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Analysis of Misconceptions and Errors Regarding Exponential and Radical Expressions Through the Theory of Reducing Abstraction

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Abstract

Investigating the reasons for misconceptions and errors is essential and important for making improvements in mathematics teaching. The idea of *reducing abstraction*, utilized as the framework for this research, can essentially be based on students' tendency to work with a lower abstraction level than the concepts they encounter in the course or what experts (mathematicians, teachers, etc.) expect from them. Since the process of reducing abstraction often occurs unconsciously, it can lead to misconceptions and errors. Exponential and radical expressions, which students first encounter in secondary school, are significant topics in mathematics, offering ease of representation and various calculations in many fields of basic sciences and engineering. Research on exponential and radical expressions, perceived by secondary and high school students as a collection of unnecessary formulas unrelated to daily life and difficult to understand, has revealed various misconceptions and errors. In this study, through the theory of reducing abstraction, the possible reasons for the misconceptions and errors revealed by research will be interpreted from an alternative viewpoint. Thus, a new perspective on student approaches regarding these subjects will be provided to teachers and mathematics educators.

Key Words

Exponential expressions • Misconception • Radical expressions • Reducing abstraction

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Introduction

Why do students have difficulty in learning mathematics? The nature of mathematics itself holds the key to this question. Mathematics is a discipline known for its abstract structure, and this abstract structure is crucial for the evolution of mathematical concepts and their interrelationships (Sfard, 1991). Mathematical concepts are generally defined in an abstract framework, and the links among these concepts form the basis of mathematical thinking. For example, basic mathematical structures such as the number sets, algebraic operations, and functions play a critical role in building more complex structures and theories (Tall, 2004). This unique abstract structure of mathematics is the biggest reason for the difficulties experienced in learning it. Ferrari (2003) expresses this fact as follows: "Although abstraction is known as the most relevant feature of mathematics from a cognitive point of view, it is one of the main reasons for failure in mathematics learning.".

Mathematics is also a discipline in which there are significant connections between distinct concepts, similar to those between elements in the load-bearing structure of a building. The delicate nature of these connections as well as their strength are among the subtleties of mathematics. When you remove any connection in a mathematical structure, the entire structure breaks down, like a building whose load-bearing elements are damaged. Therefore, the critical role of relationships between mathematical concepts in students' development of abstract mathematical thinking, and in the learning process should be considered. Otherwise, as Dreyfus (1991) stated, students learn to apply many standardized methods in mathematics classes in order to answer limited questions that are just exercises. Research also shows that learning mathematical concepts in relation to each other helps students better understand and apply these concepts (Hiebert & Carpenter, 1992).

The abstract nature of mathematical concepts and their complex relationships with each other cause students to have difficulty in learning mathematics, misunderstand the concepts and misconstruct them in their minds. "Misconceptions" and "errors" that emerge as a result of the challenges students face when learning mathematics should also be regarded in this manner (Bingölbali & Özmantar, 2012). Research shows that mathematical misconceptions and errors have significant effects on students' long-term mathematics learning (Nesher, 1987). Therefore, it is vital that educators identify and correct these misconceptions and errors.

Pointing out the role of the term *conception* in making sense of *misconception*, Smith, diSessa and Roschelle (1993) defined misconception as "student understanding that systematically produces errors". This definition also implies that the terms *misconception* and *error* are connected to one another. At this point, the difference between *calculation error*, *misconception* and *error* must be expressed in order to avoid conceptual confusion. *Calculation errors* are errors that can be made carelessly and rarely by both expert and inexperienced people during the solution process of an exercise or problem, and result in wrong answers to the calculations. These types of errors are easily detected and corrected immediately. From this explanation, it is understood that the systematic errors made by students are different from an ordinary calculation error, indicating the presence of a deep understanding, an understanding system (Nesher, 1987), a cognitive structure (Oliver, 1989), or a misconception that reveals and controls itself. In other words, errors done by students are the surface image, and there is a misconception that controls and causes the formation of this image (Nesher, 1987). A *misconception* is not an error or an incorrect answer owing to lack of knowledge. *Misconception* means an understanding that fits a concept in the mind but is fundamentally distinct from its formal definition. If a student can explain that the mistake he made on a subject is "correct" with reasons based on his own belief and says that he is confident about it, we can conclude that he has a misconception. Given these clarifications, it can be stated that misconceptions may result in errors, but not all errors are misconceptions (Yenilmez & Yaşa, 2008).

