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ON THE POLAR DERIVATIVE OF LACUNARY TYPE POLYNOMIALS

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Abstract. Let $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$, where $1 \leq \nu \leq n$, be a polynomial of degree n having all its zeros in $|z| \leq k \leq 1$. For polar derivative $D_\alpha p(z)$, it is known that for each $|\alpha| \leq 1$ on $|z| = 1$,

$$|D_\alpha p(z)| \leq \frac{n}{1+k^\nu}\Big\{(|\alpha| + k^\nu)\|p\|_\infty - \frac{1-|\alpha|}{k^{n-\nu}} \min_{|z|=k} |p(z)|\Big\}.$$

In this paper, we obtain the L_q mean extension and a refinement of the above and other related results for the polar derivative of polynomials.

1. Introduction

Let $\mathcal{P}_n$ be the set of polynomials of degree n with complex coefficients. If $p \in \mathcal{P}_n$, denote by

$$\|p\|_q := \Big\{\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^q d\theta\Big\}^{1/q}, \quad 0 < q < \infty,$$
$$\|p\|_\infty := \max_{|z|=1} |p(z)|.$$

For $p \in \mathcal{P}_n$, Bernstein [1], proved that

$$\|p'\|_\infty \leq n\|p\|_\infty. \tag{1}$$

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In the case $q \geq 1$ the following inequality proved by Zygmund [2] and in the case $0 < q < 1$, it is due to Arestov [3],

$$\|p'\|_q \leq n\|p\|_q, \quad 0 < q < \infty. \tag{2}$$

Erdös conjectured and later Lax [4] proved that if $p(z)$ having no zeros in $|z| < 1$, then

$$\|p'\|_\infty \leq \frac{n}{2}\|p\|_\infty. \tag{3}$$

In the case that the polynomial has all its zeros in $|z| \leq 1$, Turán [5] proved that

$$\|p'\|_\infty \geq \frac{n}{2}\|p\|_\infty. \tag{4}$$

As a generalization of inequality (3), it is proved that

$$\|p'\|_q \leq \frac{n}{\|1+z\|_q}\|p\|_q \text{ , for } q > 0. \tag{5}$$

In the case $q \geq 1$ inequality (5) is proved by De-Brujin [6] and for the case $0 < q < 1$, it is due to Rahman and Schmeisser [7].
Malik [8] extended (3) and proved that if $p(z)$ does not any zeros in $|z| < k$, where $k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k}\|p\|_\infty, \tag{6}$$

whereas if $p(z)$ has all its zeros in $|z| \leq k \leq 1$, then

$$\|p'\|_\infty \geq \frac{n}{1+k}\|p\|_\infty. \tag{7}$$

It is proved by Govil and Rahman [9] that if $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then

$$\|p'\|_q \leq \frac{n}{\|k+z\|_q}\|p\|_q \text{ , for } q > 0. \tag{8}$$

The above inequalities were generalized for two class of polynomials. First class is lacunary type polynomials $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$, where $1 \leq \nu \leq n$, and second class is polynomials of the form $p(z) = a_n z^n + \sum_{j=\nu}^{n} a_{n-j} z^{n-j}$, where $1 \leq \nu \leq n$.
As a generalization of inequality (6), it was shown by Qazi [10] that if $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ and $p(z) \neq 0$ in $|z| < k, k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k^\nu}\|p\|_\infty, \tag{9}$$

Also, inequality (9) was extended by Gardner and Weems [11], they proved if $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ and $p(z) \neq 0$ in $|z| < k, k \geq 1$, then

$$\|p'\|_q \leq \frac{n}{\|k^\nu+z\|_q}\|p\|_q \text{ , for } q > 0. \tag{10}$$

On the other hand, for the class of polynomials of type $p(z) = a_n z^n + \sum_{j=\nu}^{n} a_{n-j} z^{n-j}$, where $1 \leq \nu \leq n$, which having all zeros in $|z| \leq k \leq 1$ it was proved by Aziz and Shah [12] that

$$\|p'\|_\infty \geq \frac{n}{1+k^\nu}\Big\{\|p\|_\infty + \frac{1}{k^{n-\nu}}\min_{|z|=k}|p(z)|\Big\}. \tag{11}$$

For a polynomial $p(z)$ of degree n, we define the so-called the polar derivative of $p(z)$ with respect to the point α as

$$D_\alpha p(z) := np(z) + (\alpha - z)p'(z).$$

The polar derivative $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and it is extension of the derivative $p'(z)$ by the following sense

