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Analysis of the Error Bounds in Numerical Integration for Log-Convex Functions

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Abstract. This study aims to investigate the application of perturbed trapezoid inequalities in the numerical integration of n-times differentiable and logarithmically convex functions. The objective is to analyze the accuracy of numerical approximations, such as the trapezoidal and Simpson's rules, by providing error bounds through these inequalities. By examining how these methods apply to *log*-convex functions, the study presents suggestions into optimizing computational approaches and understanding the properties of these functions in various areas. The obtained findings are expected to contribute to the development of more precise and efficient in numerical integration techniques such as the rectangle, the trapezoid, and Simpson rule.

Keywords: Convex functions · *log*-convex functions · Perturbed trapezoid inequality · Numerical integration.

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1 Introduction

Log-convex functions play a crucial role in mathematical analysis and optimization. A function is considered *log*-convex if its logarithm is convex. The combination of such functions with the perturbed trapezoidal inequality has wide applications in various disciplines requiring precise numerical integration and error estimation. This approach enhances the accuracy and reliability of solutions in disciplines such as numerical analysis, optimization, economics, statistical analysis, computational biology, engineering, and finance.

By taking advantage of the properties of logarithmically convex functions, it is possible to obtain higher performance and more accurate results in applications in these areas. Numerical integration methods often employ the trapezoid rule. However, for *log*-convex functions, the perturbed trapezoid inequality refines these approximations, reducing error and improving accuracy. Particularly, in solving differential equations or evaluating definite integrals numerically, *log*-convex functions can be integrated more precisely through this inequality. This is of importance in applications demanding high precision, such as computational physics or engineering simulations.

Besides, optimization problems involving *log*-convex cost or utility functions benefit from more accurate gradient and integral approximations, leading to better convergence rates and more reliable solutions. In economic models, *log*-convex utility or production functions are frequently used. Accurate integration of these functions is critical for resource allocation, cost-benefit analysis, and policy-making. In statistical analysis, *log*-convex probability density functions commonly appear, and the precise estimation of expected values and other statistical measures is essential for hypothesis testing, confidence interval construction, and inferential statistics. In engineering disciplines, particularly structural engineering and control systems, sensitive numerical integration is often required. Log-convex functions are utilized in stress-strain relationships, material properties, and control algorithms. In finance, *log*-convex functions are employed in the modeling of phenomena such as option pricing, risk assessment, and portfolio optimization, where accurate numerical integration is vital.

Logarithmically convex functions have been studied by many authors. Dragomir and Mond [1] focused on generalizing or presenting Hermite-Hadamard type integral inequalities for these functions in different forms by using the properties of logarithmically convex functions. Dragomir [2] presents two improved versions of the Hermite-Hadamard integral inequality for logarithmic convex functions. That is, he developed stronger or more sensitive forms of the Hermite-Hadamard inequality for log-convex functions. Aujla and Bourin [3] introduced new inequalities in matrix versions for both concave and *log*-convex functions and they also contributed to the extension and proof of some important existing inequalities. Alomari and Darus [4] generalized important results such as the Hadamard and Jensen inequalities obtained for coordinated *log*-convex functions for functions by defining them on a particular rectangle. Zhang and Jiang [5] investigated some properties of logarithmic convex functions and proved integral inequalities for such functions. They also presented a formula for estimating the remaining terms in the expansion of the Taylor series as an applied consequence of *log*-convex functions. Yang et al. [6] generalized the Hadamard inequality regarding log-convex functions and introduced new extended versions for these classes of functions. Niculescu [7] discussed the existence of a strengthened version of the Hermite-Hadamard inequality for *log*-convex functions. Dragomir [8] presented new inequalities of the Hermite-Hadamard type for *log*-convex functions defined on real intervals. Jain et al. [9] have discussed various estimates of the right (or left) side of the Hermite-Hadamard inequality. These estimates are considered for cases where the absolute values of the second (or first) derivatives of a function are *log*-convex under positive real exponents. Conde et al. [10] examined norm and skew angular distances in a normed space. By using convex functions, they aimed to improve and reverse some important results in the literature. This study focuses on the application of perturbed trapezoid inequalities in numerical integration of logarithmically convex and n-times differentiable functions.

The main purpose of the study is to analyze the accuracy of approximate integrals made with numerical methods such as trapezoidal and Simpson rules through these inequalities and to reveal the error limits. Additionally, by discussing how these methods can be applied to *log*-convex functions, it presents suggestions on the optimization of numerical calculation methods and a better understanding of the properties of these functions in various fields. It is expected that the findings will contribute to making numerical integration methods more precise and efficient.

2 Preliminaries

A function $\phi : I \subset \mathbb{R} \to \mathbb{R}$ is convex on I if the inequality

$$\phi\left(rx + (1-r)\, y\right) \leq r\phi\left(x\right) + (1-r)\,\phi\left(y\right), \tag{1}$$

holds for all $x, y \in I$ and $r \in [0,1]$. The function ϕ is said to be concave if $(-\phi)$ is convex. For numerical integration, trapezoid inequality is given by

$$\left|\int_x^y \phi\left(u\right) du - \tfrac{1}{2}\left(y-x\right)\left(\phi\left(x\right) + \phi\left(y\right)\right)\right| \leq \tfrac{1}{12} M_2 \left(y-x\right)^3 \tag{2}$$

where $\phi : [x, y] \to \mathbb{R}$ is assumed to be twice differentiable on $x, y \in I$ with the second derivative bounded on (x, y) by $M_2 = sup_{u\in(x,y)} \left|\phi''\left(u\right)\right| < +\infty$ [11].

