Journal of New Theory
ISSN: 2149-1402

49 (2024) 1-6
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

Graphs with Total Domination Number Double of the Matching Number

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Article Info
Received: 22 July 2024
Accepted: 23 Oct 2024
Published: 31 Dec 2024
doi:10.53570/jnt.1520557
Research Article

Abstract — A subset S of vertices of a graph G with no isolated vertex is called a total dominating set of G if each vertex of G has at least one neighbor in the set S. The total domination number $\gamma_t(G)$ of a graph G is the minimum value of the size of a total dominating set of G. A subset M of the edges of a graph G is called a matching if no two edges of M have a common vertex. The matching number $\nu(G)$ of a graph G is the maximum value of the size of a matching in G. It can be observed that $\gamma_t(G) \leq 2\nu(G)$ holds for every graph G with no isolated vertex. This paper studies the graphs satisfying the equality and proves that $\gamma_t(G) = 2\nu(G)$ if and only if every connected component of G is either a triangle or a star.

Keywords *Domination number, matching number, total domination number*

Mathematics Subject Classification (2020) 05C69, 05C70

1. Introduction

Graphs have various parameters, such as domination number, total domination number, matching number, and the minimum size of a maximal matching, denoted by γ, γ_t, ν, and ν^*, respectively. Obtaining equalities or inequalities between those parameters and classifying the graphs satisfying a given equality or inequality are widely studied. For instance, a well-known inequality is $\gamma_t(G) \leq 2\gamma(G)$. Characterization of all the graphs G with $\gamma_t(G) = 2\gamma(G)$ is still an open problem. However, the problem is solved for trees, block graphs, and chordal graphs in [1–3]. Another example of total domination numbers is that in any connected graph with at least three vertices, the total domination number is two-thirds of the graph's order [4]. The family of graphs G satisfying $\gamma_t(G) = \frac{2|V(G)|}{3}$ is completely determined in [5].

It is well known that the inequality $\gamma(G) \leq \nu(G)$ holds for every graph G. However, the inequality $\gamma_t(G) \leq \nu(G)$ is not always true. On the other hand, $\gamma_t(G) \leq \nu(G)$ is satisfied whenever G is a d-regular graph such that $d \geq 3$ or a claw-free graph with minimum degree more than two [6]. Furthermore, the inequality is also satisfied for the connected graphs with at least four vertices in which every vertex is contained in a triangle [7]. Claw-free graphs G with $\gamma_t(G) = \nu(G)$ and $\delta(G) \geq 3$ are determined in [8], whereas trees T satisfying $\gamma_t(T) \leq \nu(T)$ are characterized in [9].

Unlike the inequality $\gamma_t(G) \leq \nu(G)$, the inequality $\gamma_t(G) \leq 2\nu(G)$ is true for every graph G which does not contain any isolated vertex. Besides, $\gamma_t(G) \leq 2\nu^*(G)$ is always valid since the set of vertices

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in a maximal matching is a total dominating set. In [10], it is shown that if $\delta(G) \geq 3$, then $\gamma_t(G) \leq 2\nu^*(G) - \delta(G) + 2$ and if $\delta(G) \leq 2$, then $\gamma_t(G) \leq 2\nu^*(G)$. In the same paper, a characterization in a constructive way for the graphs G with $\gamma_t(G) = 2\nu^*(G)$ and $\delta(G) \leq 2$ is also provided.

In this paper, we focus on graphs G with $\gamma_t(G) = 2\nu(G)$. Recall that the inequality $\gamma_t(G) \leq 2\nu^*(G)$ is true when G does not include any isolated vertex. Then, since $\nu^*(G) \leq \nu(G)$ always holds, if $\gamma_t(G) = 2\nu(G)$, then $\nu^*(G) = \nu(G)$ which implies that every maximal matching in G has the same size. A graph whose maximal matchings have the same cardinality is called equimatchable. Therefore, the set of graphs we focus on is a subfamily of equimatchable graphs. For more about equimatchable graphs, see [11–15]. Furthermore, we show that if in a graph, the total domination number is equal to double the matching number, then it is a disjoint union of triangles or stars, that is, every connected component of a graph G satisfying $\gamma_t(G) = 2\nu(G)$ is either a triangle or a star.

