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Some Applications of $S - Log-$Convex Functions in Numerical Integration

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Abstract. s-logarithmically convex functions are an extension of the concept of logarithmically convex functions, which play an important role in analysis and optimization. s-logarithmically convex functions present a flexible framework to generalize traditional convex and log-convex functions. The parameter, s determines the degree and type of convexity, allowing for a more refined understanding of various phenomena. They provide more generalized bounds in various inequalities, including perturbed trapezoidal inequality, Jensen's inequality and Hölder's inequality. s-logarithmic convex functions are particularly useful in numerical integration due to their smooth and well-behaved nature. This study focuses on the role of s-logarithmically functions in determining the upper limit for error in numerical integration. The smoothness and well-behaved derivatives of s-logarithmically convex functions facilitate error analysis in numerical integration.

Keywords: $S - log-$convex Function, Perturbed Trapezoid Inequality, Numerical Integration.

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1 Introduction

Convex and logarithmic convex functions are fundamental concepts that have an important place in the field of analysis and optimization. These functions are among the frequently used tools when considering various mathematical problems and inequalities. However, s - logarithmically convex functions have been defined in order to transcend classical convexity concepts and present a more general structure. These functions appear as an extension of logarithmically convex functions and present a more flexible structure with a parameter, s, that determines the degree and type of convexity. s - logarithmically convex functions allow more general bounds to be obtained in various mathematical inequalities such as the perturbed trapezoid inequality, Jensen's inequality and Hölder's inequality. This flexibility presents a wide application potential not only in theoretical analysis but also in the fields of numerical computation and error analysis. Particularly, in numerical integration problems, the smooth and well-behaved derivatives of s -logarithmically convex functions become prominent as a powerful means in error analysis. s -logarithmically convex functions are an important extension of classical convexity, presenting greater flexibility in analysis and applications across mathematical, statistical, and economic domains. The parameter, s determines the degree and type of convexity, allowing for a more refined understanding of various phenomena. s -logarithmically convex functions are functions that generalize the classical concept of logarithmic convexity and have a wide range of applications in areas such as mathematical analysis, optimization, complex analysis and probability theory. These functions have performed significant contributions, especially in the derivation of integral and operator inequalities and in the study of geometric properties of analytic functions. These functions, which are controlled by the s - parameter in the literature, have provided a basis the way for important studies in geometric function theory, such as obtaining sharp boundaries on convex functions.

Wang and Liu [1] presented s-logarithmically preinvex functions via integral identity involving an n-times differentiable function. They present some new Hermite-Hadamard type inequalities for s-logarithmically preinvex functions. Xi and Qi [2] introduce the concept of s-logarithmically convex functions and construct some new Hermite-Hadamard type integral inequalities of these functions. The authors also obtain practical results by applying these new inequalities to various mean values. Latif and Drogomir [3] derive more general versions of the Hermite-Hadamard type inequalities using s-logarithmically convex functions. This allows obtaining inequalities valid for a wider class of functions, based on the assumption that the absolute values of the nth derivatives are s-logarithmically convex. Besides, they [4] have studied and obtained results for the Hermite-Hadamard inequalities related to the n-th derivatives of the s-logarithmically convex functions. They have also derived a version of well-known integral inequalities such as the classical trapezoid and the classical midpoint inequalities based on s-logarithmically convex functions. Zhang et al. [5] improved new integral inequalities of the Hermite-Hadamard type for products of s-logarithmic convex functions. They also obtained new results by applying these inequalities to a more general class, namely, products of s-logarithmically convex functions. In this study, the role of s-logarithmic convex functions on error analysis in numerical integration is examined and how upper error limits are determined using these functions is discussed. In this context, the advantages of s-logarithmically convex functions in numerical integration is introduced with examples and a new perspective is presented to the literature.

2 Preliminaries

A function $\phi : I^{\circ} \subset \mathbb{R} \rightarrow \mathbb{R}$ is convex on I if the inequality

$$\phi\left(rx+(1-r)\,y\right) \leq r\phi\left(x\right)+(1-r)\,\phi\left(y\right) \tag{1}$$

holds for all $x, y \in I$ and $r \in [0,1]$. It is said that ϕ is concave if $(-\phi)$ is convex. For numerical integration, trapezoid inequality is introduced as

$$\left|\int_x^y \phi\left(u\right)du-\frac{1}{2}\left(y-x\right)\left(\phi\left(x\right)+\phi\left(y\right)\right)\right| \leq \frac{1}{12}M_2\left(y-x\right)^3 \tag{2}$$

where $\phi : [x, y] \rightarrow \mathbb{R}$ is assumed to be twice differentiable on (x, y) with the second derivative bounded on (x, y) by $M_2 = sup_{u\in(x,y)}\left|\phi''\left(u\right)\right| < +\infty$ [6].

Definition 1. *[7] A positive function ϕ is called log-convex on a real interval $I = [a, b]$, if*

$$\phi\left(rx+(1-r)\,y\right) \leq \phi\left(x\right)^r\phi\left(y\right)^{1-r} \tag{3}$$

for all $x, y \in [a, b] \subset \mathbb{R}_+$ and $r \in [0, 1]$. If ϕ is a positive $log-$concave function, then it is reversed. Also, if $\phi > 0$ and ϕ'' exists on I, then ϕ is $log-$convex if and only if $\phi.\phi'' - (\phi')^2 \geq 0$.

Definition 2. *[8] The arithmetic-geometric mean inequality is*

$$\phi^r\left(x\right)\phi^{1-r}\left(y\right) \leq r\phi\left(x\right)+(1-r)\,\phi\left(y\right) \tag{4}$$

for all $x, y \in I$ and $r \in [0, 1]$. If the above inequality (3) is reversed, then ϕ is called logarithmically concave or simply log-concave .

Definition 3. *[9] A function $\phi : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$ is said to be s-logarithmically convex in the first sense if*

$$\phi\left(\alpha x+\beta y\right) \leq \left[\phi\left(x\right)\right]^{\alpha^s}\left[\phi\left(y\right)\right]^{\beta^s} \tag{5}$$

for some $s \in (0, 1]$, where $x, y, \alpha, \beta \in I$ and $\alpha^s + \beta^s = 1$.

Definition 4. *[10] A function $\phi : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$ is said to be s - logarithmically convex in the second sense if*

$$\phi\left(rx+(1-r)\,y\right) \leq \left[\phi\left(x\right)\right]^{r^s}\left[\phi\left(y\right)\right]^{(1-r)^s} \tag{6}$$

for some $s \in (0, 1]$, where $x, y \in I$ and $r \in [0, 1]$.