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha} = p'(z)$$

Aziz and Rather [13] extended inequality (5) to the polar derivative of a polynomial and proved that if $p \in \mathcal{P}_n$ and $p(z)$ does not vanish in $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, and $p \geq 1$,

$$\|D_\alpha p\|_q \leq n\frac{|\alpha|+1}{\|1+z\|_q}\|p\|_q \text{ , for } q \geq 1. \tag{12}$$

Inequality (10) is also generalized by Rather et al. [14] to the polar derivative of lacunary type polynomial, and specifically proved that if $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$, where $1 \leq \nu \leq n$ be a polynomial of degree n and $p(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then

$$\|D_\alpha p\|_q \leq n\frac{|\alpha|+k^\nu}{\|k^\nu+z\|_q}\|p\|_q \text{ , for } |\alpha| \geq 1 \textit{ and } q > 0. \tag{13}$$

Recently Dewan et al. [15] proved that if $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l},\ 1 \leq \nu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k \leq 1$, then for every complex number α with $|\alpha| \leq 1$, on $|z| = 1$

$$|D_\alpha p(z)| \leq \frac{n}{1+k^\nu}\Big\{(|\alpha|+k^\nu)\|p\|_\infty - \frac{1-|\alpha|}{k^{n-\nu}}\min_{|z|=k}|p(z)|\Big\}. \tag{14}$$

In the first theorem we obtain the L_q mean extension and a refinement of the above inequality (14), then by using of this theorem we prove the L_q mean extension for lacunary type polynomials, which proposes a generalization and refinement of inequalities (13) as well.

Theorem 1. *Let $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$, be a polynomial of degree n, has all its zeros in $|z| \leq k \leq 1$, then for every complex number α with $|\alpha| \leq 1$, $q > 0$, $\theta \in \mathbb{R}$ and $0 \leq t \leq 1$ we have*

$$\Big\||D_\alpha p(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}(1-|\alpha|)}{k^n(1+\Lambda_{\nu,t})}\Big\|_q \leq \frac{n(|\alpha|+\Lambda_{\nu,t})}{\|z+\Lambda_{\nu,t}\|_q}\|p\|_q, \tag{15}$$

where $\Lambda_{\nu,t} = \frac{n(|a_n|-\frac{tm}{k^n})k^{2\nu}+\nu|a_{n-\nu}|k^{\nu-1}}{\nu|a_{n-\nu}|+n(|a_n|-\frac{tm}{k^n})k^{\nu-1}}$, *and* $m = \min_{|z|=k} |p(z)|$.

Let $q \to \infty$ and choosing $t = 1$ then inequality (15) reduce to a following result.

Corollary 1. *If* $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$ *is a polynomial of degree* n*, has all its zeros in* $|z| \le k < 1$*, then for every complex number* α *with* $|\alpha| \le 1$*,*

$$\|D_\alpha p\|_\infty \le \frac{n(|\alpha| + \Lambda_\nu)}{1 + \Lambda_\nu}\|p\|_\infty - \frac{n\Lambda_\nu(1 - |\alpha|)}{k^n(1 + \Lambda_\nu)} \min_{|z|=k} |p(z)|, \tag{16}$$

where $\Lambda_\nu = \frac{n(|a_n|-\frac{m}{k^n})k^{2\nu}+\nu|a_{n-\nu}|k^{\nu-1}}{\nu|a_{n-\nu}|+n(|a_n|-\frac{m}{k^n})k^{\nu-1}}$, *and* $m = \min_{|z|=k} |p(z)|$.

Remark 1. *Corollary 1 is general and a refinement for inequality* (14)*. To see that, we must show*

$$\frac{n(|\alpha| + \Lambda_\nu)}{1 + \Lambda_\nu}\|p\|_\infty - \frac{n\Lambda_\nu(1 - |\alpha|)}{k^n(1 + \Lambda_\nu)} \min_{|z|=k} |p(z)| <$$
$$\frac{n}{1 + k^\nu}\Big\{(|\alpha| + k^\nu)\|p\|_\infty - \frac{1 - |\alpha|}{k^{n-\nu}} \min_{|z|=k} |p(z)|\Big\}.$$

Equivalently

$$\frac{(1 - |\alpha|)(k^\nu - \Lambda_\nu)}{k^n(1 + k^\nu)(1 + \Lambda_\nu)} \min_{|z|=k} |p(z)| < \frac{(1 - |\alpha|)(k^\nu - \Lambda_\nu)}{(1 + k^\nu)(1 + \Lambda_\nu)}\|p\|_\infty$$