This paper is organized as follows: Section 2 presents some definitions and notations to be needed for the following sections. Section 3 provides the main theorem and its proof. The final section presents a discussion and conclusions.

2. Preliminaries

In this section, we present some basic definitions, notations, and some simple observations which are frequently used throughout this paper.

A graph G is formed by two sets, namely, $V(G)$ and $E(G)$. Here, $V(G)$ is a nonempty set whose elements are called vertices, and $E(G)$ is a set consisting of unordered pairs of vertices whose elements are called edges. Whenever $\{u, v\} \in E(G)$, we say that u and v are adjacent (or neighbors). Throughout this paper, if u and v are adjacent in G, then we write $uv \in E(G)$ and say uv is an edge in G.

In a graph, the set of all the neighbors of a vertex v is denoted by $N(v)$, and the number of elements in $N(v)$ is called the degree of the vertex v. In a graph G, the minimum degree is denoted by $\delta(G)$. A vertex in a graph is isolated if it has no neighbors in the graph, i.e., its degree is zero. A vertex is called a leaf if its degree is one, i.e., it has a unique neighbor in the graph, and a vertex is said to be a support vertex whenever it is adjacent to a leaf.

A triangle, denoted by C_3, is a cycle of length three. A star is a graph in which a central vertex exists such that every other vertex is adjacent to only this central vertex. Figure 1 illustrates a triangle and two stars:

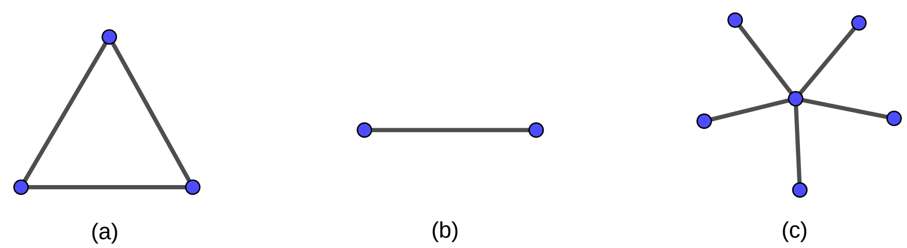

Figure 1. (a) A triangle, (b) a star with two vertices which is called K_2, and (c) a star with six vertices

A subset S of $V(G)$ is a dominating set of G if each vertex not in S has at least one neighbor belonging to S. The domination number $\gamma(G)$ of the graph G is the minimum size of a dominating set of G. If G has no isolated vertices, then a subset S of $V(G)$ is called a total dominating set of G whenever each vertex in G has at least one neighbor in S. In other words, S is a total dominating set if and only if S is a dominating set and the subgraph of G induced by S contains no isolated vertices. The total domination number of the graph G with no isolated vertices, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Notice that by definition, $\gamma(G) \leq \gamma_t(G)$. Note also that there is no total dominating set for a graph G with an isolated vertex; hence, the total domination number is undefined. Therefore, throughout this paper, we only consider graphs without isolated vertices.

If in a subset M of $E(G)$ no two edges share a common vertex, then M is a matching in G. For a matching M, the set of all the vertices serving as a vertex of an edge in M is denoted by $V(M)$. A matching is called maximal whenever it is not properly contained in another matching. The matching number of the graph G is the maximum size of a matching in G and is denoted by $\nu(G)$, $\alpha'(G)$, or $\mu(G)$. Let $\nu^*(G)$ denote the minimum cardinality of a maximal matching in G. A matching in G is maximum if its size is $\nu(G)$. Note that a maximum matching is maximal, but a maximal matching is not necessarily maximum. An example of maximal and maximum matchings is presented in Figure 2. Moreover, $\nu^*(G) \leq \nu(G)$ is always satisfied.