Definition 1. *[12] : A positive function ϕ is called log-convex on a real interval $I = [a, b]$ if*

$$\phi\left(rx + (1-r)\, y\right) \leq \phi\left(x\right)^r \phi\left(y\right)^{1-r}, \tag{3}$$

for all $x, y \in I$ and $r \in [0,1]$. If ϕ is a positive log-concave function, then the inequality is reversed. Equivalently, a function ϕ is log-convex on I if ϕ is positive and log-convex on I. Also, if $\phi > 0$ and ϕ'' exists on I, then ϕ is log-convex if and only if $\phi.\phi'' - (\phi')^2 \geq 0$ [12]. Note that if ϕ and ψ are convex functions and ψ is monotonically nondecreasing, then $\phi \circ \psi$ is convex.

Moreover, since $\phi = \exp(\log\phi)$*, it follows that a log-convex function is convex, but the converse is not true [13,14]. This fact is clear from inequality (3) by the arithmetic-geometric mean inequality, then one obtains*

$$\phi^{r}(x)\,\phi^{1-r}(y) \leq r\phi(x) + (1-r)\,\phi(y) \tag{4}$$

for all $x, y \in I$ *and* $r \in [0,1]$*. If the above inequality (3) is reversed, then* ϕ *is called logarithmically concave or simply log-concave .*

Theorem 1. *[14] Let* $\phi : [x,y] \to \mathbb{R}$ *be continuous on* $[x,y]$ *and twice differentiable on* (x,y) *and assume that the second derivative* $\phi'' : (x,y) \to \mathbb{R}$ *satisfies the condition:*

$$\upsilon \leq \phi'' \leq \varphi \tag{5}$$

for all $u \in (x,y)$*. Moreover, we have the inequality*

$$\left|\phi(u) - \left(u - \tfrac{x+y}{2}\right)\phi'(u) + \left[\tfrac{(y-x)^2}{24} + \tfrac{1}{2}\left(u - \tfrac{x+y}{2}\right)^2\right]\tfrac{\phi'(y)-\phi'(x)}{y-x} - \tfrac{1}{y-x}\int_x^y \phi(t)\,dt\right|$$
$$\leq \tfrac{1}{8}(\varphi - \upsilon)\left[\tfrac{1}{2}(y-x) + \left|u - \tfrac{x+y}{2}\right|\right]^2 \tag{6}$$

for all $u \in (x,y)$ *and the perturbed midpoint inequality is given as*

$$\left|\phi\left(\tfrac{x+y}{2}\right) + \tfrac{1}{24}(y-x)\left(\phi'(y) - \phi'(x)\right) - \tfrac{1}{y-x}\int_x^y \phi(t)\,dt\right|$$
$$\leq \tfrac{1}{32}(\varphi - \upsilon)(y-x)^2 . \tag{7}$$

Thus, we have the perturbed trapezoid inequality

$$\left|\tfrac{\phi(x)+\phi(y)}{2} - \tfrac{1}{12}(y-x)\left(\phi'(y) - \phi'(x)\right) - \tfrac{1}{y-x}\int_x^y \phi(t)\,dt\right|$$
$$\leq \tfrac{1}{8}(\varphi - \upsilon)(y-x)^2 . \tag{8}$$

2.1 Application to the midpoint and trapezoidal formulas

Let d be a division of the interval $[x,y]$, i.e., $x = u_0 < u_1 < \cdots < u_{i-1} < u_i = y$ and consider the quadrature formulas;

$$\int_x^y \phi(r)\,dr = T(\phi,d) + E(\phi,d) \tag{9}$$

and

$$\int_x^y \phi(r)\,dr = T'(\phi,d) + E'(\phi,d) \tag{10}$$

where

$$T(\phi,d) = \sum_{i=0}^{n-1}(u_{i+1} - u_i)\,\phi\left(\tfrac{u_i + u_{i+1}}{2}\right) \tag{11}$$

and

$$T'(\phi,d) = \sum_{i=0}^{n-1}(u_{i+1} - u_i)\left(\tfrac{\phi(u_i) + \phi(u_{i+1})}{2}\right) \tag{12}$$

are the midpoint and trapezoidal approximations, respectively, and $E(\phi,d)$ and $E'(\phi,d)$ are the associated errors [15]. Here, we derive some error estimates for the sum of midpoint and trapezoidal formulas.

Proposition 1. *[16] Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $\phi' \in L^1([x,y])$, where $x, y \in I^{\circ}$ with $x < y$. If $|\phi'|$ is convex on $[x,y]$, then one obtains*

$$|E(\phi,d) + E'(\phi,d)| \le \frac{1}{8}\sum_{i=0}^{n-1}(u_{i+1}-u_i)^2\left[|\phi'(u_i)| + |\phi'(u_{i+1})|\right] \tag{13}$$

for every division d of $[x,y]$.