Exponential and radical expressions, which students first encounter in middle school, are important mathematical concepts that are closely interrelated, provide ease of representation in many fields of basic sciences and engineering, and are frequently used in calculations. On the contrary, exponential and radical expressions are perceived as a collection of unnecessary formulas that are difficult to understand, have nothing to do with daily life, especially by secondary and high school students. Regarding this situation, Elstak (2007) and Erlandson (2013) state that algebraic operations involving rational exponents and radical expressions are inherently abstract and cause confusion among students. Exponential and radical expressions are among the topics where students have various misconceptions and often make mistakes while working on them. Misconceptions and mistakes regarding these subjects negatively affect students' understanding of advanced mathematics subjects and their problem-solving skills (Stacey, 2006). Research shows that the common misconceptions and mistakes students make regarding exponential and radical expressions generally arise from basic conceptual misunderstandings, failure to establish connections between concepts, and symbolic representation errors (Avcu, 2010; Aydın & Özdemir, 2010; D. Ancheta & Subia, 2020; Denbel, 2019; Elstak, 2007; Erlandson, 2013; Rushton, 2014; Sikora-Press, 2016; Şenay, 2002; Zazkis & Liljedahl, 2004). For example, many students fail to grasp that squaring a number and obtaining its square root are inverses of one another. (Zazkis & Liljedahl, 2004). In addition, important misconceptions are observed about how to handle negative numbers in exponential and radical expressions (Aydın & Özdemir, 2010).

This study seeks to analyze the misconceptions and errors regarding exponential and radical expressions determined by previous research through the *theory of reducing abstraction.* The reason underlying this approach is to offer teachers and mathematics educators a view about student understanding by interpreting the possible causes of misconceptions and errors revealed as a result of research on these subjects from a different perspective.

Theory of Reducing Abstraction

Although *abstraction* is simply known as "the process of transition from concrete to abstract", it is a multifaceted and complex concept, specifically within the context of mathematics education. Consequently, researchers have not reached an agreement on a unique definition of *abstraction*. However, it's thoroughly acknowledged that the concept of *abstraction* can be examined from multiple viewpoints, such as “some concepts are more abstract than others” and “abstraction ability is important for meaningful mathematics learning” (Hazzan, 1999).

Responses to questions such as how a mathematical concept is understood, through what processes it becomes meaningful, or what is effective in the process of structuring mathematical knowledge lie within the concept of mathematical abstraction. Because of the importance of abstraction in mathematics learning, mathematics educators have examined and discussed this issue from many different perspectives and developed various theories about abstraction. One of these is the *theory of reducing abstraction* (*reducing the level of abstraction*) which was first developed and used by Hazzan (1999) to explain undergraduate students' understanding of concepts of abstract algebra and is generally associated with fields related to advanced mathematical thought (such as computer science) and mathematics subjects at the undergraduate level (Hazzan, 2001, 2003a, 2003b; Raychaudhuri, 2014; Şenay & Özdemir, 2014). Hazzan and Zaskis (2005) and Eraslan (2008) showed that this notion can also be utilized to explain student understandings at the school (primary and secondary school) mathematics level. *Reducing the level of abstraction* (shortly, *reducing abstraction*) idea can be based mainly on students' tendency to work with a lower abstraction level than the concepts they encounter in the course or what experts (mathematicians, teachers, etc.) expect from them. The process of reducing the abstraction level is indicative of the student finding ways to cope with the new concepts he has learned. In this way, students make concepts mentally accessible so they can think and handle with them. *Theory of reducing abstraction* consists of three subthemes developed based on different interpretations of abstraction in the literature. Nonetheless, it must be said that these themes of reducing the abstraction level are neither mutually excluding nor completely inclusive. (Hazzan, 1999, 2001). Below, the subthemes of the *theory of reducing abstraction* will be explained with examples.

Reducing abstraction in terms of the quality of the relationships between the object of thought and the thinking person.

This point of view of *reducing abstraction* is based on Wilensky 's (1991) claim that "whether something is abstract, or concrete is not a natural feature of that thing, on the contrary, it arises from the characteristics of the relationship between the person and the object." To put it another way, for every *concept* and *person* we can sight a different level of abstraction that reflects the prior relationship between the two. The closer and more connected somebody is to an *object*, the more tangible (and less abstract) he or she feels that *object*. The basis of this perspective lies in the tendency of some students' mental processes to make an unfamiliar thought more familiar or to make the abstract concrete (Hazzan, 1999). For example, the operation $(1001)_2 + (11)_2$ can be directly calculated in the binary system using the 'carry over' approach we have known since elementary school. However, a student who first converts the given numbers to the decimal system he is used to and then performs addition has reduced the level of abstraction.

