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Some Properties between Lichtenberg, Jacobsthal and Mersenne Gaussian Numbers

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Abstract

We define the Gauss Lichtenberg numbers. Then we give a formula for the Gauss Lichtenberg numbers by using the Lichtenberg numbers. We show that there is a relation between the Gauss Lichtenberg numbers, Lichtenberg, Jacobsthal and Mersenne numbers. Their Binet's formulas are obtained. We also define the matrices of the Gauss Lichtenberg numbers. We examine properties of the matrices.

Keywords: *Gauss numbers, Jacobsthal numbers, Lichtenberg numbers, Mersenne numbers.*
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1. Introduction

The investigation of Gaussian numbers is a research topic of great interest in recent years. If $\mathbb{Z}$ is the set of integers, the set of these numbers is denoted by $\mathbb{Z}[i] = \{a+bi : a,b \in \mathbb{Z}, i^2 = -1\}$. Gaussian numbers were investigated by Gauss in 1832. In 1963, Horadam [11], introduced the concept of Gaussian Fibonacci numbers. And then Jordan [12] considered the two different sequences of Gaussian numbers and extended some relationships which are known about the common Fibonacci sequences.
In [2], Aşci and Gürel introduced the concept of Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas numbers. They also studied also the complex Jacobsthal polynomials [3]. Polynomials that can be defined by Jacobsthal-like recursion relations are called Jacobsthal Polynomials and they were studied in 1997 by Horadam and in 2000 by Djordjević (see [6, 9]). More mathematicians were involved in the study of Jacobsthal polynomials such as Djordjević and Srivastava [7], among others. In [18, 19], Morales defined and studied the Gaussian third-order Jacobsthal numbers and Gaussian third-order Jacobsthal-Lucas polynomials.
Recently, in [5], Daşdemir and Bilgici introduced the Gaussian Mersenne numbers and studied some of their properties. In [17], Kumari et al. defined the generalization of the Mersenne and Gaussian Mersenne polynomials with arbitrary initial values and studied their properties. Further, in [23], Stockmeyer listed several characterizations of the new sequence called Lichtenberg sequence relative to Mersenne and Jacobsthal numbers, and numerous examples of cases in which the Lichtenberg sequence occurs. Also, in [8], Hinz introduced some fundamental characterizations of the Lichtenberg sequence, without placing emphasis on its properties.
Lichtenberg numbers are named after Georg Christoph Lichtenberg, who studied these numbers in the 17th century. Lichtenberg numbers are denoted by ℓ_n, defined mathematically by the recurrence $\ell_n + \ell_{n-1} = 2^n - 1$ and have the form

$$\ell_n = \frac{1}{6}\left[(-1)^{n+1} + 2^{n+2} - 3\right].$$

The first few terms of the Lichtenberg sequence are:

$$0,1,2,5,10,21,42,85,170,341,682,\ldots(A000975).$$

The Lichtenberg numbers $\{\ell_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$\ell_{n+3} = 2\ell_{n+2} + \ell_{n+1} - 2\ell_n, \tag{1.1}$$

with $\ell_0 = 0$, $\ell_1 = 1$ and $\ell_2 = 2$ (see, e.g. [8, 23]). Also, the Binet formula for Lichtenberg numbers is defined in two different ways, including well-known sequences:

$$\ell_n = \frac{1}{2}\left[\frac{2^{n+2} - (-1)^{n+2}}{3} - 1\right] = \frac{1}{2}\left[J_{n+2} - 1\right] \tag{1.2}$$

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and

$$\ell_n = \frac{1}{3}\left[2^{n+1} - 1 - \frac{(-1)^n + 1}{2}\right] = \frac{1}{3}\left[M_{n+1} - \frac{(-1)^n + 1}{2}\right], \tag{1.3}$$

where J_n is the n-th Jacobsthal number (see [10]) and M_n is the n-th Mersenne number (see [4]).
In this paper, we will present a unified view on complex sequences and discuss some properties of the Gauss Lichtenberg sequence $\{g\ell_n\}_{n\geq 0}$ defined mathematically by the recurrence

$$g\ell_{n+1} + g\ell_n = 2^n(2+i) - (1+i), \quad g\ell_0 = 0,$$

and related sequences such as the Gaussian Jacobsthal sequence, which is the sequence of differences of $g\ell_n$. And, of course, at some point Mersenne and Gaussian Mersenne numbers will enter the scene.