Since $|\alpha| \le 1$ *and from* (30)*, we have* $\Lambda_\nu \le k^\nu$*, the above inequality becomes*

$$\frac{\min_{|z|=k} |p(z)|}{k^n} < \|p\|_\infty \tag{17}$$

the inequality (17) *is true by the Lemma 2, so we get the result.*

If we take $\alpha = 0$ in Corollary 1, we have

Corollary 2. *If* $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$ *is a polynomial of degree* n*, has all its zeros in* $|z| \le k < 1$*, then for* $\theta \in \mathbb{R}$*, we have*

$$|np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})| \le \frac{n\Lambda_\nu}{1 + \Lambda_\nu}\{\|p\|_\infty - \frac{1}{k^n} \min_{|z|=k} |p(z)|\}. \tag{18}$$

Suppose $e^{i\theta_0}$ is such that $|p(e^{i\theta_0})| = \|p\|_\infty$, then by using the inequality $n\|p\|_\infty - |e^{i\theta_0}p'(e^{i\theta_0})| = |np(e^{i\theta_0})| - |e^{i\theta_0}p'(e^{i\theta_0})| \le |np(e^{i\theta_0}) - e^{i\theta_0}p'(e^{i\theta_0})|$ in (18), it becomes to following refinement and generalization of (11).

Corollary 3. *If* $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$ *is a polynomial of degree* n*, has all its zeros in* $|z| \le k < 1$*, then*

$$\|p'\|_\infty \ge \frac{n}{1 + \Lambda_\nu}\{\|p\|_\infty + \frac{\Lambda_\nu}{k^n} \min_{|z|=k} |p(z)|\}. \tag{19}$$

Remark 2. *Corollary 3 is general and refinement to inequality* (11)*. To see that, we using again the method used in Remark 1, it follows that inequality* (19) *is better than inequality* (11)*.*

In the second case by using Theorem 1, we can prove the following theorem that provides a refinement and generalization of (13) and related many results.

Theorem 2. *Let $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ be a polynomial of degree n, does not vanish in $|z| < k$, $k \geq 1$, then for every complex number α with $|\alpha| \geq 1$, $q > 0$, $\theta \in \mathbb{R}$ and $0 \leq t \leq 1$, we have*

$$\left\| |D_\alpha p(e^{i\theta})| + \frac{nmt(|\alpha|-1)}{1+A_{\nu,t}} \right\|_q \leq \frac{n(|\alpha| + A_{\nu,t})}{\|z + A_{\nu,t}\|_q} \|p\|_q, \tag{20}$$

where $A_{\nu,t} = \frac{n(|a_0|-tm)k^{\nu+1}+\nu|a_\nu|k^{2\nu}}{\nu|a_\nu|k^{\nu+1}+n(|a_0|-tm)}$, and $m = \min\limits_{|z|=k} |p(z)|$.

Remark 3. *By using inequality* (30) *from Lemma 2, we have $A_{\nu,t} \geq k^\nu \geq 1$, resulting* (20) *to be a generalization and refinement of* (13)*.*

Let $q \to \infty$ and by choosing $t = 1$, the inequality (20) reduce to a following result that recently proved by Dewan et al. [15].

Corollary 4. *If $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ be a polynomial of degree n, does not vanish in $|z| < k$, $k \geq 1$, then for every complex number α with $|\alpha| \geq 1$, we have*

$$\|D_\alpha p\|_\infty \leq \frac{n}{1+A_\nu} \{(|\alpha| + A_\nu)\|p\|_\infty - (|\alpha|-1) \min_{|z|=k} |p(z)|\}, \tag{21}$$

where $A_\nu = \frac{n(|a_0|-m)k^{\nu+1}+\nu|a_\nu|k^{2\nu}}{\nu|a_\nu|k^{\nu+1}+n(|a_0|-m)}$, and $m = \min\limits_{|z|=k} |p(z)|$.

By dividing both sides of (20) by $|\alpha|$ and let $|\alpha| \to \infty$, we have the following result that is an refinement and generalization of (10) .