Theorem 1. *[9] Let $\phi : (x, y) \to \mathbb{R}$ be continuous on (x, y) and twice differentiable on (x, y) and assume that the second derivative $\phi'' : (x, y) \to \mathbb{R}$ satisfies the condition*

$$\upsilon \leq \phi'' \leq \varphi$$

for all $u \in (x, y)$. Thus, we have the inequality

$$\left|\phi(u) - \left(u - \frac{x+y}{2}\right)\phi'(u) + \left[\frac{(y-x)^2}{24} + \frac{1}{2}\left(u - \frac{x+y}{2}\right)^2\right]\frac{\phi'(y) - \phi'(x)}{y-x} - \frac{1}{y-x}\int_x^y \phi(r)\,dr\right|$$
$$\leq \frac{1}{8}(\varphi - \upsilon)\left[\frac{1}{2}(y-x) + \left|u - \frac{x+y}{2}\right|\right]^2$$

for all $u \in (x, y)$ where the perturbed midpoint inequality:

$$\left|\phi\left(\tfrac{x+y}{2}\right) + \tfrac{1}{24}(y-x)(\phi'(y) - \phi'(x)) - \tfrac{1}{y-x}\int_x^y \phi(r)\,dr\right| \leq \tfrac{1}{32}(\varphi - \upsilon)(y-x)^2$$

and perturbed trapezoid inequality is

$$\left|\frac{\phi(x) + \phi(y)}{2} - \frac{1}{12}(y-x)(\phi'(y) - \phi'(x)) - \frac{1}{y-x}\int_x^y \phi(r)\,dr\right| \leq \frac{1}{8}(\varphi - \upsilon)(y-x)^2.$$

2.1 Application to the midpoint and trapezoidal formula

Let d be a division of the interval $[x, y]$, i.e. $x = u_0 < u_1 < \cdots < u_{n-1} < u_n = y$ and consider the quadrature formula

$$\int_x^y \phi(r)\,dr = T(\phi, d) + E(\phi)$$

and

$$\int_x^y \phi(r)\,dr = T'(\phi, d) + E'(\phi)$$

where

$$T(\phi, d) = \sum_{j=0}^{n-1}(u_{j+1} - u_j)\,\phi\left(\tfrac{u_j + u_{j+1}}{2}\right)$$

and

$$T'(\phi, d) = \sum_{j=0}^{n-1}(u_{j+1} - u_j)\left(\tfrac{\phi(u_j) + \phi(u_{j+1})}{2}\right)$$

are the midpoint and trapezoidal versions of the associated errors, $E(\phi, d)$, $E'(\phi, d)$, respectively where we derive some error estimates for the sum of midpoint and trapezoidal formula.

Proposition 1. *[11] Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $\phi' \in L([x,y])$, where $x, y \in I^{\circ}$ with $x < y$. If $|\phi'|$ is convex on $[x,y]$, then:*

$$|E(\phi,d) + E'(\phi,d)| \leq \tfrac{1}{8}\textstyle\sum_{j=0}^{n-1}(u_{j+1}-u_j)^2\left[|\phi'(u_j)| + |\phi'(u_{j+1})|\right]$$

for every division d of $[x,y]$.

Theorem 2. *[11] Let $s \in (0,1]$ and $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $\phi'' \in L[x,y]$, where $x, y \in I^{\circ}$ with $x < y$. If $|\phi''|^q$ is s-convex on $[x,y]$ for $q \geq 1$, then we have*

$$\begin{aligned}
&|E(\phi,d)| \leq \frac{1}{2^{4+\frac{s}{q}}}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\\
&\left\{\sum_{j=0}^{n-1}\left(\frac{s+4}{(s+2)(s+3)}|\phi''(u_j)|^q + \left(\frac{2}{3} + \frac{2}{(s+1)(s+3)}\beta(s+1,n)\right)|\phi''(u_{j+1})|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{s+4}{(s+2)(s+3)}|\phi''(u_{j+1})|^q + \left(\frac{2}{3} + \frac{2}{(s+1)(s+3)}\beta(s+1,n)\right)|\phi''(u_j)|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

for every division d of $[x,y]$.

Let $I_n : x = u_1 < u_2 < \cdots < u_{n-1} < u_n = y$ be a division of the interval $[x,y]$, $\xi_j \in \left[u_j + \delta\frac{h_j}{2}, u_{j+1} - \delta\frac{h_j}{2}\right]$; $j = 0, 1, \cdots, n-1$ a sequence of intermediate points and $h_j = u_{j+1} - u_j$; $j = 0, 1, \cdots, n-1$, then, the quadrature rule is given by theorem in the following.

Theorem 3. *[11] Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable on (x,y) whose second derivative and $\phi'' : (x,y) \to \mathbb{R}$ belongs to $L_1(x,y)$ i.e $||\phi''||_1 := \int_x^y ||\phi''||\, dr < \infty$. Then, the perturbed Riemann's quadrature formula holds:*

$$\textstyle\int_x^y \phi(r)\, dr = A(\phi, \phi', I_n, \xi, \delta) + R(\phi, \phi', I_n, \xi, \delta)$$

where

$$\begin{aligned}
A(\phi, \phi', I_n, \xi, \delta) &= (1-\delta)\sum_{j=0}^{n-1} h_j \phi(\xi_j)\\
&\quad - (1-\delta)\sum_{j=0}^{n-1} h_j\left(\xi_j - \frac{u_j + u_{j+1}}{2}\right)\phi'(\xi_j)\\
&\quad + \frac{\delta}{2}\sum_{j=0}^{n-1} h_j\left(\phi(u_j) + \phi(u_{j+1})\right)\\
&\quad - \frac{\delta^2}{2}\sum_{j=0}^{n-1} h_j^2\left(\phi'(u_{j+1}) - \phi'(u_j)\right)
\end{aligned}$$

and the remainder $R(\phi,\phi',I_n,\xi,\delta)$ *satisfies the estimation*

$$\begin{aligned}|R(\phi,\phi',I_n,\xi,\delta)| &\le \tfrac{1}{2}\textstyle\sum_{j=0}^{n-1}\left[\frac{h_j(1-\delta)}{2}+\left|\xi_j-\frac{u_j+u_{j+1}}{2}\right|\right]^2 \|\phi''\|_1\\ &\le \left(1-\tfrac{\delta}{2}\right)^2\textstyle\sum_{j=0}^{n-1}\frac{h_j^2}{2}\|\phi''\|_1\end{aligned}$$

where $\delta\in[0,1]$ *and* $u_j+\delta\frac{h_j}{2}\le\xi_j\le u_{j+1}-\delta\frac{h_j}{2}$. *The following perturbed midpoint rule holds:*

$$\int_x^y \phi(r)\,dr = M(\phi,\phi',I_n)+R_M(\phi,\phi',I_n)$$

where

$$M(\phi,\phi',I_n)=\textstyle\sum_{j=0}^{n-1}h_j\phi\left(\frac{u_j+u_{j+1}}{2}\right)$$

and the remainder term $R_M(\phi,\phi',I_n)$ *satisfies the estimation:*

$$|R_M(\phi,\phi',I_n)|\le\|\phi''\|_1\textstyle\sum_{j=0}^{n-1}\frac{h_j^2}{8}$$

The following perturbed trapezoidal rule holds:

$$\textstyle\int_x^y\phi(r)\,dr=T(\phi,\phi',I_n)+R_T(\phi,\phi',I_n)$$

where

$$\begin{aligned}T(\phi,\phi',I_n)=&\frac{1}{2}\sum_{j=0}^{n-1}h_j(\phi(u_j)+\phi(u_{j+1}))\\&-\frac{1}{8}\sum_{j=0}^{n-1}h_j{}^2(\phi'(u_{j+1})-\phi'(u_j))\end{aligned}$$

and the residual term

$$|R_T(\phi,\phi',I_n)|\le\sum_{j=0}^{n-1}\frac{h_j^2}{2}\|\phi''\|_1\,.$$

Theorem 4. *Let* $s\in(0,1]$ *and* $\phi:I\subset\mathbb{R}_0\to\mathbb{R}_+$ *be* n *times differentiable mapping on* I°, $x,y\in I^\circ$ *with* $x<y$ *where* n *is even number. If* $\left|\phi^{(n)}\right|$ *is* $s-$*log-convex on* $[x,y]$, *then the inequality in the following holds:*

$$\begin{aligned}
&\Big|\frac{1}{y-x}\int_x^y \phi(r)\,dr-\frac{\phi(x)+\phi(y)}{2}+\cdots\\
&-\frac{(y-x)^{(n-4)}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)\right]\\
&+\frac{(y-x)^{(n-3)}\left[n.(n-1).a_n+\cdots+4.3a_4+3.2a_3+4a_2\right]}{2.n!.a_n}\left[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)\right]\\
&-\frac{(y-x)^{(n-2)}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)\right]\\
&+\frac{(y-x)^{(n-1)}[a_n+\cdots+a_1+2a_0]}{2.n!.a_n}[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)]\Big|\\
&\leq\frac{(y-x)^n}{2.n!.\left|a_n\right|}
\end{aligned}$$