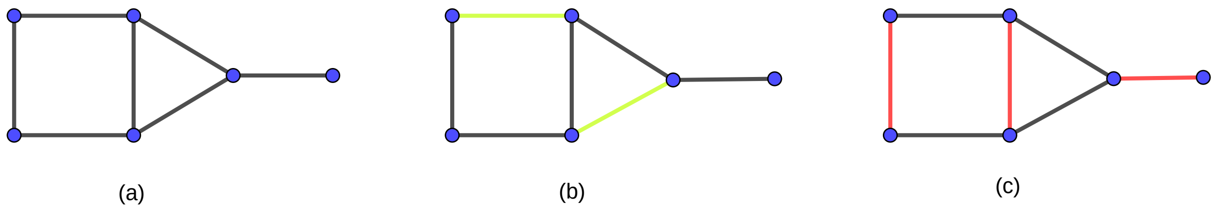

Figure 2. (a) A graph, (b) its maximal matching in yellow, and (c) its maximum matching in red

A path between vertices u and v of a graph G is a sequence of edges $v_1v_2, v_2v_3, \cdots, v_{k-1}v_k$ in G for some $k \geq 2$ where $v_1 = u$ and $v_k = v$. A graph is called connected if, for every pair of vertices, there exists a path between them. A connected component of a graph is a connected subgraph that is not contained in another connected subgraph. A subset of vertices in a graph is called independent if it has no two adjacent vertices.

Finally, we provide a simple observation frequently used in proofs: Let M be a maximal matching in a graph G. Then, since M is maximal, there is no edge in the subgraph of G induced by $V(G) \setminus V(M)$, that is, $V(G) \setminus V(M)$ is either empty or an independent set. In other words, $N(w) \subseteq V(M)$ for every $w \in V(G) \setminus V(M)$. Moreover, let $G_1, G_2, \cdots, G_n$ be all connected components of a graph G. Then,

$$\gamma_t(G) = \sum_{i=1}^{n} \gamma_t(G_i) \quad \text{and} \quad \nu(G) = \sum_{i=1}^{n} \nu(G_i)$$

As $\gamma_t(G_i) \leq 2\nu(G_i)$ is true for every $i \in \{1, 2, \cdots, n\}$, we see that $\gamma_t(G) = 2\nu(G)$ holds if and only if $\gamma_t(G_i) = 2\nu(G_i)$ is valid for every $i \in \{1, 2, \cdots, n\}$. Therefore, characterizing all the connected graphs G with $\gamma_t(G) = 2\nu(G)$ is sufficient to solve our main problem.

3. Main Result

In this section, we determine all the graphs G with $\gamma_t(G) = 2\nu(G)$. Characterizations of such graphs are presented in the following theorem.

Theorem 3.1. Let G be a graph. Then, $\gamma_t(G) = 2\nu(G)$ holds if and only if every connected component of G is a triangle or a star.

Throughout this section, we provide the proof of Theorem 3.1. We first present a lemma, which is frequently used in the rest of this section.

Lemma 3.2. Let G be a graph with $\gamma_t(G) = 2\nu(G)$, M be a maximum matching in G, and $ab \in M$. If a is not a support vertex, then a is the unique neighbor of b among the vertices in $V(M)$.

PROOF. We prove the claim by contradiction. Let $S = V(M) \setminus \{a\}$ and assume that b is adjacent to a vertex c in S. An illustration of G, M, and S is given in Figure 3:

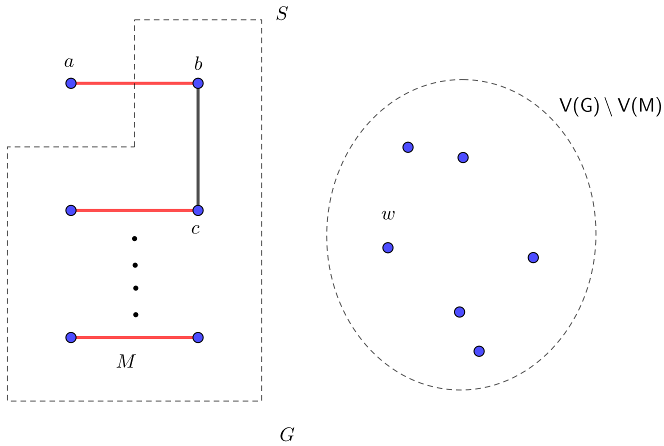

Figure 3. A graph G and a matching M, shown by red edges. The sets S and $V(G) \setminus V(M)$ consist of vertices inside the dashed polygonal region and elliptic region, respectively