Theorem 2. *[16] Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $\phi'' \in L^1[x,y]$, where $x, y \in I^{\circ}$ with $x < y$. If $|\phi''|^q$ is s-convex on $[x,y]$ for $q \ge 1$, then we have*

$$\begin{aligned} |E(\phi,d)| &\le \frac{1}{2^{4+\frac{s}{q}}}\left(\frac{2}{3}\right)^{1-\frac{1}{q}} \\ &\times \Bigg\{\sum_{i=0}^{n-1}\left(\frac{s+4}{(s+2)(s+3)}|\phi''(u_i)|^q + \left(\frac{2}{3} + \frac{2}{(s+1)(s+3)}\beta(s+1,n)\right)|\phi''(u_{i+1})|^q\right)^{\frac{1}{q}} \\ &+ \left(\frac{s+4}{(s+2)(s+3)}|\phi''(u_{i+1})|^q + \left(\frac{2}{3} + \frac{2}{(s+1)(s+3)}\beta(s+1,n)\right)|\phi''(u_i)|^q\right)^{\frac{1}{q}}\Bigg\}. \end{aligned} \tag{14}$$

for every division d of $[x,y]$.

Let $I_n : x = u_1 < u_2 < \cdots < u_{n-1} < u_n = y$ be a division of the interval $[x,y]$, $\xi_j \in \left[u_j + \delta\frac{h_j}{2}, u_{j+1} - \delta\frac{h_j}{2}\right]$; $j = 0, 1, \cdots, n-1$ a sequence of intermediate points and $h_j = u_{j+1} - u_j$; $j = 0, 1, \cdots, n-1$, then, the quadrature is given by the following theorem.

Theorem 3. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable on (x,y) whose second derivative $\phi'' : (x,y) \to \mathbb{R}$ belongs to $L^1(x,y)$ i.e $||\phi''||_1 := \int_x^y ||\phi''||\,dr < \infty$. Then, the perturbed Riemann's quadrature formula holds*

$$\int_x^y \phi(r)\,dr = A(\phi,\phi',I_n,\xi,\delta) + R(\phi,\phi',I_n,\xi,\delta) \tag{15}$$

where

$$\begin{aligned} A(\phi,\phi',I_n,\xi,\delta) &= (1-\delta)\sum_{j=0}^{n-1} h_j\phi(\xi_j) - (1-\delta)\sum_{j=0}^{n-1} h_j\left(\xi_j - \frac{u_j+u_{j+1}}{2}\right)\phi'(\xi_j) \\ &+ \frac{\delta}{2}\sum_{j=0}^{n-1} h_j\left(\phi(u_j) + \phi(u_{j+1})\right) - \frac{\delta^2}{2}\sum_{j=0}^{n-1} h_j^2\left(\phi'(u_{j+1}) - \phi'(u_j)\right) \end{aligned} \tag{16}$$

and the residual term $R(\phi,\phi',I_n,\xi,\delta)$ *satisfies the estimation;*

$$\begin{aligned}|R(\phi,\phi',I_n,\xi,\delta)| &\leq \frac{1}{2}\sum_{j=0}^{n-1}\left[\frac{h_j(1-\delta)}{2}+\left|\xi_j-\frac{u_j+u_{j+1}}{2}\right|\right]^2\|\phi''\|_1 \\ &\leq \left(1-\frac{\delta}{2}\right)^2\sum_{j=0}^{n-1}\frac{h_j^2}{2}\|\phi''\|_1\end{aligned} \tag{17}$$

where $\delta\in[0,1]$ *and* $u_j+\delta\frac{h_j}{2}\leq\xi_j\leq u_{j+1}-\delta\frac{h_j}{2}$. *The perturbed midpoint rule in the following holds:*

$$\int_x^y \phi(r)\,dr = M(\phi,\phi',I_n)+R_M(\phi,\phi',I_n) \tag{18}$$

where

$$M(\phi,\phi',I_n)=\sum_{j=0}^{n-1}h_j\phi\left(\frac{u_j+u_{j+1}}{2}\right) \tag{19}$$

and the remainder term $R_M(\phi,\phi',I_n)$ *satisfies the estimation:*

$$|R_M(\phi,\phi',I_n)|\leq\|\phi''\|_1\sum_{j=0}^{n-1}\frac{h_j^2}{8} \tag{20}$$

The perturbed trapezoidal rule is given by the following theorem (see [16]).

$$\int_x^y \phi(r)\,dr = T(\phi,\phi',I_n)+R_T(\phi,\phi',I_n) \tag{21}$$

where

$$T(\phi,\phi',I_n)=\frac{1}{2}\sum_{j=0}^{n-1}h_j(\phi(u_j)+\phi(u_{j+1}))-\frac{1}{8}\sum_{j=0}^{n-1}{h_j}^2(\phi'(u_{j+1})-\phi'(u_j)) \tag{22}$$

and the remainder term $R_T(\phi,\phi',I_n)$ *satisfies the estimation:*

$$|R_T(\phi,\phi',I_n)|\leq\sum_{j=0}^{n-1}\frac{h_j^2}{2}\|\phi''\|_1\,. \tag{23}$$