2. Gauss Lichtenberg numbers, Gauss modified Lichtenberg numbers and their some properties

2.1. Gauss Lichtenberg numbers

Now, we introduce Gauss Lichtenberg numbers $g\ell_n$ and present some of their basic properties.

Definition 2.1. *Gauss Lichtenberg numbers $g\ell_n$ are defined by*

$$g\ell_{n+3} = 2g\ell_{n+2} + g\ell_{n+1} - 2g\ell_n, \quad n \geq 0, \tag{2.1}$$

with $g\ell_0 = 0$, $g\ell_1 = 1$ and $g\ell_2 = 2+i$.

The first few terms are as follows:

$$\begin{aligned}
g\ell_3 &= 2g\ell_2 + g\ell_1 - 2g\ell_0 \\
&= 2(2+i) + (1) - 2(0) = 5+2i, \\
g\ell_4 &= 2g\ell_3 + g\ell_2 - 2g\ell_1 \\
&= 2(5+2i) + (2+i) - 2(1) = 10+5i, \\
g\ell_5 &= 2g\ell_4 + g\ell_3 - 2g\ell_2 \\
&= 2(10+5i) + (5+2i) - 2(2+i) = 21+10i.
\end{aligned}$$

Theorem 2.2. *For $n \geq 0$, we have*

$$g\ell_{n+1} = \ell_{n+1} + i\ell_n$$

where ℓ_n is the n-th Lichtenberg number defined mathematically by the recurrence $\ell_{n+1} + \ell_n = 2^{n+1} - 1$ and $\ell_0 = 0$.

Proof. We can prove the theorem by the induction method on n. For $n = 3$, we have

$$g\ell_3 = 2g\ell_2 + g\ell_1 - 2g\ell_0 = 5+2i = \ell_3 + i\ell_2.$$

Now, assume that the theorem holds for $n \leq k$, that is $g\ell_{k+1} = \ell_{k+1} + i\ell_k$. Then, for $n = k+1$, we have

$$\begin{aligned}
g\ell_{k+2} &= 2g\ell_{k+1} + g\ell_k - 2g\ell_{k-1} \\
&= 2(\ell_{k+1} + i\ell_k) + (\ell_k + i\ell_{k-1}) - 2(\ell_{k-1} + i\ell_{k-2}) \\
&= 2\ell_{k+1} + \ell_k - 2\ell_{k-1} + i(2\ell_k + \ell_{k-1} - 2\ell_{k-2}) \\
&= \ell_{k+2} + i\ell_{k+1}.
\end{aligned}$$

Thus, the result follows. □

Here, we define Binet's formulas for the Gauss Lichtenberg numbers. Let $\{-1,1,2\}$ be the solutions of the equation $x^3 - 2x^2 - x + 2 = 0$ associated with Eq. (2.1). So, we obtain for all $n \geq 0$:

$$g\ell_n = \frac{1}{6}\left[(-1)^n(-1+i) + 2^{n+1}(2+i) - 3(1+i)\right].$$

Using the definitions of Gaussian Jacobsthal numbers $\{J^g_{n+1}\}_{n\geq 0}$ in [2] and Gaussian Mersenne numbers $\{M^g_n\}_{n\geq 0}$ in [5], we have

$$\begin{aligned}
J^g_{n+1} &= J_{n+1} + J_n i = \frac{1}{3}\left[2^n(2+i) - (-1)^n(-1+i)\right], \\
M^g_{n+1} &= M_{n+1} + M_n i = 2^n(2+i) - (1+i),
\end{aligned} \tag{2.2}$$

where J_n is the n-th Jacobsthal number and M_n is the n-th Mersenne number, respectively. Finally, we can write

$$g\ell_n = \frac{1}{2}\left[J^g_{n+2} - (1+i)\right] = \frac{1}{6}\left[2M^g_{n+1} + (-1)^n(-1+i) - (1+i)\right]. \tag{2.3}$$

In particular, we obtain

$$g\ell_n = \frac{1}{3}\begin{cases} M^g_{n+1} - 1 & \text{if} \quad n \equiv 0 \pmod 2 \\ M^g_{n+1} - i & \text{if} \quad n \equiv 1 \pmod 2 \end{cases}.$$

Furthermore, we easily obtain the identities stated in the following result:

Theorem 2.3. *For $n \geq 0$, we have*

$$(g\ell_n)^2 + (g\ell_{n+1})^2 = \frac{1}{9}\left[5\left(M^g_{n+1}\right)^2 + 2\left(M^g_{n+1} + (1+i)\right)A_n - 2i\right],$$

where $\{A_n\}_{n\geq 0}$ is the sequence defined by $A_{n+1} = -A_n + (1+i)$ and $A_0 = 1$.