$$\times\begin{cases}
\Big[\left|\phi^{(n)}(y)\right|^s\sum_{i=0}^n|a_i|\left(-s\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)\right]\\
+\left|\phi^{(n)}(x)\right|^s\sum_{i=0}^n|a_i|\left(-s\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(y)}\right|\right)\right]\Big]\\
\textit{where}\qquad 0<\left|\phi^{(n)}(x)\right|,\left|\phi^{(n)}(y)\right|\leq 1\\
\Big[\left|\phi^{(n)}(y)\right|\sum_{i=0}^n|a_i|\left(-s\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)\right]\\
+\left|\phi^{(n)}(y)\right|^{1-s}\left|\phi^{(n)}(x)\right|^s\sum_{i=0}^n|a_i|\left(-s\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)^{-i-1}\\
\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)\right]\Big]\qquad \textit{where}\qquad 0<\left|\phi^{(n)}(x)\right|\leq 1<\left|\phi^{(n)}(y)\right|\\
\Big[\left|\phi^{(n)}(x)\right|^{1-s}\left|\phi^{(n)}(y)\right|\sum_{i=0}^n|a_i|\left(-s\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)\right]\\
+\left|\phi^{(n)}(x)\right|\left|\phi^{(n)}(y)\right|^{1-s}\sum_{i=0}^n|a_i|\left(-s\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)\right]\Big]\\
\qquad\textit{where}\qquad 1<\left|\phi^{(n)}(x)\right|,\left|\phi^{(n)}(y)\right|
\end{cases}$$

Proof. See [12] for proof.

Theorem 5. *Let $s\in(0,1]$ and $\phi: I\subset\mathbb{R}_0\to\mathbb{R}_+$ be n times differentiable mapping on I°, $x,y\in I^\circ$ with $x<y$, $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$ where n is even number. If $\left|\phi^{(n)}\right|^q$ is $s-log-$convex on the interval $[x,y]$, thus one obtains*

$$\Big|\frac{1}{y-x}\int_x^y \phi(r)\,dr - \frac{\phi(x)+\phi(y)}{2} + \cdots$$
$$- \frac{(y-x)^{(n-4)}\left[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n}\left[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)\right]$$
$$+ \frac{(y-x)^{(n-3)}\left[n.(n-1).a_n + \cdots + 4.3a_4 + 3.2a_3 + 4a_2\right]}{2.n!.a_n}\left[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)\right]$$
$$- \frac{(y-x)^{(n-2)}\left[n.a_n + \cdots + 2.a_2\right]}{2.n!.a_n}\left[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)\right]$$
$$+ \frac{(y-x)^{(n-1)}\left[a_n + \cdots + a_1 + 2a_0\right]}{2.n!.a_n}\left[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)\right]\Big|$$
$$\leq \frac{(y-x)^n}{2.n!.\left|a_n\right|}\sum_{i=0}^{n}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}$$

$$\times\left\{\begin{array}{l}
\left[\left|\phi^{(n)}(x)\right|^s\left(\frac{\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|^{sq}-1}{sq\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|}\right)^{\frac{1}{q}} + \left|\phi^{(n)}(y)\right|^s\left(\frac{\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|^{sq}-1}{sq\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|}\right)^{\frac{1}{q}}\right] \\
\quad \text{where} \quad 0<\left|\phi^{(n)}(x)\right|,\left|\phi^{(n)}(y)\right|\leq 1 \\
\left[\left|\phi^{(n)}(x)\right|^s\left|\phi^{(n)}(y)\right|^{1-s}\left(\frac{\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|^{sq}-1}{sq\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|}\right)^{\frac{1}{q}} + \left|\phi^{(n)}(y)\right|\left(\frac{\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|^{sq}-1}{sq\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|}\right)^{\frac{1}{q}}\right] \\
\quad \text{where} \quad 0<\left|\phi^{(n)}(x)\right|\leq 1<\left|\phi^{(n)}(y)\right| \\
\left[\left|\phi^{(n)}(x)\right|\left|\phi^{(n)}(y)\right|^{1-s}\left(\frac{\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|^{sq}-1}{sq\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|}\right)^{\frac{1}{q}} + \left|\phi^{(n)}(x)\right|^{1-s}\left|\phi^{(n)}(y)\right|\left(\frac{\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|^{sq}-1}{sq\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|}\right)^{\frac{1}{q}}\right] \\
\quad \text{where} \quad 1<\left|\phi^{(n)}(x)\right|,\left|\phi^{(n)}(y)\right|
\end{array}\right.$$

Proof. See [12] for proof.

Theorem 6. *Let $s\in(0,1]$ and $\phi : I\subset\mathbb{R}_0\to\mathbf{R}_+$ be n times differentiable mapping on I°, $x,y\in I^\circ$ with $x<y$ and $p>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, where n is even number. If the mapping $\left|\phi^{(n)}\right|^p$ is log-convex on $[x,y]$, thus the inequality in the following holds:*

$$\Big|\frac{1}{y-x}\int_x^y \phi(r)\,dr - \frac{\phi(x)+\phi(y)}{2} + \cdots$$
$$- \frac{(y-x)^{(n-4)}[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4]}{2.n!.a_n}\left[\phi^{(n-4)}(x) + \phi^{(n-4)}(y)\right]$$
$$+ \frac{(y-x)^{(n-3)}[n.(n-1).a_n + \cdots + 4.3a_4 + 3.2a_3 + 4a_2]}{2.n!.a_n}\left[\phi^{(n-3)}(y) - \phi^{(n-3)}(x)\right]$$
$$- \frac{(y-x)^{(n-2)}[n.a_n + \cdots + 2.a_2]}{2.n!.a_n}\left[\phi^{(n-2)}(x) + \phi^{(n-2)}(y)\right]$$
$$+ \frac{(y-x)^{(n-1)}[a_n + \cdots + a_1 + 2a_0]}{2.n!.a_n}\left[\phi^{(n-1)}(y) - \phi^{(n-1)}(x)\right]\Big|$$
$$+ \frac{(y-x)^{(n-1)}[a_n + \cdots + a_1 + 2a_0]}{2.n!.a_n}\left[\phi^{(n-1)}(y) - \phi^{(n-1)}(x)\right]\Big|$$
$$\le \frac{(y-x)^n}{2.n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}$$

$$\times \begin{cases} \Bigg[\left|\phi^{(n)}(y)\right|^s \left(\sum_{i=0}^n |a_i| \left(-sp\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -sp\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)\right]\right)^{\frac{1}{p}} \\ \quad + \left|\phi^{(n)}(x)\right|^s \left(\sum_{i=0}^n |a_i| \left(-sp\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -sp\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)\right]\right)^{\frac{1}{p}}\Bigg] \\ \qquad \textit{where} \quad 0 < \left|\phi^{(n)}(x)\right|, \left|\phi^{(n)}(y)\right| \le 1 \\ \Bigg[\left|\phi^{(n)}(y)\right| \left(\sum_{i=0}^n |a_i| \left(-sp\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -sp\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)\right]\right)^{\frac{1}{p}} \\ \quad + \left|\phi^{(n)}(x)\right|^s \left|\phi^{(n)}(y)\right|^{1-s} \left(\sum_{j=0}^n |a_i| \left(-sp\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -sp\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)\right]\right)^{\frac{1}{p}}\Bigg] \\ \qquad \textit{where} \quad 0 < \left|\phi^{(n)}(x)\right| \le 1 < \left|\phi^{(n)}(y)\right| \\ \Bigg[\left|\phi^{(n)}(y)\right| \left(\sum_{i=0}^n |a_i| \left(-sp\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -sp\ln\left|\frac{\phi^{(n)}(x)}{\phi^{(n)}(y)}\right|\right)\right]\right)^{\frac{1}{p}} \\ \quad + \left|\phi^{(n)}(x)\right|^{1-s} \left|\phi^{(n)}(y)\right| \left(\sum_{i=0}^n |a_i| \left(-sp\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)^{-i-1}\left[\Gamma(i+1) - \Gamma\left(i+1, -sp\ln\left|\frac{\phi^{(n)}(y)}{\phi^{(n)}(x)}\right|\right)\right]\right)^{\frac{1}{p}}\Bigg] \\ \qquad \textit{where} \quad 1 < \left|\phi^{(n)}(x)\right|, \left|\phi^{(n)}(y)\right| \end{cases}$$

Proof. See [12] for proof.