Theorem 4. *Let* $I\subseteq\mathbb{R}_0\to\mathbb{R}_+$ *be an* n*-times differentiable mapping on* I°, $x,y\in I^\circ$ *with* $x<y$, $a_0,a_1,\cdots,a_n\in\mathbb{R}$ *where* n *is even number. If* $\left|\phi^{(n)}\right|$ *is log-convex on* $[x,y]$, *then the following inequality holds:*

$$\left|\frac{1}{y-x}\int_x^y \phi(r)\,dr - \frac{\phi(x)+\phi(y)}{2} + \cdots\right.$$
$$-\frac{(y-x)^{(n-4)}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)\right]$$
$$+\frac{(y-x)^{(n-3)}\left[n.(n-1).a_n+\cdots+4.3a_4+3.2a_3+4a_2\right]}{2.n!.a_n}\left[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)\right]$$
$$-\frac{(y-x)^{(n-2)}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)\right]$$
$$\left.+\frac{(y-x)^{(n-1)}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)\right]\right|$$
$$\leq \frac{(y-x)^n}{2.n!.\left|a_n\right|}\left\{\left|\phi^{(n)}(y)\sum_{i=0}^{n}\left[-\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)\right]\right|\right.$$
$$\left.+\left|\phi^{(n)}(x)\sum_{i=0}^{n}\left[-\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)\right]\right|\right\}. \tag{24}$$

Proof. See [17] for proof.

Theorem 5. *Let $I \subseteq \mathbb{R}_0 \to \mathbb{R}_+$ be an n-times differentiable mapping on I°, $x, y \in I^\circ$ with $x < y$, $p > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, $a_0, a_1, \cdots, a_n \in \mathbb{R}$ and n is an even number. If $\left|\phi^{(n)}\right|^q$ is log-convex on the interval $[x, y]$, thus one obtains*

$$\left|\frac{1}{y-x}\int_x^y \phi(r)\,dr - \frac{\phi(x)+\phi(y)}{2} + \cdots\right.$$
$$-\frac{(y-x)^{(n-4)}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)\right]\Bigg)$$
$$+\frac{(y-x)^{(n-3)}\left[n.(n-1).a_n+\cdots+4.3a_4+3.2a_3+4a_2\right]}{2.n!.a_n}\left[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)\right]$$
$$-\frac{(y-x)^{(n-2)}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)\right]$$
$$\left.+\frac{(y-x)^{(n-1)}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)\right]\right|$$
$$\leq \frac{(y-x)^n}{2.n!.\left|a_n\right|}\left(\sum_{i=0}^{n}\frac{\left|a_i\right|}{(ip+1)^{\frac{1}{p}}}\right)\left\{\left[\left|\phi^{(n)}(y)\right|^q\frac{\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|^q-1}{q\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|}\right]^{\frac{1}{q}}+\left[\left|\phi^{(n)}(x)\right|^q\frac{\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|^q-1}{q\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|}\right]^{\frac{1}{q}}\right\}. \tag{25}$$

Proof. See [17] for proof.

Theorem 6. *Let $I \subseteq \mathbb{R}_0 \to \mathbb{R}_+$ be an n-times differentiable mapping on I°, $x, y \in I^\circ$ with $x < y$ and $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $a_0, a_1, \cdots, a_n \in \mathbb{R}$, where n is an even number. If the mapping $\left|\phi^{(n)}\right|^p$ is log-convex on $[x, y]$, then the following inequality holds:*

$$\left|\frac{1}{y-x}\int_x^y \phi(r)\,dr - \frac{\phi(x)+\phi(y)}{2} + \cdots \right.$$
$$\left. - \frac{(y-x)^{(n-4)}\left[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n}\left[\phi^{(n-4)}(x) + \phi^{(n-4)}(y)\right]\right)$$
$$+ \frac{(y-x)^{(n-3)}\left[n.(n-1).a_n + \cdots + 4.3a_4 + 3.2a_3 + 4a_2\right]}{2.n!.a_n}\left[\phi^{(n-3)}(y) - \phi^{(n-3)}(x)\right]$$
$$- \frac{(y-x)^{(n-2)}\left[n.a_n + \cdots + 2.a_2\right]}{2.n!.a_n}\left[\phi^{(n-2)}(x) + \phi^{(n-2)}(y)\right]$$
$$\left. + \frac{(y-x)^{(n-1)}\left[a_n + \cdots + a_1 + 2a_0\right]}{2.n!.a_n}\left[\phi^{(n-1)}(y) - \phi^{(n-1)}(x)\right]\right|$$
$$\leq \frac{(y-x)^n}{2.n!.\left|a_n\right|}\left[\sum_{i=0}^{n}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}$$
$$\times \left[\left|\phi^{(n)}(y)\right|\left(\sum_{i=0}^{n}|a_i|\left[-p\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(x)}\right|\right]^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -p\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)\right]\right)^{\frac{1}{p}}\right.$$
$$\left. + \left|\phi^{(n)}(x)\right|\left(\sum_{i=0}^{n}|a_i|\left[-p\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right]^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -p\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)\right]\right)^{\frac{1}{p}}\right]. \tag{26}$$

Proof. See [17] for proof.