Proof. Using Eq. (2.3) and the notation $A_n = \frac{1}{2}\left[(-1)^{n+1}(-1+i) + (1+i)\right]$, we can write $g\ell_n = \frac{1}{3}\left[M^g_{n+1} - A_n\right]$, where M^g_{n+1} is the $(n+1)$-th Gaussian Mersenne number and

$$A_n = \begin{cases} 1 & \text{if} \quad n \equiv 0 \pmod 2 \\ i & \text{if} \quad n \equiv 1 \pmod 2 \end{cases}. \tag{2.4}$$

Then, using the identities $A_n^2 + A_{n+1}^2 = 0$ and $M^g_{n+1} = 2M^g_n + (1+i)$, we have

$$\begin{aligned} 9\cdot\left[(g\ell_n)^2 + (g\ell_{n+1})^2\right] &= \left(M^g_{n+1} - A_n\right)^2 + \left(M^g_{n+2} - A_{n+1}\right)^2 \\ &= \left(M^g_{n+1}\right)^2 - 2M^g_{n+1}A_n + A_n^2 + \left(M^g_{n+2}\right)^2 - 2M^g_{n+2}A_{n+1} + A^2_{n+1} \\ &= \left(M^g_{n+1}\right)^2 + \left(M^g_{n+2}\right)^2 - 2\left(M^g_{n+1}A_n + M^g_{n+2}A_{n+1}\right) \\ &= 5\left(M^g_{n+1}\right)^2 + 2i + 2M^g_{n+1}A_n + 2(1+i)A_n - 4i \\ &= 5\left(M^g_{n+1}\right)^2 + 2\left(M^g_{n+1} + (1+i)\right)A_n - 2i, \end{aligned}$$

where $\{A_n\}_{n\geq 0}$ is the sequence defined in Eq. (2.4). □

Remark 2.4. *For $n \geq 1$, using $A_{n+1} = -A_n + (1+i)$, we have*

$$\begin{aligned} g\ell_{n+2} - 3g\ell_{n+1} + 2g\ell_n &= \frac{1}{3}\left[M^g_{n+3} - A_{n+2} - 3M^g_{n+2} + 3A_{n+1} + 2M^g_{n+1} - 2A_n\right] \\ &= -\frac{1}{3}\left[A_{n+2} - 3A_{n+1} + 2A_n\right] \\ &= -\frac{1}{3}\left[A_n + 3A_n - 3(1+i) + 2A_n\right] \\ &= (1+i) - 2A_n. \end{aligned}$$

Then, $g\ell_{n+2} - 3g\ell_{n+1} + 2g\ell_n = (1+i) - 2A_n = \begin{cases} -1+i & \textit{if} \quad n \equiv 0 \pmod 2 \\ 1-i & \textit{if} \quad n \equiv 1 \pmod 2 \end{cases}.$

Definition 2.5. *The Gauss Lichtenberg number matrix $Q\ell_n$ is defined by*

$$Q\ell_n = \begin{bmatrix} g\ell_{n+2} & g\ell_{n+3} - 2g\ell_{n+2} & -2g\ell_{n+1} \\ g\ell_{n+1} & g\ell_{n+2} - 2g\ell_{n+1} & -2g\ell_n \\ g\ell_n & g\ell_{n+1} - 2g\ell_n & -2g\ell_{n-1} \end{bmatrix}, \quad n \geq 1,$$

where $Q\ell_0 = \begin{bmatrix} 2+i & 1 & -2 \\ 1 & i & 0 \\ 0 & 1 & i \end{bmatrix}$ *for convenience.*

Note that the matrix $Q\ell_n$ has order 3×3 and its coefficients are gaussian integers. Now, we study important properties of this new matrix, whose coefficients are Gauss Lichtenberg numbers.