3 Main Results

Corollary 1. *Under the assumptions of Theorem 4 and $n = 2$, we have the following inequality:*

$$\left|\frac{1}{y-x}\int_x^y \phi(u)\,du - (\phi(x)+\phi(y)) + \frac{(y-x)[a_2+a1+2a_0]}{4.a_2}[\phi'(y)-\phi'(x)]\right| \le \frac{(y-x)^2}{4.|a_2|}$$

$$\times \begin{cases} \Big[|\phi''(y)|^s \sum_{i=0}^2 |a_i| \left(-s\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)\right] \\ +|\phi''(x)|^s \sum_{i=0}^2 |a_i| \left(-s\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)\right]\Big] \\ \quad \text{where} \quad 0<|\phi''(x)|,|\phi''(y)|\le 1 \\ \Big[|\phi''(y)| \sum_{i=0}^2 |a_i| \left(-s\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)\right] \\ +|\phi''(y)|^{1-s}|\phi''(x)|^s \sum_{i=0}^2 |a_i| \left(-s\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)\right]\Big] \\ \quad \text{where} \quad 0<|\phi''(x)|\le 1<|\phi''(y)| \\ \Big[|\phi''(x)|^{1-s}|\phi''(y)| \sum_{i=0}^2 |a_i| \left(-s\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)\right] \\ +|\phi''(x)||\phi''(y)|^{1-s} \sum_{i=0}^2 |a_i| \left(-s\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)\right]\Big] \\ \quad \text{where} \quad 1<|\phi''(x)|,|\phi''(y)| \end{cases}$$

Corollary 2. *Under the assumptions of Theorem 5 and $n = 2$, we have the following inequality:*

$$\left|\frac{1}{y-x}\int_x^y \phi(u)\,du - (\phi(x)+\phi(y)) + \frac{(y-x)[a_2+a1+2a_0]}{4.a_2}[\phi'(y)-\phi'(x)]\right| \le \frac{(y-x)^2}{4.|a_2|}\sum_{i=0}^{2}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}$$

$$\times\begin{cases}\left[|\phi''(x)|^{s}\left(\frac{\left|\frac{\phi''(y)}{\phi''(x)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|}\right)^{\frac{1}{q}}+|\phi''(y)|^{s}\left(\frac{\left|\frac{\phi''(x)}{\phi''(y)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|}\right)^{\frac{1}{q}}\right] \\ \quad \textit{where} \quad 0<|\phi''(x)|,|\phi''(y)|\leq 1 \\ \left[|\phi''(x)|^{s}|\phi''(y)|^{1-s}\left(\frac{\left|\frac{\phi''(y)}{\phi''(x)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|}\right)^{\frac{1}{q}}+|\phi''(y)|\left(\frac{\left|\frac{\phi''(x)}{\phi''(y)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|}\right)^{\frac{1}{q}}\right] \\ \quad \textit{where} \quad 0<|\phi''(x)|\leq 1<|\phi''(y)| \\ \left[|\phi''(x)||\phi''(y)|^{1-s}\left(\frac{\left|\frac{\phi''(y)}{\phi''(x)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|}\right)^{\frac{1}{q}}+|\phi''(x)|^{1-s}|\phi''(y)|\left(\frac{\left|\frac{\phi''(x)}{\phi''(y)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|}\right)^{\frac{1}{q}}\right] \\ \quad \textit{where} \quad 1<|\phi''(x)|,|\phi''(y)|\end{cases}$$

Corollary 3. *Under the assumptions of Theorem 6 and $n=2$, we have the following inequality:*

$$\left|\frac{1}{y-x}\int_{x}^{y}\phi(u)\,du-(\phi(x)+\phi(y))+\frac{(y-x)\left[a_2+a1+2a_0\right]}{4.a_2}\left[\phi'(y)-\phi'(x)\right]\right|$$
$$\leq\frac{(y-x)^2}{4.\left|a_2\right|}\left[\sum_{i=0}^{2}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}$$

$$\times\begin{cases}\left[|\phi''(y)|^{s}\left(\sum_{i=0}^{2}|a_i|\left(-sp\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)\right]\right)^{\frac{1}{p}}\right. \\ \left.+|\phi''(x)|^{s}\left(\sum_{i=0}^{2}|a_i|\left(-sp\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)\right]\right)^{\frac{1}{p}}\right] \\ \quad \textit{where} \quad 0<|\phi''(x)|,|\phi''(y)|\leq 1 \\ \left[|\phi''(y)|\left(\sum_{i=0}^{2}|a_i|\left(-sp\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)\right]\right)^{\frac{1}{p}}\right. \\ \left.+|\phi''(x)|^{s}|\phi''(y)|^{1-s}\left(\sum_{i=0}^{2}|a_i|\left(-sp\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)\right]\right)^{\frac{1}{p}}\right] \\ \quad \textit{where} \quad 0<|\phi''(x)|\leq 1<|\phi''(y)| \\ \left[|\phi''(y)|\left(\sum_{i=0}^{2}|a_i|\left(-sp\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(x)}{\phi''(y)}\right|\right)\right]\right)^{\frac{1}{p}}\right. \\ \left.+|\phi''(x)|^{1-s}|\phi''(y)|\left(\sum_{i=0}^{2}|a_i|\left(-sp\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(y)}{\phi''(x)}\right|\right)\right]\right)^{\frac{1}{p}}\right] \\ \quad \textit{where} \quad 1<|\phi''(x)|,|\phi''(y)|\end{cases}$$

4 Applications in Numerical Integration

$s-logarithmically$-convex functions are valuable instruments in the analysis and minimization of numerical errors. The distinctive properties and associated inequalities of these functions enable more accurate error estimations, facilitating of optimization of numerical integration methods. Let d be a division of the interval $[x,y]$, i.e. $x=u_0<u_1<\cdots<u_{n-1}<u_n=y$, $h_j=u_{j+1}-u_j$, $(j=1,2,3,\cdots,n-1)$ and consider perturbed trapezoidal rule

$$\int_x^y \phi(r)\,dr = T(\phi,\phi',I_h)+\tilde{R}_T(\phi,\phi',I_h) \tag{7}$$

where

$$T(\phi,\phi',I_h)=\sum_{j=0}^{n-1}\left|\phi(u_j)+\phi(u_{j+1})\right|h_j+\frac{[a_2+a_1+2a_0]}{4.a_2}\sum_{j=0}^{n-1}\left[\phi'(u_{j+1})-\phi'(u_j)\right]h_j^2 \tag{8}$$

is the trapezoidal variants and $\tilde{R}_T(\phi,\phi',I_h)$ is the associated error.