3 Main Results

Corollary 1. *Under the assumptions of Theorem 4, for $n = 2$, we have the inequality*

$$\left|\frac{1}{y-x}\int_x^y \phi(r)\,dr - (\phi(x)+\phi(y)) + \frac{(y-x)\left[a_2 + a_1 + 2a_0\right]}{4a_2}\left[\phi'(y) - \phi'(x)\right]\right|$$
$$\leq \frac{(y-x)^2}{4.\left|a_2\right|}\left[\left|\phi''(y)\sum_{i=0}^{2}\left[-\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right]^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)\right]\right|\right.$$
$$\left. + \left|\phi''(x)\sum_{i=0}^{2}\left[-\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right]^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -\ln\left|\frac{\phi''(x)}{\phi''(x)}\right|\right)\right]\right|\right]. \tag{27}$$

Corollary 2. *Under the assumptions of Theorem 5, for $n = 2$, one obtains*

$$\left|\frac{1}{y-x}\int_x^y \phi(r)\,dr - (\phi(x)+\phi(y)) + \frac{(y-x)[a_2+a_1+2a_0]}{4a_2}[\phi'(y)-\phi'(x)]\right|$$
$$\leq \frac{(y-x)^2}{4.|a_2|}\left(\sum_{i=0}^{2}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right)\left\{\left[|\phi''(y)|^q\frac{\left|\frac{\phi''(x)}{\phi''(y)}\right|^q-1}{q\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|}\right]^{\frac{1}{q}}+\left[|\phi''(x)|^q\frac{\left|\frac{\phi''(y)}{\phi''(x)}\right|^q-1}{q\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|}\right]^{\frac{1}{q}}\right\}. \tag{28}$$

Corollary 3. *Under the assumptions of Theorem 6, for $n = 2$,*

$$\left|\frac{1}{y-x}\int_x^y \phi(r)\,dr - (\phi(x)+\phi(y)) + \frac{(y-x)[a_2+a_1+2a_0]}{4a_2}[\phi'(y)-\phi'(x)]\right|$$
$$\leq \frac{(y-x)^2}{4.|a_2|}\left[\sum_{i=0}^{2}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}$$
$$\times\left[|\phi''(y)|\left(\sum_{i=0}^{2}|a_i|\left[-p\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-p\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)\right]\right)^{\frac{1}{p}}\right.$$
$$\left.+|\phi''(y)|\left(\sum_{i=0}^{2}|a_i|\left[-p\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-p\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)\right]\right)^{\frac{1}{p}}\right] \tag{29}$$

is obtained.

4 Applications in Numerical Integration

Let d be a division of the interval $[x, y]$, i.e., $x = u_0 < u_1 < \cdots < u_{n-1} < u_n = y$, with $h_j = u_{j+1} - u_j$ for $j = 1, 2, 3, \cdots, n-1$. Consider perturbed trapezoidal rule given by:

$$\int_x^y \phi(r)\,dr = T(\phi,\phi',I_h) + \tilde{R}_T(\phi,\phi',I_h) \tag{30}$$

where

$$T(\phi,\phi',I_h) = \sum_{j=0}^{n-1}|\phi(u_j)+\phi(u_{j+1})|\,h_j + \frac{[a_2+a_1+2a_0]}{4.a_2}\sum_{j=0}^{n-1}[\phi'(u_{j+1})-\phi'(u_j)]\,h_j^2 \tag{31}$$

is the trapezoidal approximation and $\tilde{R}_T(\phi,\phi',I_h)$ is the associated error term.

Theorem 7. *Assume that $\phi : I = [x, y] \to (0, \infty)$ is a function with second-order derivatives on I° such that $\phi'' \in L^1([x, y])$, where $x, y \in I^\circ$ with $x < y$.*

If $|\phi''|$ is log-convex on $[x,y]$, then one obtains

$$\begin{aligned}
&\left|\tilde{R}_T\left(\phi,\phi',I_h\right)\right| \\
&\leq \frac{1}{4.\left|a_2\right|}\sum_{j=0}^{n-1}\left[\left|\phi''\left(u_{j+1}\right)\sum_{i=0}^{2}\left[-\ln\left|\frac{\phi''\left(u_j\right)}{\phi''\left(u_{j+1}\right)}\right|\right]^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-\ln\left|\frac{\phi''\left(u_j\right)}{\phi''\left(u_{j+1}\right)}\right|\right)\right]\right|\right. \\
&\left.+\left|\phi''\left(u_{j+1}\right)\sum_{i=0}^{2}\left[-\ln\left|\frac{\phi''\left(u_{j+1}\right)}{\phi''\left(u_j\right)}\right|\right]^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-\ln\left|\frac{\phi''\left(u_{j+1}\right)}{\phi''\left(u_j\right)}\right|\right)\right]\right|\right]
\end{aligned} \tag{32}$$

for every division d of $[x,y]$, i.e., $x=u_0<u_1<\cdots<u_{n-1}<u_n=y$.