Corollary 5. *Let $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ be a polynomial of degree n, does not vanish in $|z| < k$, $k \geq 1$, then for $q > 0$, $\theta \in \mathbb{R}$ and $0 \leq t \leq 1$, we have*

$$\left\| |p'(e^{i\theta})| + \frac{nmt}{1+A_{\nu,t}} \right\|_q \leq \frac{n}{\|z + A_{\nu,t}\|_q} \|p\|_q. \tag{22}$$

By dividing both sides of (21) by $|\alpha|$ and let $|\alpha| \to \infty$, we have

Corollary 6. *If $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ be a polynomial of degree n, does not vanish in $|z| < k$, $k \geq 1$, then*

$$\|p'\|_\infty \leq \frac{n}{1+A_\nu} \{(\|p\|_\infty - \min_{|z|=k} |p(z)|\}. \tag{23}$$

Remark 4. *Inequality* (23) *has been studied by Gardner et al. [16].*

2. Lemmas

The following lemmas are needed for proof of the theorems. The first lemma is due to Aziz et al. [17].

Lemma 1. *Let $p(z) \in \mathcal{P}_n$ and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for each γ, $0 \leq \gamma < 2\pi$, and $q > 0$,*

$$\int_0^{2\pi}\int_0^{2\pi} |q'(e^{i\theta}) + e^{i\gamma}p'(e^{i\theta})|^q d\theta d\gamma \leq 2\pi n^q \int_0^{2\pi} |p(e^{i\theta})|^q d\theta.$$

Lemma 2. *If $p(z) = \sum_{i=0}^{n} a_i z^i$ is a polynomial of degree n, having all its zeros in $|z| \leq k \leq 1$, then*

$$\min_{|z|=k} |p(z)| < k^n \max_{|z|=1} |p(z)|, \tag{24}$$

and in particular $\min_{|z|=k} |p(z)| < k^n|a_n|$.

The above lemma is due to Zireh [18].

Lemma 3. *The function*

$$S(x) = \frac{nxk^{2\nu} + \nu|a_{n-\nu}|k^{\nu-1}}{nxk^{\nu-1} + \nu|a_{n-\nu}|}$$

for $k \leq 1$ is a non-increasing function of x.

Proof. The proof follows by considering the first derivative test for $S(x)$. □

The following lemma is due to Aziz and Rather [13].

Lemma 4. *If $p(z) = a_nz^n + \sum_{l=\nu}^{n} a_{n-l}z^{n-l}$, has all its zeros in $|z| \leq k \leq 1$, and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then on $|z| = 1$,*

$$|q'(z)| \leq L_\nu |p'(z)|, \tag{25}$$

where

$$L_\nu = \frac{n|a_n|k^{2\nu} + \nu|a_{n-\nu}|k^{\nu-1}}{\nu|a_{n-\nu}| + n|a_n|k^{\nu-1}}, \tag{26}$$

and

$$\frac{\nu}{n}|\frac{a_{n-\nu}}{a_n}| \leq k^\nu. \tag{27}$$

Lemma 5. *If $p(z) = a_nz^n + \sum_{l=\nu}^{n} a_{n-l}z^{n-l}$, has all its zeros in $|z| \leq k \leq 1$, and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for $0 \leq t \leq 1$ and $|z| = 1$, we have*

$$|q'(z)| \leq \Lambda_{\nu,t}|p'(z)| - \frac{nmt\Lambda_{\nu,t}}{k^n}, \tag{28}$$

where

$$\Lambda_{\nu,t} = \frac{n(\mid a_n \mid - \frac{tm}{k^n})k^{2\nu} + \nu \mid a_{n-\nu} \mid k^{\nu-1}}{\nu \mid a_{n-\nu} \mid + n(\mid a_n \mid - \frac{tm}{k^n})k^{\nu-1}} \tag{29}$$

and

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{|a_n|-\frac{tm}{k^n}} \leq k^{\nu}. \tag{30}$$

where $m = \min\limits_{|z|=k} |p(z)|$.