Proposition 1. *For $n \geq 0$, we have*

$$\det\left(Q\ell_{n+1}\right) = (-1)^n 2^{n+2}(2+i).$$

Proof. First, using $g\ell_{n+1} = \ell_{n+1} + i\ell_n$, note that

$$\begin{aligned} Q\ell_{n+1} &= \begin{bmatrix} \ell_{n+3} & \ell_{n+4} - 2\ell_{n+3} & -2\ell_{n+2} \\ \ell_{n+2} & \ell_{n+3} - 2\ell_{n+2} & -2\ell_{n+1} \\ \ell_{n+1} & \ell_{n+2} - 2\ell_{n+1} & -2\ell_n \end{bmatrix} + i\begin{bmatrix} \ell_{n+2} & \ell_{n+3} - 2\ell_{n+2} & -2\ell_{n+1} \\ \ell_{n+1} & \ell_{n+2} - 2\ell_{n+1} & -2\ell_n \\ \ell_n & \ell_{n+1} - 2\ell_n & -2\ell_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n+2} + i\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n+1} \\ &= \begin{bmatrix} 2+i & 1 & -2 \\ 1 & i & 0 \\ 0 & 1 & i \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n+1} = Q\ell_0\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n+1}. \end{aligned}$$

So we use multiplicative property of the determinant and we get

$$\begin{aligned} \det\left(Q\ell_{n+1}\right) &= \begin{vmatrix} 2+i & 1 & -2 \\ 1 & i & 0 \\ 0 & 1 & i \end{vmatrix}\begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}^{n+1} \\ &= (-2)^{n+2}(2+i). \end{aligned}$$

Thus, the result follows. □

Using Proposition 1 and properties of the determinant we can deduce that.

Proposition 2. *For $n,m \geq 1$, we have*

$$\det(Q\ell_n)\det(Q\ell_m)+\det(Q\ell_{n+1})\det(Q\ell_{m+1}) = 5\cdot(-2)^{n+m+2}(3+4i).$$

Proof. From definition of determinant, we have

$$\begin{aligned}\det(Q\ell_n)\det(Q\ell_m)+\det(Q\ell_{n+1})\det(Q\ell_{m+1}) &= (-2)^{n+1}(2+i)\cdot(-2)^{m+1}(2+i)+(-2)^{n+2}(2+i)\cdot(-2)^{m+2}(2+i)\\ &= (-2)^{n+m+2}(2+i)^2+(-2)^{n+m+4}(2+i)^2\\ &= 5\cdot(-2)^{n+m+2}(2+i)^2\\ &= 5\cdot(-1)^{n+m}2^{n+m+2}(3+4i).\end{aligned}$$

Then, the result is obtained. □

Using $n=m$ in Proposition 2, we get

Corollary 2.6. *For $n \geq 1$, we have*

$$\det\left((Q\ell_n)^2\right)+\det\left((Q\ell_{n+1})^2\right) = 5\cdot 2^{2(n+1)}(3+4i).$$

Proposition 3. *For $n,m \geq 0$, we have*

$$4(2+i)\det(Q\ell_{n+m+1}) = \det(Q\ell_{n+1})\det(Q\ell_{m+1}).$$

Proof. From definition of determinant, we have

$$\begin{aligned}\det(Q\ell_{n+1})\det(Q\ell_{m+1}) &= \left((-2)^{n+2}(2+i)\right)\left((-2)^{m+2}(2+i)\right)\\ &= (-2)^{n+m+4}(2+i)^2\\ &= (-2)^2(2+i)\cdot(-2)^{n+m+2}(2+i)\\ &= 4(2+i)\det(Q\ell_{n+m+1}).\end{aligned}$$

□

Proposition 4. *For $n \geq 0$, we have*

$$\det\left((Q\ell_{n+2})^n\right)-\det\left((Q\ell_{n+1})^n\right) = (-2)^{n^2+2n}(2+i)^n((-2)^n+1).$$

Proof.

$$\begin{aligned}\det\left((Q\ell_{n+2})^n\right)-\det\left((Q\ell_{n+1})^n\right) &= \left((-2)^{n+3}(2+i)\right)^n+\left((-2)^{n+2}(2+i)\right)^n\\ &= (-2)^{n^2+3n}(2+i)^n+(-2)^{n^2+2n}(2+i)^n\\ &= (-2)^{n^2+2n}(2+i)^n((-2)^n+1).\end{aligned}$$

Then, the result follows. □

2.2. Gauss modified Lichtenberg numbers

In this subsection, we introduce Gauss modified Lichtenberg numbers $\{g\ell_n^*\}_{n\geq 0}$ and present some of their basic properties.