Theorem 7. *Let assume that $s\in(0,1]$ and $\phi: I\subset\mathbb{R}_0\to\mathbb{R}_+$ is a function with second-order derivatives on I° such that $\phi''\in L([x,y])$, where $x,y\in I^\circ$ with $x<y$. If $|\phi''|$ is $s-log$-convex on $[x,y]$, then one obtains*

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right|\leq\frac{1}{4.\,|a_2|}$$

$$\times\begin{cases}\sum_{j=0}^{n-1}\Bigg[\left|\phi''\left(u_{j+1}\right)\right|^{s}\sum_{i=0}^{2}\left|a_{i}\right|\left(-s\ln\left|\frac{\phi''(u_{j})}{\phi''(u_{j+1})}\right|\right)^{-i-1}\\ \left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_{j})}{\phi''(u_{j+1})}\right|\right)\right]+\left|\phi''\left(u_{j}\right)\right|^{s}\sum_{i=0}^{2}\left|a_{i}\right|\left(-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_{j})}\right|\right)^{-i-1}\\ \left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_{j})}\right|\right)\right]\Bigg]h_{j}^{3}\quad where\quad 0<\left|\phi''\left(u_{j}\right)\right|,\left|\phi''\left(u_{j+1}\right)\right|\leq 1\\ \sum_{j=0}^{n-1}\Bigg[\left|\phi''\left(u_{j+1}\right)\right|\sum_{i=0}^{2}\left|a_{i}\right|\left(-s\ln\left|\frac{\phi''(u_{j})}{\phi''(u_{j+1})}\right|\right)^{-i-1}\\ \left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_{j})}{\phi''(u_{j+1})}\right|\right)\right]\\ +\left|\phi''\left(u_{j+1}\right)\right|^{1-s}\left|\phi''\left(u_{j}\right)\right|^{s}\sum_{i=0}^{2}\left|a_{i}\right|\left(-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_{j})}\right|\right)^{-i-1}\\ \left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_{j})}\right|\right)\right]\Bigg]h_{j}^{3}\quad where\quad 0<\left|\phi''\left(u_{j}\right)\right|\leq 1<\left|\phi''\left(u_{j+1}\right)\right|\\ \sum_{j=0}^{n-1}\Bigg[\left|\phi''\left(u_{j}\right)\right|^{1-s}\left|\phi''\left(u_{j+1}\right)\right|\sum_{i=0}^{2}\left|a_{i}\right|\left(-s\ln\left|\frac{\phi''(u_{j})}{\phi''(u_{j+1})}\right|\right)^{-i-1}\\ \left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_{j})}{\phi''(u_{j+1})}\right|\right)\right]\\ +\left|\phi''\left(u_{j}\right)\right|\left|\phi''\left(u_{j+1}\right)\right|^{1-s}\sum_{i=0}^{2}\left|a_{i}\right|\left(-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_{j})}\right|\right)^{-i-1}\\ \left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_{j})}\right|\right)\right]\Bigg]h_{j}^{3}\quad where\quad 1<\left|\phi''\left(u_{j}\right)\right|,\left|\phi''\left(u_{j+1}\right)\right|\end{cases}$$

for every division d of $[x,y]$, i.e. $x=u_0<u_1<\cdots<u_{n-1}<u_n=y$.

Proof. By applying Corollary (1) on the subinterval $[u_j,u_{j+1}]$, $(j=1,2,3,\cdots,n-1)$ of the division d, we have

$$\begin{aligned}\left|\tilde{R}_{T}\left(\phi,\phi',I_{h}\right)\right| &= \left|\sum_{j=0}^{n-1}\left[\phi\left(u_{j}\right)+\phi\left(u_{j+1}\right)\right]h_{j}-\frac{a_{2}+a_{1}+2a_{0}}{4.a_{2}}\sum_{j=0}^{n-1}\left[\phi'\left(u_{j}\right)+\phi'\left(u_{j+1}\right)\right]h_{j}^{2}-\int_{u_{j}}^{u_{j+1}}\phi\left(t\right)dt\right|\\ &\leq\sum_{j=0}^{n-1}h_{j}\left|\left[\phi\left(u_{j}\right)+\phi\left(u_{j+1}\right)\right]-\frac{a_{2}+a_{1}+2a_{0}}{4.a_{2}}\left[\phi'\left(u_{j}\right)+\phi'\left(u_{j+1}\right)\right]h_{j}-\frac{1}{h_{j}}\int_{u_{j}}^{u_{j+1}}\phi\left(t\right)dt\right|\\ &\leq\frac{1}{4.\left|a_{2}\right|}\end{aligned}$$

$$\times\begin{cases}\sum_{j=0}^{n-1}\left[\left|\phi''(u_{j+1})\right|^{s}\sum_{i=0}^{2}|a_i|\left(-s\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)^{-i-1}\right.\\ \left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]+\left|\phi''(u_j)\right|^{s}\sum_{i=0}^{2}|a_i|\left(-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)^{-i-1}\\ \left.\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right]h_j^3 \quad \text{where} \quad 0<\left|\phi''(u_j)\right|,\quad \left|\phi''(u_{j+1})\right|\leq 1\\ \sum_{j=0}^{n-1}\left[\left|\phi''(u_{j+1})\right|\sum_{i=0}^{2}|a_i|\left(-s\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)^{-i-1}\right.\\ \left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]+\left|\phi''(u_{j+1})\right|^{1-s}\left|\phi''(u_j)\right|^{s}\sum_{i=0}^{2}|a_i|\left(-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)^{-i-1}\\ \left.\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right]h_j^3 \quad \text{where} \quad 0<\left|\phi''(u_j)\right|\leq 1<\left|\phi''(u_{j+1})\right|\\ \sum_{j=0}^{n-1}\left[\left|\phi''(u_j)\right|^{1-s}\left|\phi''(u_{j+1})\right|\sum_{i=0}^{2}|a_i|\left(-s\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)^{-i-1}\right.\\ \left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]+\left|\phi''(u_j)\right|\left|\phi''(u_{j+1})\right|^{1-s}\sum_{i=0}^{2}|a_i|\left(-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)^{-i-1}\\ \left.\left[\Gamma(i+1)-\Gamma\left(i+1,-s\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right]h_j^3 \quad \text{where} \quad 1<\left|\phi''(u_j)\right|,\left|\phi''(u_{j+1})\right|\end{cases}$$

Theorem 8. *Let us assume that $s\in(0,1]$ and $\phi: I\subset\mathbb{R}_0\to\mathbb{R}_+$ is a function with second-order derivatives on I° such that $\phi''\in L([x,y])$, where $x,y\in I^{\circ}$ with $x<y$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$. If $|\phi''|^q$ is $s-log-$convex on $[x,y]$, then one obtains*

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right|\leq\frac{1}{4.\,|a_2|}\sum_{i=0}^{2}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}$$

$$\times\begin{cases}\sum_{j=0}^{n-1}\left[\left|\phi''(u_j)\right|^{s}\left(\frac{\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|}\right)^{\frac{1}{q}}+\left|\phi''(u_{j+1})\right|^{s}\left(\frac{\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|}\right)^{\frac{1}{q}}\right]h_j^3\\ \quad \textit{where} \quad 0<\left|\phi''(u_j)\right|,\left|\phi''(u_{j+1})\right|\leq 1\\ \sum_{j=0}^{n-1}\left[\left|\phi''(u_j)\right|^{s}\left|\phi''(u_{j+1})\right|^{1-s}\left(\frac{\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|}\right)^{\frac{1}{q}}+\left|\phi''(u_{j+1})\right|\left(\frac{\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|}\right)^{\frac{1}{q}}\right]h_j^3\\ \quad \textit{where} \quad 0<\left|\phi''(u_j)\right|\leq 1<\left|\phi''(u_{j+1})\right|\\ \sum_{j=0}^{n-1}\left[\left|\phi''(u_j)\right|\left|\phi''(u_{j+1})\right|^{1-s}\left(\frac{\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|}\right)^{\frac{1}{q}}+\left|\phi''(u_j)\right|^{1-s}\left|\phi''(u_{j+1})\right|\left(\frac{\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|}\right)^{\frac{1}{q}}\right]h_j^3\\ \quad \textit{where} \quad 1<\left|\phi''(u_j)\right|,\left|\phi''(u_{j+1})\right|\end{cases}$$

for every division d of $[x,y]$, i.e. $x=u_0<u_1<\cdots<u_{n-1}<u_n=y$.

