{w_i^2} - 1 \right) \right]^{\frac{1}{q}}$$

$$= \frac{1}{2^{1-\frac{1}{q}} (\sigma_2 - \sigma_1)^{\frac{1}{q}} \sigma_1 \sigma_2} \left(\sum_{i=1}^{n} \frac{1}{w_i (i+\theta)^{2s} M_{n,\theta,s}^2} - 1 \right)^{1-\frac{1}{q}}$$

$$\times \left[\frac{\sigma_2^{q+1} - \sigma_1^{q+1}}{2} \left(\sum_{i=1}^{n} \frac{1}{w_i (i+\theta)^{2s} M_{n,\theta,s}^2} - 1 \right) - \frac{\sigma_2^q - \sigma_1^q}{6} \left(\sum_{i=1}^{n} \frac{1}{w_i^2 (i+\theta)^{3s} M_{n,\theta,s}^3} - 1 \right) \right]^{\frac{1}{q}}. \tag{62}$$

Now using (61) and (62) in (57), we get (60). □

Corollary 24. *Let $0 < \sigma_1 < \sigma_2$, $\theta_1, \theta_2 \geq 0$, $s_1, s_2 > 0$, then for $q > 1$ the following inequality holds*

$$-Z(M, \theta_1, s_1) + \sum_{i=1}^{n} \frac{\log(i+\theta_2)^{s_2} M_{n,\theta_2,s_2}}{(i+\theta_1)^{s_1} M_{n,\theta_1,s_1}}$$

$$\leq \frac{1}{2^{1-\frac{1}{q}} (\sigma_2 - \sigma_1)^{\frac{1}{q}} \sigma_1 \sigma_2} \left(\sum_{i=1}^{n} \frac{(i+\theta_2)^{s_2} M_{n,\theta_2,s_2}}{(i+\theta_1)^{2s_1} M_{n,\theta_1,s_1}^2} - 1 \right)^{1-\frac{1}{q}}$$

$$\times \left[\frac{\sigma_2^{q+1} - \sigma_1^{q+1}}{2} \left(\sum_{i=1}^{n} \frac{(i+\theta_2)^{s_2} M_{n,\theta_2,s_2}}{(i+\theta_1)^{2s_1} M_{n,\theta_1,s_1}^2} - 1 \right) - \frac{\sigma_2^q - \sigma_1^q}{6} \left(\sum_{i=1}^{n} \frac{(i+\theta_2)^{2s_2} M_{n,\theta_2,s_2}^2}{(i+\theta_1)^{3s_1} M_{n,\theta_1,s_1}^3} - 1 \right) \right]^{\frac{1}{q}}. \tag{63}$$

Proof. For $r_i = \frac{1}{(i+\theta_1)^{s_1} M_{n,\theta_1,s_1}}$, $w_i = \frac{1}{(i+\theta_2)^{s_2} M_{n,\theta_2,s_2}}$, $i = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^{n} r_i \log \frac{r_i}{w_i} = \sum_{i=1}^{n} \frac{1}{(i+\theta_1)^{s_1} M_{n,\theta_1,s_1}} \left(\log(i+\theta_2)^{s_2} M_{n,\theta_2,s_2} - \log(i+\theta_1)^{s_1} M_{n,\theta_1,s_1} \right)$$

$$= -Z(M, \theta_1, s_1) + \sum_{i=1}^{n} \frac{\log(i+\theta_2)^{s_2} M_{n,\theta_2,s_2}}{(i+\theta_1)^{s_1} M_{n,\theta_1,s_1}}. \tag{64}$$

Also,

$$\frac{1}{2^{1-\frac{1}{q}} (\sigma_2 - \sigma_1)^{\frac{1}{q}} \sigma_1 \sigma_2} \left(\sum_{i=1}^{n} \frac{r_i^2}{w_i} - 1 \right)^{1-\frac{1}{q}}$$

$$\times\left[\frac{\sigma_2^{q+1}-\sigma_1^{q+1}}{2}\left(\sum_{i=1}^{n}\frac{r_i^2}{w_i}-1\right)-\frac{\sigma_2^q-\sigma_1^q}{6}\left(\sum_{i=1}^{n}\frac{r_i^3}{w_i^2}-1\right)\right]^{\frac{1}{q}}$$

$$=\frac{1}{2^{1-\frac{1}{q}}(\sigma_2-\sigma_1)^{\frac{1}{q}}\sigma_1\sigma_2}\left(\sum_{i=1}^{n}\frac{(i+\theta_2)^{s_2}M_{n,\theta_2,s_2}}{(i+\theta_1)^{2s_1}M_{n,\theta_1,s_1}^2}-1\right)^{1-\frac{1}{q}}$$

$$\times\left[\frac{\sigma_2^{q+1}-\sigma_1^{q+1}}{2}\left(\sum_{i=1}^{n}\frac{(i+\theta_2)^{s_2}M_{n,\theta_2,s_2}}{(i+\theta_1)^{2s_1}M_{n,\theta_1,s_1}^2}-1\right)-\frac{\sigma_2^q-\sigma_1^q}{6}\left(\sum_{i=1}^{n}\frac{(i+\theta_2)^{2s_2}M_{n,\theta_2,s_2}^2}{(i+\theta_1)^{3s_1}M_{n,\theta_1,s_1}^3}-1\right)\right]^{\frac{1}{q}}. \quad (65)$$

Now utilizing (64) and (65) in (57), we get (63). □

4. Conclusion

Jensen's inequality has an important role in the literature of applied analysis. It enables us to provide some tools for: estimation of certain parameters in optimization problems, systematic development of a qualitative theoretical framework concerning differential and integral equations, facilitates the integration of Rao-Blackwell estimates into the process of parameter estimation within a probability space, estimation of Zipf-Mandelbrot entropy and various divergences etc. We have established a Jensen type inequality in terms of Green functions for a class of functions $f \in C^2$. After that we have utilized this result for a function f such that $|f''|^q$ for $q > 1$, either convex or concave, and obtained some related interpolating inequalities. Then we have utilized these interpolating inequalities and deduced some estimates for the Jensen and the Jensen-Steffensen differences. Also, we have obtained some new variants of the reverse Jensen-Steffensen and Jensen's inequalities. Drawing upon the outcomes stemming around Jensen's inequality, we have innovated various extensions to the Hölder inequality, defined improved bounds for the discrepancy of the the Hermite-Hadamard inequality, and proposed supplementary inequalities for the power mean, geometric mean, and quasi-arithmetic mean. In this work,we have also geometrically interpreted the results for power mean and the Hermite-Hadamard inequality. This interpretation directly verifies the authenticity and sharpness of the results about Jensen's inequality. Finally, we have presented some applications involving Zipf-Mandelbrot entropy and various cases of Csiszár divergence.
It may be noted that some related results to this work for twice differentiable functions are published in [5]. These ideas may encourage many other mathematicians to produce various important results around Jensen's, the Jensen-Steffensen and some other inequalities.

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