Recall that for any vertex w in $V(G) \setminus V(M)$, $N(w) \subseteq V(M)$. Since a is not a support vertex, any vertex adjacent to a is not a leaf and has a neighbor other than a. Therefore, w has at least one neighbor in S. Further, it can be observed that any vertex in $V(M)$ has at least one neighbor in S. Therefore, S is a total dominating set and

$$\gamma_t(G) \leq |S| = |V(M)| - 1 = 2\nu(G) - 1$$

which contradicts with $\gamma_t(G) = 2\nu(G)$. □

We study the graphs concerning their minimum degrees. We begin with the case when the minimum degree is more than one.

Proposition 3.3. C_3 is the unique connected graph G satisfying $\gamma_t(G) = 2\nu(G)$ and $\delta(G) \geq 2$.

PROOF. Let G be a connected graph satisfying the conditions $\gamma_t(G) = 2\nu(G)$ and $\delta(G) \geq 2$. We first show that $\nu(G) = 1$. Assume that $\nu(G) \geq 2$. Let $M = \{e_1, e_2, \cdots, e_k\}$ be a maximum matching where $k = \nu(G) \geq 2$. Since the minimum degree in G is at least 2, there is no leaf in G. Therefore, there is no support vertex in G either. Thus, by Lemma 3.2, any vertex in $V(M)$ has exactly one neighbor in $V(M)$. In other words, the subgraph of G induced by $V(M)$ is a disjoint union of k edges. Since G is connected, a vertex w in $V(G) \setminus V(M)$ must exist such that w has neighbors from different edges in M. Without loss of generality, suppose that $e_1 = xy$, $e_2 = zt$, and w is adjacent to y and t. Consider the edge set $M' = (M \setminus \{xy\}) \cup \{yw\}$. Then, M' matches, and because of its size, it is a maximum matching. However, as w is adjacent to t, we get a contradiction when we apply Lemma 3.2 for M', $a = y$, and $b = w$. Consequently, we see that the matching number of G is 1.

Let uv be any edge of G. Then, $\{uv\}$ is a maximum matching. Hence, since the minimum degree is two, any vertex different than u and v is a common neighbor of u and v. Thereby, G must have at least three vertices. If G has three vertices, then G has to be C_3 and thus $\gamma_t(C_3) = 2\nu(C_3) = 2$. Otherwise, let w_1 and w_2 be two distinct vertices other than u and v. Then, $\{uw_1, vw_2\}$ is a matching which yields $\nu(G) \geq 2$, a contradiction. Thereby, C_3 is the unique (up to isomorphism) connected graph G with $\gamma_t(G) = 2\nu(G)$ and $\delta(G) \geq 2$. □

We next analyze the graphs with a minimum degree of one.

Proposition 3.4. Let G be a connected graph with $\delta(G) = 1$. Then, $\gamma_t(G) = 2\nu(G)$ if and only if G is a star.

PROOF. Let G be a connected graph satisfying $\gamma_t(G) = 2\nu(G)$ and $\delta(G) = 1$. First, note that if $G = K_2$, then it is a star. Suppose that G is not K_2. Observe that K_2 is the unique connected graph containing a vertex, a leaf, and a support vertex. Thus, G has no such a vertex. Let M be a maximum matching in G and $uv \in M$. We first show that at least one of u and v is not a support vertex. Assume that u and v are support vertices. Then, u is adjacent to a leaf x, and v is adjacent to a leaf y. By the observation above, $\{x, y\} \cap \{u, v\} = \emptyset$. Moreover, since they are leaves, $x \neq y$ and none of x and y can be another vertex in $V(M)$. Then, note that $(M \setminus \{uv\}) \cup \{ux, vy\}$ is matching whose size is greater than the size of M contradicting with the fact that M is a maximum matching.