Proof. Applying Corollary 1 on the subinterval $[u_j,u_{j+1}]$, $(j=1,2,3,\cdots,n-1)$ of the division d yields

$$\begin{aligned}
&\left|\tilde{R}_T\left(\phi,\phi',I_h\right)\right| \\
&=\left|\sum_{j=0}^{n-1}\left[\phi\left(u_j\right)+\phi\left(u_{j+1}\right)\right]h_j-\frac{a_2+a_1+2a_0}{4.a_2}\sum_{j=0}^{n-1}\left[\phi'\left(u_j\right)+\phi'\left(u_{j+1}\right)\right]h_j^2-\int_{u_j}^{u_{j+1}}\phi\left(t\right)dt\right| \\
&\leq\sum_{j=0}^{n-1}h_j\left|\left[\phi\left(u_j\right)+\phi\left(u_{j+1}\right)\right]-\frac{a_2+a_1+2a_0}{4.a_2}\left[\phi'\left(u_j\right)+\phi'\left(u_{j+1}\right)\right]h_j-\frac{1}{h_j}\int_{u_j}^{u_{j+1}}\phi\left(t\right)dt\right| \\
&\leq \frac{1}{4.\left|a_2\right|}\sum_{j=0}^{n-1}\left[\left|\phi''\left(u_{j+1}\right)\sum_{i=0}^{2}\left[-\ln\left|\frac{\phi''\left(u_j\right)}{\phi''\left(u_{j+1}\right)}\right|\right]^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-\ln\left|\frac{\phi''\left(u_j\right)}{\phi''\left(u_{j+1}\right)}\right|\right)\right]\right|\right. \\
&\left.+\left|\phi''\left(u_{j+1}\right)\sum_{i=0}^{2}\left[-\ln\left|\frac{\phi''\left(u_j\right)}{\phi''\left(u_{j+1}\right)}\right|\right]^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-\ln\left|\frac{\phi''\left(u_j\right)}{\phi''\left(u_{j+1}\right)}\right|\right)\right]\right|\right].
\end{aligned} \tag{33}$$

Theorem 8. *Assume that $\phi : I=[x,y]\to(0,\infty)$ is a function with second-order derivatives on I° such that $\phi''\in L\left([x,y]\right)$, where $x,y\in I^\circ$ with $x<y$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$. If $|\phi''|^q$ is log-convex on $[x,y]$, then one obtains*

$$\begin{aligned}
&\left|\tilde{R}_T\left(\phi,\phi',I_h\right)\right| \\
&\leq \frac{1}{4.\left|a_2\right|}\left(\sum_{i=0}^{2}\frac{\left|a_i\right|}{\left(ip+1\right)^{\frac{1}{p}}}\right) \\
&\times\sum_{j=0}^{n-1}\left\{\left[\left|\phi''\left(u_{j+1}\right)\right|^q\frac{\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|^q-1}{q\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|}\right]^{\frac{1}{q}}+\left[\left|\phi''\left(u_j\right)\right|^q\frac{\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|^q-1}{q\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|}\right]^{\frac{1}{q}}\right\}\times h_j^3
\end{aligned} \tag{34}$$

for every division d of $[x,y]$, i.e., $x=u_0<u_1<\cdots<u_{n-1}<u_n=y$.

Proof. Applying Corollary 2 on the subinterval $[u_j, u_{j+1}]$, $(j = 1, 2, 3, \cdots, n-1)$ of the division d gives

$$\begin{aligned}
&\left|\tilde{R}_T(\phi, \phi', I_h)\right| \\
&= \left|\sum_{j=0}^{n-1}\left[\phi(u_j) + \phi(u_{j+1})\right] h_j - \frac{a_2 + a_1 + 2a_0}{4.a_2}\sum_{j=0}^{n-1}\left[\phi'(u_j) + \phi'(u_{j+1})\right] h_j^2 - \int_{u_j}^{u_{j+1}} \phi(t)\, dt\right| \\
&\leq \sum_{j=0}^{n-1} h_j \left|\left[\phi(u_j) + \phi(u_{j+1})\right] - \frac{a_2 + a_1 + 2a_0}{4.a_2}\left[\phi'(u_j) + \phi'(u_{j+1})\right] h_j - \frac{1}{h_j}\int_{u_j}^{u_{j+1}} \phi(t)\, dt\right| \\
&\leq \frac{1}{4.\left|a_2\right|}\left(\sum_{i=0}^{2} \frac{\left|a_i\right|}{(ip+1)^{\frac{1}{p}}}\right) \\
&\times \sum_{j=0}^{n-1}\left\{\left[\left|\phi''(u_{j+1})\right|^q \frac{\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|^q - 1}{q\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|}\right]^{\frac{1}{q}} + \left[\left|\phi''(u_j)\right|^q \frac{\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|^q - 1}{q\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|}\right]^{\frac{1}{q}}\right\} \times h_j^3.
\end{aligned} \tag{35}$$

Theorem 9. *Assume that $\phi : I = [x, y] \to (0, \infty)$ is a function with second-order derivatives on I° such that $\phi'' \in L^1([x, y])$, where $x, y \in I^\circ$ with $x < y$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $\left|\phi''\right|^p$ is log-convex on $[x, y]$, then one obtains*

$$\begin{aligned}
&\left|\tilde{R}_T(\phi, \phi', I_h)\right| \\
&\leq \frac{1}{4.\left|a_2\right|}\left[\sum_{i=0}^{2}\frac{\left|a_i\right|}{i+1}\right]^{1-\frac{1}{p}} \\
&\times \left\{\left|\phi''(u_{j+1})\right| \sum_{j=0}^{n-1}\left(\sum_{i=0}^{2}\left|a_i\right|\left[-p\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right]^{-i-1}\left[\mathit{\Gamma}(i+1) - \mathit{\Gamma}\left(i+1, -p\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]\right)^{\frac{1}{p}}\right. \\
&\left. + \left|\phi''(u_j)\right|\left(\sum_{i=0}^{2}\left[p\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right]^{-i-1}\left[\mathit{\Gamma}(i+1) - \mathit{\Gamma}\left(i+1, p\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right)^{\frac{1}{p}}\right\} .h_j^3
\end{aligned} \tag{36}$$

for every division d of $[x, y]$, i.e., $x = u_0 < u_1 < \cdots < u_{n-1} < u_n = y$.