Proof. Let $m = \min\limits_{|z|=k} |p(z)|$. If $m = 0$, then inequality (28) reduce to inequality (25) in Lemma 4, which is trivial. Therefore, we suppose that the polynomial $p(z)$ having all its zeros in $|z| < k$, hence for every $\beta \in \mathbb{C}$ with $|\beta| < 1$, we have $|\frac{\beta m z^n}{k^n}| < |p(z)|$ for $|z| = k$. Now the Rouche's theorem implies that the polynomial $p(z)-\frac{\beta m z^n}{k^n}$, has all its zeros in $|z| < k < 1$. By applying Lemma 4 to the polynomial $p(z) - \frac{\beta m z^n}{k^n}$, for $|z| = 1$ we get

$$|q'(z)| \leq S_\nu |p'(z) - \frac{\beta n m z^{n-1}}{k^n}|, \tag{31}$$

where

$$S_\nu = \frac{n(\mid a_n - \frac{\beta m}{k^n} \mid)k^{2\nu} + \nu \mid a_{n-\nu} \mid k^{\nu-1}}{\nu \mid a_{n-\nu} \mid + n(\mid a_n - \frac{\beta m}{k^n} \mid)k^{\nu-1}}.$$

By applying Lemma 2 we get $\mid a_n \mid > \dfrac{m}{k^n}$, then we can substituted $\mid a_n - \frac{\beta\, m}{k^n} \mid$ by $|a_n| - \frac{|\beta| m}{k^n}$, since we have that

$$\mid a_n - \frac{\beta\, m}{k^n} \mid \geq \mid a_n \mid - \frac{\mid \beta \mid m}{k^n}. \tag{32}$$

By applying Lemma 3 for (32) and taking $t = |\beta|$, we get

$$S_\nu \leq \Lambda_{\nu,t}. \tag{33}$$

Combining (31) and (33), one can obtain

$$|q'(z)| \leq \Lambda_{\nu,t} |p'(z) - \frac{\beta m n z^{n-1}}{k^n}|. \tag{34}$$

Again since $|\frac{\beta m z^n}{k^n}| < |p(z)|$, by choosing the suitable argument of β, we have

$$|p'(z) - \frac{\beta m n z^{n-1}}{k^n}| = |p'(z)| - |\frac{\beta m n z^{n-1}}{k^n}|, \tag{35}$$

from (34) and (35) we get,

$$|q'(z)| \leq \Lambda_{\nu,t} |p'(z)| - \frac{n m t \Lambda_{\nu,t}}{k^n}.$$

To prove (30), we use (27) for the polynomial $p(z) - \frac{\beta m z^n}{k^n}$, as a result we have

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{|a_n - \frac{\beta m}{k^n}|} \leq k^{\nu},$$

or

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{k^{\nu}} \leq |a_n - \frac{\beta m}{k^n}|. \tag{36}$$

This means $\frac{\nu}{n}\frac{|a_{n-\nu}|}{k^{\nu}}$ is lower bound for $|a_n - \frac{\beta m}{k^n}|$ for every β, it implies that $\frac{\nu}{n}\frac{|a_{n-\nu}|}{k^{\nu}}$ is less than $\min\limits_{|\beta|\leq 1}|a_n - \frac{\beta m}{k^n}|$, hence from (32) we have

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{k^{\nu}} \leq |a_n| - \frac{|\beta| m}{k^n}.$$

or

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{|a_n| - \frac{tm}{k^n}} \leq k^{\nu}.$$

□

The next lemma is due to Aziz et. al [19].

Lemma 6. *Let A, B, C are positive real numbers which that $B + C \leq A$, then for any real γ,*

$$|(A - C) + e^{i\gamma}(B + C)| \leq |B + e^{i\gamma}A|. \tag{37}$$

We also need the following lemma is due to Rather et al. [14].

Lemma 7. *If a, b are two non-negative real numbers which that $a \geq bc$ where $c \geq 1$, then for every $x \geq 1, q > 0$ and $0 \leq \gamma < 2\pi$*

$$(a + bx)^q \int_0^{2\pi} |c + e^{i\gamma}|^q d\gamma \leq (c + x)^q \int_0^{2\pi} |a + be^{i\gamma}|^q d\gamma \tag{38}$$

3. Proof of the theorems

Proof of the Theorem 1. By the assumptions, $p(z)$ having all its zeros in $|z| \leq k \leq 1$, therefore by Lemma 5, for $|z| = 1$, we have

$$|q'(z)| \leq \Lambda_{\nu,t}\big(|p'(z)| - \frac{nmt}{k^n}\big).$$

This inequality can be rewritten as

$$|q'(z)| + \frac{nmt\Lambda_{\nu,t}}{k^n(1 + \Lambda_{\nu,t})} \leq \Lambda_{\nu,t}\{|p'(z)| - \frac{nmt\Lambda_{\nu,t}}{k^n(1 + \Lambda_{\nu,t})}\}. \tag{39}$$