Proof. By applying Corollary (2) on the subinterval $[u_j, u_{j+1}]$, $(j = 1, 2, 3, \cdots, n-1)$ of the division d, we have

$$\begin{aligned}
&\left|\tilde{R}_T\left(\phi, \phi', I_h\right)\right| \\
&= \left|\sum_{j=0}^{n-1}\left[\phi\left(u_j\right)+\phi\left(u_{j+1}\right)\right] h_j - \frac{a_2+a_1+2a_0}{4.a_2}\sum_{j=0}^{n-1}\left[\phi'\left(u_j\right)+\phi'\left(u_{j+1}\right)\right] h_j^2 - \int_{u_j}^{u_{j+1}}\phi\left(t\right)dt\right| \\
&\leq \sum_{j=0}^{n-1} h_j\left|\left[\phi\left(u_j\right)+\phi\left(u_{j+1}\right)\right] - \frac{a_2+a_1+2a_0}{4.a_2}\left[\phi'\left(u_j\right)+\phi'\left(u_{j+1}\right)\right] h_j - \frac{1}{h_j}\int_{u_j}^{u_{j+1}}\phi\left(t\right)dt\right| \\
&\leq \frac{1}{4.\left|a_2\right|}\sum_{i=0}^{2}\frac{\left|a_i\right|}{\left(ip+1\right)^{\frac{1}{p}}}
\end{aligned}$$

$$\times\begin{cases}
\sum_{j=0}^{n-1}\left[\left|\phi''\left(u_j\right)\right|^s\left(\frac{\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|}\right)^{\frac{1}{q}} + \left|\phi''\left(u_{j+1}\right)\right|^s\left(\frac{\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|}\right)^{\frac{1}{q}}\right] h_j^3 \\
\text{where}\quad 0<\left|\phi''\left(u_j\right)\right|,\quad \left|\phi''\left(u_{j+1}\right)\right|\leq 1 \\
\sum_{j=0}^{n-1}\left[\left|\phi''\left(u_j\right)\right|^s\left|\phi''\left(u_{j+1}\right)\right|^{1-s}\left(\frac{\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|}\right)^{\frac{1}{q}} + \left|\phi''\left(u_{j+1}\right)\right|\left(\frac{\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|}\right)^{\frac{1}{q}}\right] h_j^3 \\
\text{where}\quad 0<\left|\phi''\left(u_j\right)\right|\leq 1<\left|\phi''\left(u_{j+1}\right)\right| \\
\sum_{j=0}^{n-1}\left[\left|\phi''\left(u_j\right)\right|\left|\phi''\left(u_{j+1}\right)\right|^{1-s}\left(\frac{\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|}\right)^{\frac{1}{q}} + \left|\phi''\left(u_j\right)\right|^{1-s}\left|\phi''\left(u_{j+1}\right)\right|\left(\frac{\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|^{sq}-1}{sq\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|}\right)^{\frac{1}{q}}\right] h_j^3 \\
\text{where}\quad 1<\left|\phi''\left(u_j\right)\right|,\left|\phi''\left(u_{j+1}\right)\right|
\end{cases}$$

Theorem 9. *Let's assume that $s \in (0,1]$ and $\phi : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is a function with second-order derivatives on I° such that $\phi'' \in L([x,y])$, where $x, y \in I^\circ$ with $x < y$, $p, q > 1$, $\frac{1}{p}+\frac{1}{q}=1$. If $|\phi''|^p$ is $s-$log-convex on $[x,y]$, then one obtains*

$$\left|\tilde{R}_T\left(\phi, \phi', I_h\right)\right| \leq \frac{1}{4.\left|a_2\right|}\left[\sum_{i=0}^{2}\frac{\left|a_i\right|}{i+1}\right]^{1-\frac{1}{p}}$$

$$\times\begin{cases}\sum_{j=0}^{n-1}\left[\left|\phi''\left(u_{j+1}\right)\right|^{s}\left(\sum_{i=0}^{2}\left|a_{i}\right|\left(-sp\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)^{-i-1}\right.\right.\\ \left.\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]\right)^{\frac{1}{p}}+\left|\phi''\left(u_{j}\right)\right|^{s}\left(\sum_{i=0}^{2}\left|a_{i}\right|\left(-sp\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)^{-i-1}\right.\\ \left.\left.\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right)^{\frac{1}{p}}\right]h_{j}^{3}\quad where\quad 0<\left|\phi''\left(u_{j}\right)\right|,\left|\phi''\left(u_{j+1}\right)\right|\leq 1\\ \sum_{j=0}^{n-1}\left[\left|\phi''\left(u_{j+1}\right)\right|\left(\sum_{i=0}^{2}\left|a_{i}\right|\left(-sp\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]\right)^{\frac{1}{p}}\right.\\ +\left|\phi''\left(u_{j}\right)\right|^{s}\left|\phi''\left(u_{j+1}\right)\right|^{1-s}\left(\sum_{i=0}^{2}\left|a_{i}\right|\left(-sp\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)^{-i-1}\right.\\ \left.\left.\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right)^{\frac{1}{p}}\right]h_{j}^{3}\quad where\quad 0<\left|\phi''\left(u_{j}\right)\right|\leq 1<\left|\phi''\left(u_{j+1}\right)\right|\\ \sum_{j=0}^{n-1}\left[\left|\phi''\left(u_{j+1}\right)\right|\left(\sum_{i=0}^{2}\left|a_{i}\right|\left(-sp\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]\right)^{\frac{1}{p}}\right.\\ +\left|\phi''\left(u_{j}\right)\right|^{1-s}\left|\phi''\left(y\right)\right|\left(\sum_{i=0}^{2}\left|a_{i}\right|\left(-sp\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)^{-i-1}\right.\\ \left.\left.\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right)^{\frac{1}{p}}\right]h_{j}^{3}\quad where\quad 1<\left|\phi''\left(u_{j}\right)\right|,\left|\phi''\left(u_{j+1}\right)\right|\end{cases}$$

for every division d of $[x,y]$, i.e. $x=u_0<u_1<\cdots<u_{n-1}<u_n=y$.