Reducing abstraction in terms of reflecting the process-object duality.

This interpretation of the reduction of abstraction rests on the *process-object duality* proposed in some theories related to the mental construction of concepts in mathematics education. Theories based on this duality (such as APOS theory) distinguish between the understanding of mathematical ideas as *processes* and those as *objects*, and despite their differences, they agree that the understanding of a mathematical concept as a *process* (sequential operations) is prior to and *less abstract* than its understanding as an *object*. Therefore, understanding a mathematical concept as if it were a *process* can be conceived of as a lower level of abstraction (i.e., *reducing the level of abstraction)* than its understanding as an *object* (Hazzan, 1999).

This interpretation of reducing the abstraction level is also related to the subtheme we explained previously, because the more a person deals with a concept that he initially perceives as a *process*, that concept begins to become more *familiar* to him and eventually he can perceive it as an *object*. In addition, we can develop this interpretation of reducing abstraction from two different perspectives grounded in the understanding of a concept as if it were a *process*:

(i) Personalizing formal expressions and logical arguments by the student using *first-person language*

(ii) Student's tendency to use *canonical procedures* when solving problems.

Employing first-person language such as '*I do something*' shows how students perceive the processes related to the concepts they are thinking about in a simplistic manner. The tendency of students to use *first-person language* to make the *unfamiliar* language of mathematics more recognizable and understandable to them is an indication of the reducing the abstraction level.

Employing *first-person language* as an indicator of the understanding of a concept as if it were a *process*, that is, the reducing abstraction, can be illustrated with the following example: Regarding commutativity which is a algebraic property of a set and of an operation defined on the set elements; we can say that someone who uses the expression "If the set of real numbers holds the commutativity property according to the operation *, this property also valid for every subset of real numbers.", he comprehends the property of commutativity as an *object.* However, a person who tries to explain the commutativity with his own verbs, that is, using first-person language, such as "If *I get* a different result when *I change* the place of the numbers in the multiplication, it is not commutative.", has reduced the level of abstraction.

Canonical procedures, on the other hand, are procedures that are automatically triggered by more or less asked questions. This situation may occur due to the problem's nature, or it may arise from the student's association with the method applied to solve a similar problem he has encountered before. *Canonical procedures* are preferred by students because they provide the chance to solve the problem without examining the properties of the mathematical concepts in the problem, that is, without considering the concept as an *object* or even without the need to understand the mathematical ideas behind that method, but they can also cause errors (Hazzan, 1999).

Reducing abstraction in terms of the degree of complexity of the concept of thought.

We can explain this interpretation of reducing abstraction with an example as follows: The real number set is a more *complex* mathematical structure than any element in the set, that is, a real number. This does not, however, imply that thinking with complex objects will always be more difficult. Here, the assumption is that "the more complex a mathematical structure is, the more abstract it is"; because, when a structure is analyzed as a whole, more details should be ignored. In this respect, this interpretation of abstraction focuses on how students tend to work with a simpler object that is related or thought to be related to it instead of a *complex mathematical object* they encounter, thus reducing the abstraction level by working with a *less complex object* (Hazzan, 1999).

To handle the new concepts, they have encountered, to make them mentally accessible and to think with them, students tend to work with less (lower level) of abstraction than experts expect from them, that is, they "reduce abstraction" but this situation should not be understood as a cognitive process that inevitably leads to misconceptions or mathematical errors (Hazzan, 1999). However, since the process of reducing abstraction does not occur consciously most of the time, it can also cause misconceptions and errors. In fact, although the major goal of the research carried out within the basis of the *theory of reducing abstraction* was not to examine misconceptions and errors, it was observed in these research that students had various misconceptions and made mistakes in most of their answers. In addition, Şenay and Özdemir (2019) revealed that this theoretical framework is a useful tool for analyzing the reasons for misconceptions and errors with their study on misconceptions and errors in number theory. Consequently, based on these considerations, the theory of reduction of abstraction is adopted as the theoretical framework of this study.

Method

Research Design

In this research, document analysis, one of the qualitative research methodologies, was used. Document analysis is a systematic process for examining and evaluating documents, which can be in printed or digital formats. Similar to other qualitative research approaches, document analysis involves examining and interpreting data to derive meaning and build a comprehension of the associated subject (Corbin and Strauss, 2008).