Taking $A = |p'(z)|,\ B = |q'(z)|$ and $C = \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}$ in Lemma 6, and attend that $\Lambda_{\nu,t} \leq k^{\nu} \leq 1$, by (30), so $B + C \leq A - C \leq A$. Then for any real γ, we get

$$\Big|\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1 + \Lambda_{\nu,t})}\} + e^{i\gamma}\{|q'(e^{i\theta})| + \{\frac{nmt\Lambda_{\nu,t}}{k^n(1 + \Lambda_{\nu,t})}\}\Big| \leq \Big||q'(e^{i\theta})| + e^{i\gamma}|p'(e^{i\theta})|\Big|. \tag{40}$$

This implies for each $q > 0$, that

$$\begin{aligned}\int_0^{2\pi} \Big|\{| p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}+e^{i\gamma}\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\theta \\ \leq \int_0^{2\pi} \Big||q'(e^{i\theta})| + e^{i\gamma}|p'(e^{i\theta})|\Big|^q d\theta,\end{aligned} \tag{41}$$

From every side of (41), we integrate with respect to γ from 0 to 2π, which gives

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} \Big|\{| p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + e^{i\gamma}\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\theta d\gamma \\ \leq \int_0^{2\pi}\int_0^{2\pi} \Big|e^{i\gamma}|p'(e^{i\theta})| + |q'(e^{i\theta})|\Big|^q d\theta d\gamma \\ = \int_0^{2\pi}\Big\{\int_0^{2\pi} \Big|e^{i\gamma}|p'(e^{i\theta})| + |q'(e^{i\theta})|\Big|^q d\gamma\Big\} d\theta \\ = \int_0^{2\pi}\Big\{\int_0^{2\pi} \Big|e^{i\gamma}p'(e^{i\theta}) + q'(e^{i\theta})\Big|^q d\gamma\Big\} d\theta \\ = \int_0^{2\pi}\Big\{\int_0^{2\pi} \Big|e^{i\gamma}p'(e^{i\theta}) + q'(e^{i\theta})\Big|^q d\theta\Big\} d\gamma.\end{aligned}$$

From the Lemma 1 and above result, we conclude that

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} \Big|\{| p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + e^{i\gamma}\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\theta d\gamma \\ \leq 2\pi n^q \int_0^{2\pi} |p(e^{i\theta})|^q d\theta.\end{aligned} \tag{42}$$

For $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and using the fact that

$$|np(z) - zp'(z)| = |q'(z)| \text{ for}, |z| = 1,$$

we have

$$\begin{aligned}|D_\alpha p(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})} &= |np(e^{i\theta}) + (\alpha - e^{i\theta})p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})} \\ &\leq |np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})| + |\alpha||p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})} \\ &= |q'(e^{i\theta})| + |\alpha||p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})} \\ &=\Big\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\Big\} + |\alpha|\Big\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\Big\}\end{aligned}$$

By integrating both sides of above inequality with respect to θ from 0 to 2π, for each $q>0$, we have

$$\int_0^{2\pi}\Big\{|D_\alpha p(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})}\Big\}^q d\theta$$
$$\leq \int_0^{2\pi}\Big\{\{|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}+|\alpha|\{|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big\}^q d\theta$$

Multiply both sides of above inequality by

$$\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^p d\gamma$$

we have

$$\Big\{\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^q d\gamma\Big\}\int_0^{2\pi}\Big\{|D_\alpha p(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})}\Big\}^q d\theta$$
$$\leq \Big\{\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^q d\gamma\Big\}\int_0^{2\pi}\Big\{\{|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}+|\alpha|\{|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big\}^q d\theta \tag{43}$$

By taking

$$a=|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})},\ b=|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})},\ c=\frac{1}{\Lambda_{\nu,t}},\ x=\frac{1}{|\alpha|},$$

the conditions of Lemma 7 are established (since the inequality (39) implies $a>bc$). Then Lemma 7 implies that for every α with $|\alpha|\leq 1$, we have

$$\Big\{(|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})})+\frac{1}{|\alpha|}(|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})})\Big\}^q\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^q d\gamma$$
$$\leq \Big|\frac{1}{\Lambda_{\nu,t}}+\frac{1}{|\alpha|}\Big|^q\int_0^{2\pi}\Big|\{|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}+e^{i\gamma}\{|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\gamma$$