Proof. By applying Corollary (3) on the subinterval $[u_j,u_{j+1}]$, $(j=1,2,3,\cdots,n-1)$ of the division d, we have

$$\begin{aligned}&\left|\tilde{R}_T\left(\phi,\phi',I_h\right)\right|\\ &=\left|\sum_{j=0}^{n-1}\left[\phi\left(u_j\right)+\phi\left(u_{j+1}\right)\right]h_j-\frac{a_2+a_1+2a_0}{4.a_2}\sum_{j=0}^{n-1}\left[\phi'\left(u_j\right)+\phi'\left(u_{j+1}\right)\right]h_j^2-\int_{u_j}^{u_{j+1}}\phi\left(t\right)dt\right|\\ &\leq\sum_{j=0}^{n-1}h_j\left|\left[\phi\left(u_j\right)+\phi\left(u_{j+1}\right)\right]-\frac{a_2+a_1+2a_0}{4.a_2}\left[\phi'\left(u_j\right)+\phi'\left(u_{j+1}\right)\right]h_j-\frac{1}{h_j}\int_{u_j}^{u_{j+1}}\phi\left(t\right)dt\right|\\ &\leq\frac{1}{4.\left|a_2\right|}\left[\sum_{i=0}^{2}\frac{\left|a_i\right|}{i+1}\right]^{1-\frac{1}{p}}\end{aligned}$$

$$\times\begin{cases}\sum_{j=0}^{n-1}\Bigg[\left|\phi''\left(u_{j+1}\right)\right|^{s}\left(\sum_{i=0}^{2}\left|a_i\right|\left(-sp\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)^{-i-1}\right. \\ \left.\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]\right)^{\frac{1}{p}}+\left|\phi''\left(u_j\right)\right|^{s}\left(\sum_{i=0}^{2}\left|a_i\right|\left(-sp\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)^{-i-1}\right. \\ \left.\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right)^{\frac{1}{p}}\Bigg]h_j^3 \quad \text{where} \quad 0<\left|\phi''\left(u_j\right)\right|,\left|\phi''\left(u_{j+1}\right)\right|\leq 1 \\ \sum_{j=0}^{n-1}\Bigg[\left|\phi''\left(u_{j+1}\right)\right|\left(\sum_{i=0}^{2}\left|a_i\right|\left(-sp\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)^{-i-1}\right. \\ \left.\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]\right)^{\frac{1}{p}}+\left|\phi''\left(u_j\right)\right|^{s}\left|\phi''\left(u_{j+1}\right)\right|^{1-s}\left(\sum_{i=0}^{2}\left|a_i\right|\left(-sp\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)^{-i-1}\right. \\ \left.\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right)^{\frac{1}{p}}\Bigg]h_j^3 \quad \text{where} \quad 0<\left|\phi''\left(u_j\right)\right|\leq 1<\left|\phi''\left(u_{j+1}\right)\right| \\ \sum_{j=0}^{n-1}\Bigg[\left|\phi''\left(u_{j+1}\right)\right|\left(\sum_{i=0}^{2}\left|a_i\right|\left(-sp\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_j)}{\phi''(u_{j+1})}\right|\right)\right]\right)^{\frac{1}{p}} \\ +\left|\phi''\left(u_j\right)\right|^{1-s}\left|\phi''\left(y\right)\right|\left(\sum_{i=0}^{2}\left|a_i\right|\left(-sp\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)^{-i-1}\right. \\ \left.\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\ln\left|\frac{\phi''(u_{j+1})}{\phi''(u_j)}\right|\right)\right]\right)^{\frac{1}{p}}\Bigg]h_j^3 \quad \text{where} \quad 1<\left|\phi''\left(u_j\right)\right|,\left|\phi''\left(u_{j+1}\right)\right|\end{cases}$$

5 APPLICATIONS TO SPECIAL MEANS

In this section, we introduce some examples, using the perturbed trapezoid inequalities obtained for $s-log$-convex functions presented in [12].

Proposition 2. *Let $s\in(0,1]$, $x,y\in\mathbb{R}_0$ with $0<x<y$, $n\in\mathbb{N}$ and $n>2$ where n is an even number. Then, the inequality in the following holds:*

$$\begin{aligned}&\Big|\frac{1}{4}\left(\frac{y^2\ln y-x^2\ln x}{y-x}\right)-\frac{A\left(x,y\right)}{4}+A\left(x\ln\sqrt{x},y\ln\sqrt{y}\right)+\cdots\\&+\frac{\left(-1\right)^{n-3}\left(y-x\right)^{n-4}\left(n-6\right)!\left[n.\left(n-1\right).\left(n-2\right).a_n+\cdots+4.3.2.a_4\right]}{4.n!.a_n.H\left(x^{n-5},y^{n-5}\right)}\\&+\frac{\left(-1\right)^{n-2}\left(y-x\right)^{n-3}\left(n-5\right)!\left[n.\left(n-1\right).a_n+\cdots+4.3.a_4+3.2.a_3+4.a_2\right]}{4.n!.a_n.H\left(x^{n-4},-y^{n-4}\right)}\\&+\frac{\left(-1\right)^{n-1}\left(y-x\right)^{n-2}\left(n-4\right)!\left[n.a_n+\cdots+2.a_2\right]}{4.n!.a_n.H\left(x^{n-3},y^{n-3}\right)}\\&+\frac{\left(-1\right)^{n}\left(y-x\right)^{n-1}\left(n-3\right)!\left[a_n+\cdots+a_1+2.a_0\right]}{4.n!.a_n.H\left(x^{n-2},-y^{n-2}\right)}\Big|\\&\leq\frac{\left(y-x\right)^{n}}{2.n!.\left|a_n\right|}\end{aligned}$$

$$\times \begin{cases} \left(\frac{(n-2)!}{2y^{n-1}}\right)^{s} \sum_{i=0}^{n} |a_i| \left(-s\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)^{-i-1} \left[\Gamma\left(i+1\right) - \Gamma\left(i+1, -s\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)\right] \\ + \left(\frac{(n-2)!}{2x^{n-1}}\right)^{s} \sum_{i=0}^{n} |a_i| \left(-s\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)^{-i-1} \\ \left[\Gamma\left(i+1\right) - \Gamma\left(i+1, -s\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)\right]\Bigg] \quad where \quad 0 < \left(\frac{(-1)^n (n-2)!}{2x^{n-1}}\right), \\ \left(\frac{(-1)^n (n-2)!}{2y^{n-1}}\right) \\ \left(\frac{(n-2)!}{2y^{n-1}}\right) \sum_{i=0}^{n} |a_i| \left(-s\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)^{-i-1} \left[\Gamma\left(i+1\right) - \Gamma\left(i+1, -s\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)\right] \\ + \left(\frac{y}{x}\right)^{(n-1)s} \sum_{i=0}^{n} |a_i| \left(-s\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)^{-i-1} \\ \left[\Gamma\left(i+1\right) - \Gamma\left(i+1, -s\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)\right] \quad where \quad 0 < \left(\frac{(-1)^n (n-2)!}{2x^{n-1}}\right) \leq 1 < \left(\frac{(-1)^n (n-2)!}{2y^{n-1}}\right) \\ \left(\frac{((n-2)!)^2}{4(xy)^{n-1}}\right) \left(\frac{(n-2)!}{2x^{n-1}}\right)^{-s} \Bigg[\sum_{i=0}^{n} |a_i| \left(-s\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)^{-i-1} \\ \left[\Gamma\left(i+1\right) - \Gamma\left(i+1, -s\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)\right] \\ + \sum_{i=0}^{n} |a_i| \left(-s\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)^{-i-1} \left[\Gamma\left(i+1\right) - \Gamma\left(i+1, -s\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)\right]\Bigg] \\ \quad where \quad 1 < \frac{(-1)^n (n-2)!}{2x^{n-1}}, \left(\frac{(-1)^n (n-2)!}{2y^{n-1}}\right) \end{cases}$$

Proof. The proof is obtained from Theorem (4) such that $\phi\left(u\right) = u \ln \sqrt{u}$; $u \in (4, \infty)$ and $|x - y| < 10$.