Research Sample

In document analysis, the researcher can decide on the sample size themselves and adjust it as they wish. They may include any documents they desire in their study and impose various restrictions if necessary (Corbetta, 2003). Accordingly, first of all the keywords to be used in the database searches to select the documents to be analyzed within the scope of the study were determined as “misconception”, “error”, “exponential expressions” and “radical expressions”. Searches were conducted in Google Scholar, YÖK National Thesis Center, Web of Science and Scopus databases. The studies whose full texts were obtained in the searches were re-classified according to the demographic characteristics of the sample or study group, grade, school level, etc., data collection tools and the year of the study. Among the classified studies, five studies were selected by purposive sampling method, taking into account the additional criteria that their content is appropriate to the theoretical framework, the sample or study group is different from each other in terms of demographic characteristics, grade, school level, etc., different data collection tools are used and they belong to different years. Below, information about the study groups/samples and data collection tools of the selected articles and theses are given:

(i) Şenay (2002) conducted his research on 729 students studying in the 9th grade of 9 different high schools in Konya, Turkey. In the study, a diagnostic test comprised of 20 multiple-choice questions was preferred as a data collecting tool to identify student misconceptions and errors related to exponential and radical expressions. Additionally, after analysis of the diagnostic test, semi-structured interviews were carried out with the identified students.

(ii) Avcu (2010) conducted his research with 159 students studying in the 8th grade of two different primary schools in Aydın province of Turkey. In the study, an 'Exponential Numbers Achievement Test' consisting of 18 open-ended questions was preferred as a data collecting tool to identify students' misconceptions and errors related to exponential numbers.

(iii) Sikora-Press (2016) conducted his research with 69 students enrolled in a basic mathematics course that met the graduation requirement of students studying in departments not directly related to mathematics at a university in the northeastern region of the USA. In the research, two different tests consisting of 7 isomorphic open-ended exponential and radical expression questions were given to the students at different times, and student opinions were taken on these issues.

(iv) Denbel (2019) conducted his research with 50 students studying in the geography department of a state university in Ethiopia and taking the "Mathematics for Geographers" course during the research process. In the research, a test comprised of three sections and 20 open-ended questions was preferred as a data collecting tool to identify students' misconceptions and errors related to exponential and radical expressions.

D. Ancheta and Subia (2020) conducted a study with 30 civil engineering, 10 electrical engineering and 10 mechanical engineering students studying in the engineering departments of a university in the Philippines in order to examine engineering students' misconceptions about algebra. In the study, a test was administered to the participants, including questions regarding exponential and radical expressions, and semi-structured interviews were carried out with the participants.

Data Analysis

The findings of the articles and theses in the sample of the study, explanations and interpretations of these findings were analyzed using the content analysis method within the framework of the theory of reduction of abstraction, following the systematics of Hazzan and Zaskis (2005). Accordingly, first of all, if there was content in the documents related to concepts that were not directly related to exponential and radical expressions (such as linear equations), these were identified and not included in the analysis. Afterwards, the misconceptions or errors about exponential and radical expressions and the explanations for them, which were identified in the studies included in the sample, were classified and reinterpreted according to their suitability to the subthemes of the theoretical framework by taking the opinions of a mathematics education expert. The questions in the examples used in the study and the student answers to these questions were quoted from the sources without any changes. The views of the researchers on the misconceptions and errors they identified are also stated. The students in the quoted student interviews were coded as S_1, S_2, and S_3 differently from the sources.

Findings

In this section, misconceptions and errors related to exponential and radical expressions will be interpreted based to the subthemes of the *theory of reducing abstraction* through examples selected from research sample.

Findings related to the first subtheme

Example 1. Rewrite $42^{\frac{1}{2}}$ and $28^{\frac{1}{2}}$, if possible, to an equivalent, reduced expression (Sikora-Press, 2016).

In this question, the researcher expected the students to first write the radical expressions corresponding to the given exponential expressions $\sqrt{42}$ and $\sqrt{28}$, then think about the factors of the numbers as radicands of these expressions and give the answer $\sqrt{42}$ and $2\sqrt{7}$. Consequently, it was noticed that many students had difficulty in questions involving rational exponents and gave wrong answers such as $42^{\frac{1}{2}} = 42{,}5$ and $28^{\frac{1}{2}} = 28 \div 2 = 14$. The researcher explained this situation as students having the misconception of thinking of rational exponent as addition or multiplication.