Again, integrating both sides of above inequality with respect to θ, we have

$$\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^q d\gamma\int_0^{2\pi}\Big\{|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}+\frac{1}{|\alpha|}(|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})})\Big\}^q d\theta$$
$$\leq \Big|\frac{1}{\Lambda_{\nu,t}}+\frac{1}{|\alpha|}\Big|^q\int_0^{2\pi}\int_0^{2\pi}\Big|\{|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}+e^{i\gamma}\{|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\gamma d\theta$$

Multiply both sides of above inequality by $|\alpha|^q$, we get

$$\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^q d\gamma\int_0^{2\pi}\Big\{|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}+|\alpha|(|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})})\Big\}^q d\theta$$
$$\leq \Big|\frac{|\alpha|}{\Lambda_{\nu,t}}+1\Big|^q\int_0^{2\pi}\int_0^{2\pi}\Big|\{|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}+e^{i\gamma}\{|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\gamma d\theta \tag{44}$$

By comparing second part of (43) and first part of (44) we obtain

$$\Big\{\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^q d\gamma\Big\}\int_0^{2\pi}\Big\{|D_\alpha p(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})}\Big\}^q d\theta$$
$$\leq\Big|\frac{|\alpha|}{\Lambda_{\nu,t}}+1\Big|^q\int_0^{2\pi}\int_0^{2\pi}\Big|\{|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}+e^{i\gamma}\{|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\gamma d\theta \tag{45}$$

Now by comparing inequalities (45) and (42) we get

$$\Big\{\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^q d\gamma\Big\}\int_0^{2\pi}\Big\{|D_\alpha p(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})}\Big\}^q d\theta$$
$$\leq\Big|\frac{|\alpha|}{\Lambda_{\nu,t}}+1\Big|^q 2\pi n^q\int_0^{2\pi}|p(e^{i\theta})|^q d\theta. \tag{46}$$

Multiply both sides of above inequality by $(\Lambda_{\nu,t})^q$, we get

$$\Big\{\int_0^{2\pi}|e^{i\gamma}+\Lambda_{\nu,t}|^q d\gamma\Big\}\int_0^{2\pi}\Big\{|D_\alpha p(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})})\Big\}^q d\theta$$
$$\leq\Big||\alpha|+\Lambda_{\nu,t}\Big|^q 2\pi n^q\int_0^{2\pi}|p(e^{i\theta})|^q d\theta. \tag{47}$$

Equivalently

$$\Big\{\frac{1}{2\pi}\int_0^{2\pi}|e^{i\gamma}+\Lambda_{\nu,t}|^q d\gamma\Big\}^{\frac{1}{q}}\Big\{\frac{1}{2\pi}\int_0^{2\pi}\Big||D_\alpha p(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}(1-|\alpha|)}{k^n(1+\Lambda_{\nu,t})}\Big|^q d\theta\Big\}^{\frac{1}{q}}$$
$$\leq n||\alpha|+\Lambda_{\nu,t}|\Big\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\Big\}^{\frac{1}{q}}. \tag{48}$$

This completes the proof of Theorem 1. □

Proof of the Theorem 2. By the hypothesis the polynomial $p(z)=a_0+\sum_{j=\nu}^{n}a_jz^j$, where $1\leq\nu\leq n$ does not any zeros in $|z|<k$, where $k\geq 1$. Therefore, the polynomial $q(z)=z^n\overline{p(\frac{1}{\bar z})}=a_0z^n+\sum_{j=\nu}^{n}a_jz^{n-j}$ has all its zeros in $|z|\leq\frac{1}{k}\leq 1$. By applying Theorem 1 to q(z), and replacing $\frac{1}{k}$ in equation (15), we get for every complex number α with $|\alpha|\leq 1$,

$$\Big\||D_\alpha q(e^{i\theta})|+\frac{nk^nm_1t\Lambda_{1,\nu}(1-|\alpha|)}{(1+\Lambda_{1,\nu})}\Big\|_q\leq n\frac{(|\alpha|+\Lambda_{1,\nu})}{\|\Lambda_{1,\nu}+z\|_q}\|q\|_q, \tag{49}$$

where $\Lambda_{1,\nu}=\frac{n(|a_0|-k^nm_1t)k^{-2\nu}+\nu|a_\nu|k^{1-\nu}}{\nu|a_\nu|+n(|a_0|-k^nm_1t)k^{1-\nu}}$ and $m_1=\min\limits_{|z|=\frac{1}{k}}|q(z)|$.