Let $v_1, v_2, \cdots, v_m$ be all the support vertices in G. For each v_i, choose a neighbor leaf u_i. Thus, $\{u_1v_1, \cdots, u_mv_m\}$ is a matching and can be extended to a maximal matching M. As $\nu^*(G) = \nu(G)$, then M is a maximum matching. Since u_i cannot be a support vertex, by Lemma 3.2, $N(v_i) \cap V(M) = \{u_i\}$ holds for every $i \in \{1, 2, \cdots, m\}$. Suppose that $M \setminus \{u_1v_1, \cdots, u_mv_m\}$ is not empty and equal to $\{x_1y_1, \cdots, x_ry_r\}$. By construction none of x_1, y_1, $\cdots$, x_r, and y_r is a support vertex and hence, by Lemma 3.2, $N(x_i) \cap V(M) = \{y_i\}$ and $N(y_i) \cap V(M) = \{x_i\}$, for every $i \in \{1, 2 \cdots, r\}$. Therefore, since G is connected and $V(G) \setminus V(M)$ is an independent set, there exists a vertex $w \in V(G) \setminus V(M)$ such that w is a common neighbor of a vertex from $\{u_1, v_1, \cdots, u_m, v_m\}$ and a vertex from $\{x_1, y_1, \cdots, x_r, y_r\}$. Note that w cannot be adjacent to some u_i since u_i is a leaf. Without loss of generality, suppose that w is adjacent to v_1 and y_1. Then, $M' = (M \setminus \{u_1v_1\}) \cup \{wv_1\}$ is a maximum matching because of its size. Applying Lemma 3.2 for M', $a = x_1$, and $b = y_1$ yields a contradiction. Consequently, $M = \{u_1v_1, \cdots, u_mv_m\}$ and $\{v_1, \cdots, v_m\}$ is an independent set.

We finally show that $m = 1$. Assume that $m \geq 2$. By similar ideas above, there exists a vertex $w \in V(G) \setminus V(M)$, which is adjacent to at least two of $v_1, \cdots, v_m$. Without loss of generality, suppose that v_1 and v_2 are neighbors of w. Then, $M' = (M \setminus \{u_1v_1\}) \cup \{wv_1\}$ is a maximum matching since $|M'| = |M| = \nu(G)$. Therefore, as v_2 and w are adjacent, we obtain a contradiction by applying Lemma 3.2 for M', $a = u_2$, and $b = v_2$. Thus, $m = 1$ and every vertex other than v_1 is a leaf and adjacent to v_1, which implies that G is a star.

Conversely, if G is a star graph, it is connected, has minimum degree one, and satisfies $\gamma_t(G) = 2\nu(G) = 2$. $\square$

Finally, combining Propositions 3.3 and 3.4 proves Theorem 3.1.

4. Conclusion

In this paper, we have studied the graphs G whose total domination number attains the upper bound in the inequality $\gamma_t(G) \leq 2\nu(G)$. We have shown that the family of graphs whose each connected component is a triangle or a star is the set of all the graphs G satisfying $\gamma_t(G) = 2\nu(G)$. Since $\gamma_t(G) = 2\nu(G)$ implies $\nu^*(G) = \nu(G) = \frac{\gamma_t(G)}{2}$, we have obtained an extreme condition on the graphs we study, and hence, probably that is why we have not reached an interesting or large connected graph that satisfies the equality. A potential research direction is to determine all the graphs G satisfying $2\nu(G) - 1 = \gamma_t(G)$ or $2\nu(G) - 2 = \gamma_t(G)$. Notice that the method to solve the main theorem does not work. However, if $2\nu(G) - 1 = \gamma_t(G)$, then $\nu^*(G) = \nu(G)$ since $\gamma_t(G) \leq 2\nu^*(G) \leq 2\nu(G)$ and the values $\nu^*(G)$ and $\nu(G)$ are integers. Therefore, the class of graphs G with $\gamma_t(G) = 2\nu(G) - 1$ is a subfamily of equimatchable graphs as well, and hence, results on equimatchable graphs can be helpful to determine all graphs in that class. Another research direction might be to obtain an inequality involving matching and total domination numbers on various specific graph classes, such as regular, bipartite, split, and chordal graphs.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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