From the above solutions of students who responded to the question wrongly, it appears that the expressions with rational exponents, as a mathematical *object*, are not *familiar* to these students. Students' tendencies to make the *unfamiliar* more *familiar* by associating rational exponents with addition or multiplication operations they are familiar with to overcome this situation, are indicative of reducing the abstraction level in terms of the *quality of the relationships between the object of thought and the thinking person*.

Example 2. Which of the following radicals are similar to $3\sqrt[3]{2}$?

A) $2\sqrt{5}$ B) $3\sqrt{2}$ C) $2\sqrt[3]{3}$ D) $-\sqrt[3]{2}$ (D. Ancheta & Subia, 2020)

To this question, only 10 out of 50 participants gave the correct answer of $-\sqrt[3]{2}$, while 27 gave the wrong answer of $2\sqrt[3]{3}$. During an interview with one of the students who chose the wrong answer, the student mentioned that the radical expressions $3\sqrt[3]{2}$ and $2\sqrt[3]{3}$ are similar because the signs and degrees of the roots are the same in these expressions. Researchers have stated that the cause of this misconception is students' failure to remember that for radical expressions to be similar, their root degrees and radicands need to be the same.

It has been noticed that students who choose wrong answers like the above cannot conceive of a radical expression as an *object* with its root degree and radicand, and therefore, they fail to grasp the concept of similarity of radical expressions and have misconceptions. According to the theory of reducing abstraction, in the conceptual context of the question (similarity of radical expressions), it might be asserted that there is no quality relationship between the students and the *objects* they encounter (radical expressions) or that the *objects* are not familiar to them. To overcome this situation, students showed a tendency to make the *unfamiliar familiar* by focusing on the sign and only the root degree of the given radical expressions, and thus they reduce the abstraction level in terms of the *quality of the relationships between the object of thought and the thinking person*.

Example 3. $\left(\sqrt{a^2}, \sqrt{16}\right) = ?$

A) $(\pm a, \pm 4)$ B) $(a, 4)$ C) $(\, |a|, 4\,)$ D) $(\pm a,\ 4)$ (Şenay, 2002)

For this question, students chose the wrong answer $(a, 4)$ with a rate of 53%. The researcher noted that these students have the misconception that $\sqrt{a^2} = a$ for every real number a. Below is the part of an interview with a student (S_1) that relates to this question:

R : Can you solve the question?

S_1 : The answer is $(a, 4)$.

R : How did you solve it?

S_1 : I simplified the root $\sqrt{a^2}$ with the exponent of a .

R : Then, are $\sqrt{(-2)^2}$ and $\sqrt{2^2}$ equals to each other?

S_1 : (Thinks)… Equal.

In the Turkish education system, students are first introduced to radical expressions in the 8th grade (secondary school), and at this level, only studies on the square root of positive numbers are carried out. Knowledge about square and cubic roots of negative numbers is given in the 9th grade (high school). Students who chose the wrong answer $(a\,,4)$, tried to use the knowledge they learned in secondary school about the square root of positive numbers for the square root of any real number. In other words, they tried to overcome the problem they encountered by associating the *object* $\sqrt{a^2}$ that is less familiar (or more abstract) to them with an *object* that they worked on it more in time and is more familiar (or more concrete) to them. Thus, they reduced the abstraction level in terms of the *quality of the relationships between the object of thought and the thinking person.*

Findings related to the second subtheme

Example 1. Compare the expressions $(0{,}5)^{21}$ and $(0{,}5)^{17}$ (Avcu, 2010).

The researcher stated that students mostly gave wrong answers to these and similar questions. These students made the comparison $21 > 17$ and answered $(0{,}5)^{21} > (0{,}5)^{17}$ incorrectly. According to the researcher, this misconception occurred as a result of students thinking directly about the general rule (*repeated multiplication*) regarding exponential expressions without paying attention to the base number of exponents.

When examined in terms of theoretical perspective of our study, the students who made this solution ignored the base and did not consider the exponential expression as an *object* to consider the base and exponent together, but they tended to think the most familiar *canonical procedure* namely *repeated multiplication* which is automatically triggered because of the structure of the exponential expression. It might be asserted that students with this tendency reduced the abstraction level in terms of the *process-object duality* because they understood exponential expressions as processes, which is a lower level of understanding than conceiving them as objects.

Example 2. If $a = (2^3)^2,\ b = 2^{(3^2)}\ ,\ c = (2^3)^{-2}$, what is the correct order between the numbers *a*, *b*, *c*?