Definition 2.7. *Gauss modified Lichtenberg numbers $g\ell_n^*$ are defined by*

$$g\ell_{n+3}^* = 2g\ell_{n+2}^*+g\ell_{n+1}^*-2g\ell_n^*,\ \ n\geq 0, \tag{2.5}$$

with $g\ell_0^* = 3+\frac{i}{2}$, $g\ell_1^* = 2+3i$ and $g\ell_2^* = 6+2i$.

The first few terms are as follows:

$$\begin{aligned} g\ell_3^* &= 2g\ell_2^*+g\ell_1^*-2g\ell_0^*\\ &= 2(6+2i)+(2+3i)-2\left(3+\frac{i}{2}\right) = 8+6i,\\ g\ell_4^* &= 2g\ell_3^*+g\ell_2^*-2g\ell_1^*\\ &= 2(8+6i)+(6+2i)-2(2+3i) = 18+8i,\\ g\ell_5^* &= 2g\ell_4^*+g\ell_3^*-2g\ell_2^*\\ &= 2(18+8i)+(8+6i)-2(6+2i) = 32+18i.\end{aligned}$$

Theorem 2.8. *For $n \geq 1$, we have*

$$g\ell_{n+1}^* = \ell_{n+1}^*+i\ell_n^*,$$

where ℓ_n^* is the n-th modified Lichtenberg number defined by $\ell_n^* = 2^n+\varepsilon_n$, with $\varepsilon_n = (-1)^n+1$.

Proof. We can prove the theorem by the induction method on n. For $n=3$, we have

$$g\ell_3^* = 2g\ell_2^* + g\ell_1^* - 2g\ell_0^* = 8+6i = \ell_3^* + i\ell_2^*.$$

Now, assume that the theorem holds for $n \le k$, that is $g\ell_{k+1}^* = \ell_{k+1}^* + i\ell_k^*$. Then, for $n=k+1$, we have

$$\begin{aligned} g\ell_{k+2}^* &= 2g\ell_{k+1}^* + g\ell_k^* - 2g\ell_{k-1}^* \\ &= 2\left(\ell_{k+1}^* + i\ell_k^*\right) + \left(\ell_k^* + i\ell_{k-1}^*\right) - 2\left(\ell_{k-1}^* + i\ell_{k-2}^*\right) \\ &= 2\ell_{k+1}^* + \ell_k^* - 2\ell_{k-1}^* + i\left(2\ell_k^* + \ell_{k-1}^* - 2\ell_{k-2}^*\right) \\ &= \ell_{k+2}^* + i\ell_{k+1}^*. \end{aligned}$$

Thus, the result follows. □

The Binet formula for the Gauss modified Lichtenberg numbers is given by

$$g\ell_n^* = 2^{n-1}(2+i) + (-1)^{n-1}(-1+i) + (1+i),$$

or equivalently

$$g\ell_n^* = 2^{n-1}(2+i) + 2A_n, \;\; g\ell_0^* = 3 + \frac{i}{2},$$

with A_n as in Eq. (2.4).

Theorem 2.9. *For $n \ge 2$, we have*

$$\frac{1}{2}\cdot g\ell_{n+1}^* = g\ell_{n+1} + g\ell_n - 3g\ell_{n-1},$$

Proof. Using the relation $g\ell_n = \frac{1}{3}\left[M_{n+1}^g - A_n\right]$ for the n-th Gauss Lichtenberg number, the identities $A_{n+1} = -A_n + (1+i)$ and $M_{n+1}^g = 2M_n^g + (1+i)$. Then, we have

$$\begin{aligned} g\ell_{n+1} + g\ell_n - 3g\ell_{n-1} &= \frac{1}{3}\left[M_{n+2}^g - A_{n+1} + M_{n+1}^g - A_n - 3M_n^g + 3A_{n-1}\right] \\ &= \frac{1}{3}\left[4M_{n+1}^g - 5M_n^g + 2A_{n+1} - A_n\right] \\ &= \frac{1}{3}\left[3M_n^g - 3A_n + 6(1+i)\right] \\ &= M_n^g - A_n + 2(1+i). \end{aligned}$$