Proof. Applying Corollary 3 on the subinterval $[u_j, u_{j+1}]$, $(j = 1, 2, 3, \cdots, n-1)$ of the division d yields

$$\begin{aligned}
&\left|\tilde{R}_T\left(\phi, \phi', I_h\right)\right| \\
&= \left|\sum_{j=0}^{n-1}\left[\phi\left(u_j\right)+\phi\left(u_{j+1}\right)\right] h_j - \frac{a_2+a_1+2a_0}{4.a_2}\sum_{j=0}^{n-1}\left[\phi'\left(u_j\right)+\phi'\left(u_{j+1}\right)\right] h_j^2 - \int_{u_j}^{u_{j+1}} \phi(t)\, dt\right| \\
&\leq \sum_{j=0}^{n-1} h_j \left|\left[\phi\left(u_j\right)+\phi\left(u_{j+1}\right)\right] - \frac{a_2+a_1+2a_0}{4.a_2}\left[\phi'\left(u_j\right)+\phi'\left(u_{j+1}\right)\right] h_j - \frac{1}{h_j}\int_{u_j}^{u_{j+1}} \phi(t)\, dt\right| \\
&\leq \frac{1}{4.\left|a_2\right|}\left[\sum_{i=0}^{2}\frac{\left|a_i\right|}{i+1}\right]^{1-\frac{1}{p}} \\
&\times \left\{\left|\phi''\left(u_{j+1}\right)\right| \sum_{j=0}^{n-1}\left(\sum_{i=0}^{2}\left|a_i\right|\left[-p\ln\left|\frac{\phi''\left(u_j\right)}{\phi''\left(u_{j+1}\right)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1, -p\ln\left|\frac{\phi''\left(u_j\right)}{\phi''\left(u_{j+1}\right)}\right|\right)\right]\right)^{\frac{1}{p}}\right. \\
&\left. + \left|\phi''\left(u_j\right)\right|\left(\sum_{i=0}^{2}\left[p\ln\left|\frac{\phi''\left(u_{j+1}\right)}{\phi''\left(u_j\right)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1, p\ln\left|\frac{\phi''\left(u_{j+1}\right)}{\phi''\left(u_j\right)}\right|\right)\right]\right)^{\frac{1}{p}}\right\}.h_j^3.
\end{aligned} \tag{37}$$

5 Applications to Special Means

In this section, we will apply the inequalities obtained for the *log*-convex functions presented in [17].

Proposition 2. *Let $x, y \in \mathbb{R}$ and $0 < x < y$, $n \in \mathbb{N}$ where $n > 2$ and n is an even number. Then, the inequality in the following holds*

$$\left|\frac{1}{4}\left(\frac{y^2\ln y-x^2\ln x}{y-x}\right)-\frac{A\left(x,y\right)}{4}+A\left(x\ln\sqrt{x},y\ln\sqrt{y}\right)+\cdots\right.$$
$$+\frac{\left(-1\right)^{n-3}\left(y-x\right)^{n-4}\left(n-6\right)!\left[n.\left(n-1\right).\left(n-2\right).a_n+\cdots+4.3.2.a_4\right]}{4.n!.a_n.H\left(x^{n-5},y^{n-5}\right)}$$
$$+\frac{\left(-1\right)^{n-2}\left(y-x\right)^{n-3}\left(n-5\right)!\left[n.\left(n-1\right).a_n+\cdots+4.3.a_4+3.2.a_3+4.a_2\right]}{4.n!.a_n.H\left(x^{n-4},-y^{n-4}\right)}$$
$$+\frac{\left(-1\right)^{n-1}\left(y-x\right)^{n-2}\left(n-4\right)!\left[n.a_n+\cdots+2.a_2\right]}{4.n!.a_n.H\left(x^{n-3},y^{n-3}\right)}$$
$$\left.+\frac{\left(-1\right)^{n}\left(y-x\right)^{n-1}\left(n-3\right)!\left[a_n+\cdots+a_1+2.a_0\right]}{4.n!.a_n.H\left(x^{n-2},-y^{n-2}\right)}\right|$$
$$\leq\frac{\left(y-x\right)^n}{2.n!.\left|a_n\right|}\left[\left|\left(\frac{\left(-1\right)^n\left(n-2\right)!}{2y^{n-1}}\right)\sum_{i=0}^{n}\left[-\left(n-1\right)\ln\left(\frac{y}{x}\right)\right]^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)\right]\right|\right.$$
$$\left.+\left|\left(\frac{\left(-1\right)^n\left(n-2\right)!}{2x^{n-1}}\right)\sum_{i=0}^{n}\left[\left(n-1\right)\ln\left(\frac{x}{y}\right)\right]^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)\right]\right|\right].\tag{38}$$

where L, A and H represent the logarithmically, arithmetic and harmonic means, respectively.