A) *a* < *b* < *c* B) *c* < *a* < *b* C) *b* < *c* < *a* D) *c* < *b* < *a* (Şenay, 2002)

34% of the students responded this question wrongly. It has been observed that these students tend to calculate the given expressions separately, that is perform operations instead of using the results $a = 2^6,\ b = 2^9\ ,\ c = 2^{-6}$ obtained from properties of exponential expressions, and they make various mistakes in these calculations. The following student (S_2) interview illustrates an example of this approach:

R : Can you solve the question?

S_2 : $a = 2.2.2 = 8^2 = 64,\ b = 2^2.2^2.2^2 = 4.4.4 = 64,\ c = 2.2.2 = 8^{-2} = -64$.

R : Are $(2^3)^2$ and $2^{(3^2)}$ equals to each other?

S_2 : (Thinks) … It is not.

R : But you found it equal.

S₂ : I probably did something wrong.

It might be asserted that students who make this and similar solutions tend to use the *canonical procedure*, which is automatically triggered by the nature of the encountered problem and the exponential expression, that is, to perform operations with *repeated multiplication*, without considering the given exponential expressions as an *object* to be considered with their properties. *Canonical procedures* are preferred by students because they enable the problem to be solved without examining the properties of mathematical concepts, that is, without considering the concept as an *object.* In this case, students' tendency to directly perform operations is an indication of the reduced level of abstraction in terms of the *process-object duality*.

Findings related to the third subtheme

Example 1. Simplify the expression $(-8)^{2/3}$ (Denbel, 2019).

Most of the students answered the question wrongly. Some of the students who answered wrongly solved it as $(-8)^{2/3} = -8^{2/3} = -(8)^{2/3} = -4$. The researcher stated that these students did not perceive the role of the grouping specified in parentheses or had the misconception that parentheses did not have any effect.

In regard to theoretical perspective of our study, these students, instead of dealing with an exponential expression with a negative base, which seemed *more complex* to them, made the base of the expression positive and has transformed it into an easier form to calculate. It can be said that these students reduce the abstraction level in terms of *the degree of complexity of the concept of thought*, since they tend to work with a *less complex object*.

Example 2. $\frac{\sqrt{a}.\sqrt{a}}{\sqrt{a}+\sqrt{a}}$ =? , $a \neq 0$

A) $\frac{1}{2}$ B) $\frac{\sqrt{2}}{2}$ C) $\sqrt{a}$ D) $\frac{\sqrt{a}}{2}$ (Şenay, 2002)

24% of the students answered the question wrongly as $\frac{1}{2}$. Some of these students by accepting $a = 1$ has done the operation $\frac{\sqrt{1}.\sqrt{1}}{\sqrt{1}+\sqrt{1}} = \frac{1}{2}$. However, when $a = -1$, it is easy to observe that the result will not $\frac{1}{2}$.

It might be claimed that the students who made solutions similar to the one above did not consider the expression $\sqrt{a}$ as an *object* with its properties. In addition, these students tended to avoid the *complexity* of the idea that the number *a* could be any negative or positive real number other than 0 (i.e. $a \neq 0$) by using easily calculable numerical values (e.g. $a = 1$) instead of a. This approach of students is an indication that they reduce the abstraction level in terms of *the degree of complexity of the concept of thought*.

Miscellaneous Example

Although the interpretation of the examples we have given so far is in accordance with the three subthemes of the *theory of reducing abstraction* separately, in some circumstances, it could be possible to interpret the student's approach to the same example with multiple subthemes. This also shows that the different subthemes of the *theory* are neither mutually excluding nor completely inclusive of each other. The last example we will give below is interpreted with this approach.

Example 1. Given that *a*, *b*, and *c* are nonzero real numbers, which of the following cannot be made zero as desired?

A) $a+b+c$ B) $a^2+b^2+c^3$ C) $(a^2+b+c^2)^4$ D) $a^2+b^2+c^2$ (Şenay, 2002)

66 percent of the students answered the question wrongly. The majority of these students tried to find the solution by using a numerical value instead of the letters *a*, *b*, and *c*. Rather than dealing with *objects* (letters *a*, *b*, *c*) that seem *abstract* to them, these students tended to deal with *objects* (numbers) that are *closer* to them by using numerical values instead of letters, that is, they reduced the abstraction level in terms of the *quality of the relationships between the object of thought and the thinking person.* These students also tended to calculate the expressions in the choices by giving numerical values to the letters *a*, *b*, and *c* without considering the fact that "the even power of any real number other than 0 is always positive", that is, they tried to perform operations (*canonical procedure* – *process*) without analyzing the exponential expressions they encountered as *objects* which reflects the reduction of the abstraction level in terms of the *process* - *object duality*. Some students tried to overcome the situation they encountered by using only positive numbers instead of the letters *a*, *b*, and *c*. These students have reduced the abstraction level in terms of *the degree of complexity of the concept of thought* by using the set of positive integers, which is *easier* to understand, instead of working with real numbers, which are difficult and *complex* to comprehend. Following is a student (S_3) interview regarding this point of view of reducing the level of abstraction.