Using $A_n = \frac{1}{2}\left[(-1)^{n+1}(-1+i) + (1+i)\right]$ and $M_n^g = 2^{n-1}(2+i) - (1+i)$, we obtain

$$\begin{aligned} M_n^g - A_n + 2(1+i) &= \frac{1}{2}\left[2^n(2+i) - (-1)^{n+1}(-1+i) - (1+i) + 2(1+i)\right] \\ &= \frac{1}{2}\left[2^n(2+i) + (-1)^n(-1+i) + (1+i)\right] \\ &= \frac{1}{2}\cdot g\ell_{n+1}^*. \end{aligned}$$

Thus, the result follows. □

Theorem 2.10. *For $n \ge 2$, we have*

$$9\cdot g\ell_n = 10g\ell_{n+1}^* - \frac{3}{2}g\ell_n^* - 13g\ell_{n-1}^*.$$

Proof. Using the relation $g\ell_n^* = 2^{n-1}(2+i) + 2A_n$, for the n-th Gauss modified Lichtenberg number. Then, we obtain

$$\begin{aligned} 10g\ell_{n+1}^* - \frac{3}{2}g\ell_n^* - 13g\ell_{n-1}^* &= 10\left[2^n(2+i) + 2A_{n+1}\right] - \frac{3}{2}\left[2^{n-1}(2+i) + 2A_n\right] - 13\left[2^{n-2}(2+i) + 2A_{n-1}\right] \\ &= 10\cdot 2^n(2+i) + 20A_{n+1} - 3\cdot 2^{n-2}(2+i) - 3A_n - 13\cdot 2^{n-2}(2+i) - 26A_{n+1} \\ &= 6\cdot 2^n(2+i) + 3A_n - 6(1+i) \\ &= 3\left[2^{n+1}(2+i) + A_n - 2(1+i)\right] \\ &= 9\cdot g\ell_n. \end{aligned}$$

Then, the result follows. □

Theorem 2.11. *For $n \ge 0$ and $m \ge 2$, we have*

$$g\ell_{n+m+1} = \ell_{n+1}g\ell_{m+1} + \left(\ell_{n+2} - 2\ell_{n+1}\right)g\ell_m - 2\ell_n g\ell_{m-1}.$$

Proof. Using Theorem 2.2 and formula $\ell_{n+m+1}=\ell_{n+1}\ell_{m+1}+(\ell_{n+2}-2\ell_{n+1})\ell_m-2\ell_n\ell_{m-1}$, we have

$$\begin{aligned}
&\ell_{n+1}g\ell_{m+1}+(\ell_{n+2}-2\ell_{n+1})g\ell_m-2\ell_n g\ell_{m-1}\\
&\qquad=\ell_{n+1}(\ell_{m+1}+i\ell_m)+(\ell_{n+2}-2\ell_{n+1})(\ell_m+i\ell_{m-1})-2\ell_n(\ell_{m-1}+i\ell_{m-2})\\
&\qquad=\ell_{n+1}\ell_{m+1}+(\ell_{n+2}-2\ell_{n+1})\ell_m-2\ell_n\ell_{m-1}+i[\ell_{n+1}\ell_m+(\ell_{n+2}-2\ell_{n+1})\ell_{m-1}-2\ell_n\ell_{m-2}]\\
&\qquad=\ell_{n+m+1}+i\ell_{n+m}\\
&\qquad=g\ell_{n+m+1}.
\end{aligned}$$

Then, the result is obtained. □

Definition 2.12. *The Gauss modified Lichtenberg number matrix $Q\ell_n^*$ is defined by*

$$Q\ell_n^*=\begin{bmatrix} g\ell_{n+2}^* & g\ell_{n+3}^*-2g\ell_{n+2}^* & -2g\ell_{n+1}^* \\ g\ell_{n+1}^* & g\ell_{n+2}^*-2g\ell_{n+1}^* & -2g\ell_n^* \\ g\ell_n^* & g\ell_{n+1}^*-2g\ell_n^* & -2g\ell_{n-1}^* \end{bmatrix},\ n\geq 1,$$

where $Q\ell_0^*=\begin{bmatrix} 6+2i & -4+2i & -4-6i \\ 2+3i & 2-4i & -6-i \\ 3+\frac{i}{2} & -4+2i & -1-5i \end{bmatrix}$ *for convenience.*

Note that Gaussian modified Lichtenberg number $Q\ell_n^*$ satisfies the relation $Q\ell_{n+3}^*=2Q\ell_{n+2}^*+Q\ell_{n+1}^*-2Q\ell_n^*$.