On the other hand

$$m_1 = \min_{|z|=\frac{1}{k}} |q(z)| = \min_{|z|=\frac{1}{k}} |z^n \overline{p(\frac{1}{\overline{z}})}| = \frac{\min_{|z|=k} |p(z)|}{k^n} = \frac{m}{k^n}.$$

Since $q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$, then $|q(e^{i\theta})| = |p(e^{i\theta})|$ and for $|D_\alpha q(e^{i\theta})|$,we have

$$|D_\alpha q(e^{i\theta})| = |nq(e^{i\theta}) + (\alpha - e^{i\theta})q'(e^{i\theta})| = |ne^{in\theta}\overline{p(e^{i\theta})} + (\alpha - e^{i\theta})q'(e^{i\theta})|$$

$$\left|ne^{in\theta}\overline{p(e^{i\theta})} + (\alpha - e^{i\theta})\Big(ne^{i(n-1)\theta}\overline{p(e^{i\theta})} - e^{i(n-2)\theta}\overline{p'(e^{i\theta})}\Big)\right| =$$

$$\left|ne^{in\theta}\overline{p(e^{i\theta})} + \Big(n\alpha e^{i(n-1)\theta}\overline{p(e^{i\theta})} - \alpha e^{i(n-2)\theta}\overline{p'(e^{i\theta})} - ne^{in\theta}\overline{p(e^{i\theta})} + e^{i(n-1)\theta}\overline{p'(e^{i\theta})}\Big)\right|$$

$$=\left|n\alpha e^{i(n-1)\theta}\overline{p(e^{i\theta})} - \alpha e^{i(n-2)\theta}\overline{p'(e^{i\theta})} + e^{i(n-1)\theta}\overline{p'(e^{i\theta})}\right|$$

$$=|\overline{\alpha e^{i(n-1)\theta}}|\left|np(e^{i\theta}) + (\frac{1}{\overline{\alpha}} - e^{i\theta})p'(e^{i\theta})\right| = |\alpha||D_{\frac{1}{\overline{\alpha}}}p(e^{i\theta})|$$

By replacing $m_1 = \frac{m}{k^n}$, $\|q\|_q = \|p\|_q$ and $|D_\alpha q(e^{i\theta})| = |\alpha||D_{\frac{1}{\overline{\alpha}}}p(e^{i\theta})|$ in (49) we get

$$\left\||\alpha||D_{\frac{1}{\overline{\alpha}}}p(e^{i\theta})| + \frac{nmt\Lambda_{1,\nu}(1-|\alpha|)}{(1+\Lambda_{1,\nu})}\right\|_q \leq n\frac{(|\alpha| + \Lambda_{1,\nu})}{\|\Lambda_{1,\nu} + z\|_q}\|p\|_q,$$

Or

$$|\alpha|\left\||D_{\frac{1}{\overline{\alpha}}}p(e^{i\theta})| + \frac{nmt(\frac{1}{|\alpha|} - 1)}{\frac{1}{\Lambda_{1,\nu}} + 1}\right\|_q \leq n\frac{|\alpha|\Lambda_{1,\nu}(\frac{1}{|\alpha|} + \frac{1}{\Lambda_{1,\nu}})}{\Lambda_{1,\nu}\|\frac{1}{\Lambda_{1,\nu}} + z\|_q}\|p\|_q, \tag{50}$$

where $\Lambda_{1,\nu} = \frac{n(|a_0|-tm)k^{-2\nu}+\nu|a_\nu|k^{1-\nu}}{\nu|a_\nu|+n(|a_0|-tm)k^{1-\nu}}$ and $m = \min\limits_{|z|=k}|p(z)|$. If we take $A_{\nu,t} = \frac{1}{\Lambda_{1,\nu}}$, $\gamma = \frac{1}{\overline{\alpha}}$, then $A_{\nu,t} \geq k^\nu \geq 1$ and $|\gamma| \geq 1$, then the inequality (50) becomes the following inequality

$$\left\||D_\gamma(p(e^{i\theta}))| + \frac{nmt(|\gamma|-1)}{1+A_{\nu,t}}\right\|_q \leq n\frac{(|\gamma| + A_{\nu,t})}{\|A_{\nu,t} + z\|_q}\|p\|_q, \tag{51}$$

where $A_{\nu,t} = \frac{n(|a_0|-tm)k^{1+\nu}+\nu|a_\nu|k^{2\nu}}{\nu|a_\nu|k^{1+\nu}+n(|a_0|-tm)}$ and $m = \min\limits_{|z|=k}|p(z)|$. This completes the proof of Theorem 2. □

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