R : Can you solve the question?

S_3: If we take $a=1, b=2, c=5$, $a+b+c$ cannot be made zero since $1+2+5\neq 0$.

R : Well, let's try the values you used in other choices. (Trying) As you can see, the other choices could not be made zero also. Actually, if you use all the values as positive numbers, you cannot make any choice of this question zero. For example, if we use $a=-1,\ b=-2,\ c=3$, then $a+b+c=(-1)+(-2)+3=0$.

S_3 : So, we can take negative numbers.

Discussion, Conclusion & Suggestions

Researchers have employed a variety of approaches to examine students' understanding of mathematical concepts, misconceptions, and errors. Error analyses are crucial in assisting teachers to determine what types of errors a student has made and why. To determine whether a specific error pattern exists, for example, whether the student consistently makes the same type of error or to identify procedural mistakes, a thorough analysis of errors is necessary. In addition, diagnosing errors and determining their causes also allows teachers to meet students' needs more efficiently.

In the studies examined, students' inability to make sense of rational exponents or parenthesized representations (Denbel, 2019; Sikora-Press, 2016) and not knowing the rules completely (Avcu, 2010; D. Ancheta and Subia, 2020; Şenay, 2002) were suggested as the reasons for misconceptions and errors regarding exponential and radical expressions. In this study, it was shown that misconceptions and errors can be attributed to students' tendency to work with a lower level of abstraction than the abstraction required by the concepts they encounter in the lesson or the abstraction expected by the teachers, according to the *theory of reducing abstraction*.

Although the *theory of reducing abstraction* is not a theory developed to examine student errors and misconceptions according to Hazzan (1999), the examples in our study show that unconscious reduction of the level of abstraction can lead students to errors and misconceptions, and therefore this theoretical framework can also be used to examine student errors and misconceptions. Similarly, Şenay and Özdemir (2019) showed that this theory can be used to explain the reasons for student errors and misconceptions.

Presenting the reasons for misconceptions and errors about important concepts of school mathematics such as exponential and radical expressions with a different approach from the previous studies will contribute to the development of teachers' and educators' perspectives on the subject. Conducting similar studies at different student levels and on different subjects would also be useful to identify the causes of possible misconceptions and errors.

Ethic

This study is in the category that does not require ethical approval.

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References

Avcu, R. (2010). Eight graders' capabilities in Exponents: Making mental comparisons. *Practice and Theory in Systems of Education*, *5*(1), 39-48.

Aydın, U., & Özdemir, E. (2010). Analysis of students' misconceptions concerning exponent and root expressions. *Educational Sciences: Theory & Practice*, *10*(3), 1775-1800.

Bingölbali, E. & Özmantar, M. F. (2012). Matematiksel kavram yanılgıları: Sebepleri ve çözüm arayışları [Mathematical misconceptions: Causes and solutions]. In E. Bingölbali & M. F. Özmantar (Eds.), *İlköğretimde karşılaşılan matematiksel zorluklar ve çözüm önerileri* [Mathematical difficulties encountered in primary education and solution suggestions] (pp. 1 – 30). Ankara, Turkiye: Pegem Akademi.

Corbetta, P. (2003). *Social research: Theory, methods and techniques.* Thousand Oaks: Sage.

Corbin, J. & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks: Sage.

D. Ancheta, C. M. & Subia, S. (2020). Error analysis of engineering students' misconceptions in algebra. *International Journal of Engineering Trends and Technology*, *68*(12), 66-71.

Denbel, D. G. (2019). Students' difficulties of understanding exponents and exponential expressions. *Engineering and Technology Research*, *1*(3), 083-088.

Dreyfus T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Boston, MA: Kluwer.