Proposition 5. *For $n\geq 0$, we have*

$$\det\left(Q\ell_{n+1}^*\right)=36\cdot(-1)^n2^{n+1}(2+i).$$

Proof. The usual modified Lichtenberg number ℓ_n^* satisfies the next relation $\ell_{n+3}^*=2\ell_{n+2}^*+\ell_{n+1}^*-2\ell_n^*$, Then, we have

$$\begin{bmatrix} \ell_{n+2}^* & \ell_{n+3}^*-2\ell_{n+2}^* & -2\ell_{n+1}^* \\ \ell_{n+1}^* & \ell_{n+2}^*-2\ell_{n+1}^* & -2\ell_n^* \\ \ell_n^* & \ell_{n+1}^*-2\ell_n^* & -2\ell_{n-1}^* \end{bmatrix}=\begin{bmatrix} 6 & -4 & -4 \\ 2 & 2 & -6 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n.$$

Then, we have

$$\begin{aligned}
Q\ell_{n+1}^* &= \begin{bmatrix} \ell_{n+3}^* & \ell_{n+4}^*-2\ell_{n+3}^* & -2\ell_{n+2}^* \\ \ell_{n+2}^* & \ell_{n+3}^*-2\ell_{n+2}^* & -2\ell_{n+1}^* \\ \ell_{n+1}^* & \ell_{n+2}^*-2\ell_{n+1}^* & -2\ell_n^* \end{bmatrix}+i\begin{bmatrix} \ell_{n+2}^* & \ell_{n+3}^*-2\ell_{n+2}^* & -2\ell_{n+1}^* \\ \ell_{n+1}^* & \ell_{n+2}^*-2\ell_{n+1}^* & -2\ell_n^* \\ \ell_n^* & \ell_{n+1}^*-2\ell_n^* & -2\ell_{n-1}^* \end{bmatrix}\\
&= \begin{bmatrix} 6 & -4 & -4 \\ 2 & 2 & -6 \\ 3 & -4 & -1 \end{bmatrix}\left(\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n+1}+i\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n\right)\\
&= \begin{bmatrix} 6 & -4 & -4 \\ 2 & 2 & -6 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 2+i & 1 & -2 \\ 1 & i & 0 \\ 0 & 1 & i \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n.
\end{aligned}$$

So we use multiplicative property of the determinant and we get

$$\begin{aligned}
\det\left(Q\ell_{n+1}^*\right) &= \begin{vmatrix} 6 & -4 & -4 \\ 2 & 2 & -6 \\ 3 & -4 & -1 \end{vmatrix}\begin{vmatrix} 2+i & 1 & -2 \\ 1 & i & 0 \\ 0 & 1 & i \end{vmatrix}\begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}^n\\
&= -36\cdot(-2)^n(-4-2i)\\
&= 36\cdot(-1)^n2^{n+1}(2+i).
\end{aligned}$$

Thus, the result follows. □

Proposition 6. *For $n\geq 0$, we have*

$$\det\left(Q\ell_{n+1}^*\right)=18\cdot\det\left(Q\ell_{n+1}\right).$$

Proof. The proof is easily deduced from Propositions 1 and 5. □

3. Conclusion

We defined new numbers by using definitions of Gauss sequence, the Lichtenberg number and modified Lichtenberg number are studied. The properties of those numbers were examined. Some theorems about these numbers were presented, their matrix representations are established and identities with their determinants are demonstrated. In the future, we extend these results to generalized Lichtenberg numbers, including arbitrary initial conditions. In particular, these types of numbers can be extended by applying some previous work such as binomial transform ([14, 22]), finite operators ([13]), Hessenberg determinants ([15]), third-order quaternions ([20, 21]), hybrid numbers ([1]), bicomplex numbers ([16]), and many others.

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