Proof. The proof is obtained from Theorem 4 such that $\phi\left(u\right)=u\ln\sqrt{u}$, $u\in\left(0,\infty\right)$.

Proposition 3. *Let $x,y\in\mathbb{R}$ with $0<x<y$, $\frac{1}{p}+\frac{1}{q}=1$, $\forall p,q>1$, $n\in\mathbb{N}$ where $n>2$ and n is an even number. Then, one obtains*

$$\left|\frac{1}{4}\left(\frac{y^2\ln y-x^2\ln x}{y-x}\right)-\frac{A\left(x,y\right)}{4}+A\left(x\ln\sqrt{x},y\ln\sqrt{y}\right)+\cdots\right.$$
$$+\frac{\left(-1\right)^{n-3}\left(y-x\right)^{n-4}\left(n-6\right)!\left[n.\left(n-1\right).\left(n-2\right).a_n+\cdots+4.3.2.a_4\right]}{4.n!.a_n.H\left(x^{n-5},y^{n-5}\right)}$$
$$+\frac{\left(-1\right)^{n-2}\left(y-x\right)^{n-3}\left(n-5\right)!\left[n.\left(n-1\right).a_n+\cdots+4.3.a_4+3.2.a_3+4.a_2\right]}{4.n!.a_n.H\left(x^{n-4},-y^{n-4}\right)}$$
$$+\frac{\left(-1\right)^{n-1}\left(y-x\right)^{n-2}\left(n-4\right)!\left[n.a_n+\cdots+2.a_2\right]}{4.n!.a_n.H\left(x^{n-3},y^{n-3}\right)}$$
$$\left.+\frac{\left(-1\right)^{n}\left(y-x\right)^{n-1}\left(n-3\right)!\left[a_n+\cdots+a_1+2.a_0\right]}{4.n!.a_n.H\left(x^{n-2},-y^{n-2}\right)}\right|$$
$$\leq\frac{\left(y-x\right)^n}{2.n!.\left|a_n\right|}\left(\sum_{i=0}^{n}\frac{\left|a_i\right|}{\left(ip+1\right)^{\frac{1}{p}}}\right)\left(\frac{\left(n-2\right)!}{\left(xy\right)^{n-1}}\right)\left[L\left(x^{q(n-1)},y^{q(n-1)}\right)\right]^{\frac{1}{q}}\tag{39}$$

where L represents the logarithmically mean.

Proof. The proof follows from Theorem 5 with $\phi(u) = u \ln \sqrt{u}$, $u \in (0, \infty)$.

Proposition 4. *Let $x, y \in \mathbb{R}$, $0 < x < y$, $\frac{1}{p} + \frac{1}{q} = 1$, $\forall p, q > 1$, $n \in \mathbb{N}$ where $n > 2$ and n is an even number. Then, the inequality in the following holds*

$$\left| \frac{1}{4} \left(\frac{y^2 \ln y - x^2 \ln x}{y - x} \right) - \frac{A(x,y)}{4} + A\left(x \ln \sqrt{x}, y \ln \sqrt{y}\right) + \cdots \right.$$
$$+ \frac{(-1)^{n-3} (y-x)^{n-4} (n-6)! \left[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4\right]}{4.n!.a_n.H\left(x^{n-5}, y^{n-5}\right)}$$
$$+ \frac{(-1)^{n-2} (y-x)^{n-3} (n-5)! \left[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 4.a_2\right]}{4.n!.a_n.H\left(x^{n-4}, -y^{n-4}\right)}$$
$$+ \frac{(-1)^{n-1} (y-x)^{n-2} (n-4)! \left[n.a_n + \cdots + 2.a_2\right]}{4.n!.a_n.H\left(x^{n-3}, y^{n-3}\right)}$$
$$\left. + \frac{(-1)^{n} (y-x)^{n-1} (n-3)! \left[a_n + \cdots + a_1 + 2.a_0\right]}{4.n!.a_n.H\left(x^{n-2}, -y^{n-2}\right)} \right|$$
$$\leq \frac{(y-x)^n}{2.n!.\left|a_n\right|} \left[\sum_{i=0}^{n} \frac{|a_i|}{i+1} \right]^{1-\frac{1}{p}}$$
$$\times \left[\frac{(n-2)!}{2y^{n-1}} \left(\sum_{i=0}^{n} |a_i| \left[-p(n-1) \ln\left(\frac{y}{x}\right) \right]^{-i-1} \left[\Gamma(i+1) - \Gamma\left(i+1, -p(n-1)\ln\left(\frac{y}{x}\right)\right) \right] \right)^{\frac{1}{p}} \right.$$
$$\left. + \frac{(n-2)!}{2y^{n-1}} \left(\sum_{i=0}^{n} |a_i| \left[-p(n-1) \ln\left(\frac{y}{x}\right) \right]^{-i-1} \left[\Gamma(i+1) - \Gamma\left(i+1, -p(n-1)\ln\left(\frac{y}{x}\right)\right) \right] \right)^{\frac{1}{p}} \right]. \tag{40}$$

Proof. The proof follows from Theorem 6 with $\phi(u) = u \ln \sqrt{u}$, $u \in (0, \infty)$.

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