Elstak, I.R. (2007). *College student's understanding of rational exponents: A teaching experiment* (Doctoral Dissertation). Retrieved from https://etd.ohiolink.edu/acprod/odb_etd/ws/send_file/send?accession=osu1186505864&disposition=inline

Eraslan, A. (2008): The notion of reducing abstraction in quadratic functions. *International Journal of Mathematical Education in Science and Technology,* 39(8), 1051-1060. http://dx.doi.org/10.1080/00207390802136594

Erlandson, K. M. (2013). *A study of college students' misconceptions of radical expressions* (Master's Thesis). Retrieved from https://soar.suny.edu/handle/20.500.12648/211

Ferrari, P. L. (2003). Abstraction in mathematics. *Philosophical Transactions: Biological Sciences*, 358(1435), 1225-1230.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics, 40*, 71–90.

Hazzan, O. (2001). Reducing abstraction: The case of constructing an operation table for a group. *Journal of Mathematical Behaviour, 20*(2), 163-172.

Hazzan, O. (2003a). How students attempt to reduce abstraction in the learning of mathematics and in the learning of computer science. *Computer Science Education, 13*(2), 95-122.

Hazzan, O. (2003b). Reducing abstraction when learning computability theory. *Journal of Computers in Mathematics and Science Teaching, 22*(2), 95-117.

Hazzan, O. and Zaskis, R. (2005). Reducing abstraction: The case of school mathematics. *Educational Studies in Mathematics, 58***,** 101–119.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Nesher, P. (1987). Towards an instructional theory: The role of student's misconceptions. *For the Learning of Mathematics, 7*(3), 33-39.

Oliver, A. (1989). Handling Pupils' Misconceptions. Presidential address delivered at the Thirteenth National Convention on Mathematics, Physical Science and Biology Education, Pretoria, 3-7 July 1989. Retrieved from http://academic.sun.ac.za/mathed /Malati/Misconceptions.htm

Raychaudhuri, D. (2014). Adaptation and extension of the framework of reducing abstraction in the case of differential equations. *International Journal of Mathematical Education in Science and Technology*, *45*(1), 35-57, http://dx.doi.org/10.1080/0020739X.2013.790503

Rushton, N. (2014). Common errors in Mathematics. *Research Matters: A Cambridge Assessment publication, 17*(1), 8-17.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Sikora-Press, C. M. (2016). *Writing Radical Wrongs: A Study of Students' Misconceptions with Radicals and Rational Exponents* (Master's Thesis). Retrieved from https://soar.suny.edu/handle/20.500.12648/403

Smith, J. P., diSessa, A. A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences, 3*(2), 115-163.

Stacey, K. (2006). Addressing mathematical misconceptions: Using research to guide teaching. *Australian Primary Mathematics Classroom*, *11*(2), 14-18.

Şenay, Ş. C. (2002). Üslü ve köklü sayıların öğretiminde öğrencilerin yaptıkları hatalar ve yanılgıları üzerine bir araştırma [A study on students' mistakes and misconceptions in teaching exponential and radical numbers] (Master's thesis, Selçuk University, Konya, Turkiye). Retrieved from https://tez.yok.gov.tr/UlusalTezMerkezi/

Şenay, Ş. C. ve Özdemir, A. Ş. (2014). Matematik öğretmen adaylarının lineer kongrüanslar ile ilgili soyutlamayı indirgeme eğilimleri [Pre-Service Mathematics Teachers' Tendencies of Reducing Abstraction about Linear Congruence]. *Eğitim ve İnsani Bilimler Dergisi: Teori ve Uygulama* [Journal of Education and Humanities: Theory and Practice], 5(10), 59 – 72.

Şenay, Ş. C. and Özdemir, A. Ş. (2019). Analysis of Errors and Misconceptions about Number Theory Through the Notion of Reducing Abstraction. In Abstract Book of *International Conference on Mathematics and Mathematics Education* (ICMME 2019), p. 359 http://www.icmme2019.selcuk.edu.tr/ICMME2019_Abstract_Book.pdf

Tall, D. (2004). Building concepts in calculus: The role of intuitive concept images and formal concept definitions. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 121-128).

Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematical education. In I. Harel & S. Papert (eds.), *Constructionism* (pp. 193-203). Norwood, NJ: Ablex Publishing Corporation.

Yenilmez, K. ve Yaşa, E. (2008). İlköğretim Öğrencilerinin Geometrideki Kavram Yanılgıları [Primary School Students' Misconceptions in Geometry]. *Uludağ Üniversitesi Eğitim Fakültesi Dergisi* [Journal of Uludag University Faculty of Education], XXI (2), 461-483.

Zazkis, R., & Liljedahl, P. (2004). Understanding the structure of powers and roots in secondary school mathematics. *Journal of Mathematical Behavior*, *23*(2), 179-202.