Proposition 3. *Let* $s \in (0, 1]$, $x, y \in \mathbb{R}_0$ *with* $0 < x < y$, $\frac{1}{p} + \frac{1}{q} = 1$, $\forall p, q > 1$, $n \in \mathbb{N}$ *and* $n > 2$ *where* n *is an even number. Then, the inequality in the following holds:*

$$\Bigg| \frac{1}{4}\left(\frac{y^2 \ln y - x^2 \ln x}{y - x}\right) - \frac{A\left(x, y\right)}{4} + A\left(x \ln \sqrt{x}, y \ln \sqrt{y}\right) + \cdots$$

$$+ \frac{(-1)^{n-3} \left(y - x\right)^{n-4} \left(n - 6\right)! \left[n.\left(n-1\right).\left(n-2\right).a_n + \cdots + 4.3.2.a_4\right]}{4.n!.a_n.H\left(x^{n-5}, y^{n-5}\right)}$$

$$+ \frac{(-1)^{n-2} \left(y - x\right)^{n-3} \left(n - 5\right)! \left[n.\left(n-1\right).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 4.a_2\right]}{4.n!.a_n.H\left(x^{n-4}, -y^{n-4}\right)}$$

$$+ \frac{(-1)^{n-1} \left(y - x\right)^{n-2} \left(n - 4\right)! \left[n.a_n + \cdots + 2.a_2\right]}{4.n!.a_n.H\left(x^{n-3}, y^{n-3}\right)}$$

$$+ \frac{(-1)^{n} \left(y - x\right)^{n-1} \left(n - 3\right)! \left[a_n + \cdots + a_1 + 2.a_0\right]}{4.n!.a_n.H\left(x^{n-2}, -y^{n-2}\right)} \Bigg|$$

$$\leq \frac{\left(y - x\right)^n}{2.n!.\left|a_n\right|} \sum_{i=0}^{n} \frac{|a_i|}{\left(ip + 1\right)^{\frac{1}{p}}}$$

$$\times\begin{cases}\left[\left(\frac{(n-2)!}{(xy)^{n-1}}\right)^{s}\left(L\left(x^{(n-1)sq},y^{(n-1)sq}\right)\right)^{\frac{1}{q}}\right] \\ 0<\left(\frac{(-1)^{n}(n-2)!}{2x^{n-1}}\right),\left(\frac{(-1)^{n}(n-2)!}{2y^{n-1}}\right) \\ \left(\frac{(n-2)!}{2(y)^{n-1}}\right)\left[\left(\frac{1}{x}\right)^{s(n-1)}+1\right]\left(L\left(x^{(n-1)sq},y^{(n-1)sq}\right)\right)^{\frac{1}{q}} \\ 0<\left(\frac{(-1)^{n}(n-2)!}{2x^{n-1}}\right)\leq 1<\left(\frac{(-1)^{n}(n-2)!}{2y^{n-1}}\right) \\ \left(\frac{((n-2)!)^{2-s}}{2^{2-s}x^{n-1}}\right)\left[\frac{\left(L\left(x^{(n-1)sq},y^{(n-1)sq}\right)\right)^{\frac{1}{q}}}{H(x^{n-1},y^{n-1})}\right] \\ 1<\frac{(-1)^{n}(n-2)!}{2x^{n-1}},\left(\frac{(-1)^{n}(n-2)!}{2y^{n-1}}\right)\end{cases}$$

where A, H and L denote the arithmetic, harmonic mean and logarithmic mean, respectively.

Proof. The proof is obtained from Theorem (5) such that $\phi(u)=u\ln\sqrt{u}$; $u\in(4,\infty)$ and $|x-y|<10$.

Proposition 4. *Let $s\in(0,1]$, $x,y\in\mathbb{R}_0$ with $0<x<y$, $\frac{1}{p}+\frac{1}{q}=1$, $\forall p,q>1$, $n\in\mathbb{N}$ and $n>2$ where n is an even number. Then, the inequality in the following holds:*

$$\begin{aligned}&\Bigg|\frac{1}{4}\left(\frac{y^{2}\ln y-x^{2}\ln x}{y-x}\right)-\frac{A(x,y)}{4}+A\left(x\ln\sqrt{x},y\ln\sqrt{y}\right)+\cdots\\&+\frac{(-1)^{n-3}(y-x)^{n-4}(n-6)!\left[n.(n-1).(n-2).a_{n}+\cdots+4.3.2.a_{4}\right]}{4.n!.a_{n}.H\left(x^{n-5},y^{n-5}\right)}\\&+\frac{(-1)^{n-2}(y-x)^{n-3}(n-5)!\left[n.(n-1).a_{n}+\cdots+4.3.a_{4}+3.2.a_{3}+4.a_{2}\right]}{4.n!.a_{n}.H\left(x^{n-4},-y^{n-4}\right)}\\&+\frac{(-1)^{n-1}(y-x)^{n-2}(n-4)!\left[n.a_{n}+\cdots+2.a_{2}\right]}{4.n!.a_{n}.H\left(x^{n-3},y^{n-3}\right)}\\&+\frac{(-1)^{n}(y-x)^{n-1}(n-3)!\left[a_{n}+\cdots+a_{1}+2.a_{0}\right]}{4.n!.a_{n}.H\left(x^{n-2},-y^{n-2}\right)}\Bigg|\\&\leq\frac{(y-x)^{n}}{2.n!.|a_{n}|}\sum_{i=0}^{n}\frac{|a_{i}|}{(i+1)^{1-\frac{1}{p}}}\end{aligned}$$

$$\times\begin{cases}\left[\left(\frac{(n-2)!}{(y)^{n-1}}\right)^{s}\left(\sum_{i=0}^{n}|a_i|\left(-sp\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)\right]\right)^{\frac{1}{p}}\right.\\ \left.+\left(\frac{(n-2)!}{(x)^{n-1}}\right)^{s}\left(\sum_{i=0}^{n}|a_i|\left(-sp\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)\right]\right)^{\frac{1}{p}}\right]\\ \quad \textit{where}\quad 0<\left(\frac{(-1)^n(n-2)!}{2x^{n-1}}\right),\left(\frac{(-1)^n(n-2)!}{2y^{n-1}}\right)\\ \left(\frac{(n-2)!}{(y)^{n-1}}\right)\left[\left(\sum_{i=0}^{n}|a_i|\left(-sp\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)\right]\right)^{\frac{1}{p}}\right.\\ \left.+\left(\frac{y}{x}\right)^{(n-1)s}\left(\sum_{i=0}^{n}|a_i|\left(-sp\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)\right]\right)^{\frac{1}{p}}\right]\\ \quad \textit{where}\quad 0<\left(\frac{(-1)^n(n-2)!}{2x^{n-1}}\right)\leq 1<\left(\frac{(-1)^n(n-2)!}{2y^{n-1}}\right)\\ \left(\frac{(n-2)!}{(y)^{n-1}}\right)\left[\left(\sum_{i=0}^{n}|a_i|\left(-sp\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\left(n-1\right)\ln\left(\frac{y}{x}\right)\right)\right]\right)^{\frac{1}{p}}\right.\\ \left.+\left(\frac{(n-2)!}{(y)^{n-1}}\right)^{1-s}\left(\sum_{i=0}^{n}|a_i|\left(-sp\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)^{-i-1}\left[\Gamma\left(i+1\right)-\Gamma\left(i+1,-sp\left(n-1\right)\ln\left(\frac{x}{y}\right)\right)\right]\right)^{\frac{1}{p}}\right]\\ \quad \textit{where}\quad 1<\frac{(-1)^n(n-2)!}{2x^{n-1}},\left(\frac{(-1)^n(n-2)!}{2y^{n-1}}\right)\end{cases}$$

where A, H and L denote the arithmetic, harmonic mean and logarithmic mean, respectively.

Proof. The proof is obtained from Theorem (5) such that $\phi(u) = u\ln\sqrt{u}$; $u \in (4,\infty)$ and $|x-y| < 10$.

6 Conclusion

s-logarithmically convex functions generalize classical convex and *log*-convex functions by introducing a parameter, s which controls the degree and type of convexity. This also allows for a deeper understanding of various mathematical phenomena, as well as more generalized bounds in inequalities. The focus of this study is the use of s-logarithmically convexity in deriving more precise upper limits for errors in numerical integration. The smoothness and well-behaved structure of these functions, together with their well-behaved derivatives, become them particularly suitable for error analysis in this context.

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