Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 13.07.2025
DOI : 10.15672/hujms.1522471

Research Article

A study on life insurance premiums under asymmetric dependence using Canadian insurance data

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Abstract

This study evaluates the impact of symmetric and asymmetric dependence on premium calculations for various annuity and life insurance products across different age groups. Initially, we determined the marginal survival probabilities for individual lifetimes at specific ages using the Gompertz mortality model. Subsequently, joint survival probabilities were derived, considering independent and dependent future lifetimes for individuals within a group. The dependency structure was examined using Archimedean copulas for symmetric models and Khoudraji copulas for asymmetric models, which are widely referenced in the literature. In addition, actuarial calculations were conducted using real data on dependent lifetimes sourced from a Canadian insurance company. The data set is divided into three different populations based on age differences between married couples: the entire population without considering age differences, the population where males are older, and the population where females are older. The symmetric and asymmetric dependence structures of these populations were determined using an asymmetry test. The best-fitting models were identified using maximum likelihood estimation and goodness-of-fit tests. Finally, actuarial calculations were performed on the data set. Our findings showed that there were no significant differences between symmetric and asymmetric premium calculations for the whole population. However, when the population is disaggregated by age, the asymmetry becomes evident in the data structures, which increases the differences in the premium calculations. For example, the Kho-Fr model selected for the population of older female exhibiting asymmetric dependency was generally found to produce higher premiums than the Gumbel model. These findings reveal the importance of determining the dependency structure and working with age-based sub-populations rather than treating the whole population as a homogenous structure in model selection.

Mathematics Subject Classification (2020). 62H05, 62P05, 91G05, 91G70, 62H20

Keywords. Dependent lifetimes, copula, annuity, insurance, premium, dependence, asymmetry, multiple life products, actuarial calculations

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Received: 25.07.2024; Accepted: 03.07.2025

1. Introduction

In actuarial science, including the dependency factor in models is important for premium calculations for multiple annuities and insurance products covering more than a single life. Standard practice assumes for simplicity that future lifetimes among a group of people are independent. However, numerous studies show that this simplified assumption is not valid for real insurance data. As is well known, married couples can be exposed to similar risks because they have the same life circumstances. Therefore, individuals' future lifetimes may affect each other, and modeling should account for dependence.

The main methods used for modeling dependent lives include copula models, multiple state models, and common shock models. Copula models are a function that describes the joint distribution of multiple random variables. These models allow modeling dependence structures between marginal distributions. Multiple state models are another important approach to modeling the interdependence of individuals' future lifetimes, considering multiple life cases. These models are used to represent transitions between various life states (e.g., active, disabled, dead) and to capture the dependence structure arising from such transitions. They are particularly useful in insurance products involving joint lives or multiple risks, where changes in one individual's state can influence the state transition probabilities of another. Common shock models describe a particular event or shock as a situation that affects all individuals in the same way. For example, a common shock between spouses, such as health status, financial crisis or social factors, may have similar effects on the lifetime of both individuals. Such models relate dependent lives not only by time but also by the joint impact of external factors. In this study, copula models are preferred for modeling the dependence of individuals' future lifetimes in joint lives due to their suitability for symmetric and asymmetric analysis.

Several studies have shown that there are significant differences between risk-related quantities such as premiums under dependent and independent assumptions [13, 15, 19]. In this sense, Frees et al. [19] is the first study in the literature to model joint lives with copula functions using the Canadian insurance dataset, which is a good example of a situation where joint lives are highly correlated, and then applied to actuarial annuities. On the other hand, as explained in the motivation section of the study, there may be asymmetry in real-life data due to gender differences, age differences, health status, social and economic factors. Inspired by this idea, the study aims to measure the impact of asymmetric dependence on the pricing of various annuities and insurance products in the actuarial field. In this context, the marginal survival probabilities of individual lifetimes are calculated using the Gompertz mortality model, and the interactions of the future lifetimes of individuals within a group are analyzed using various dependent models.

The paper examines symmetric and asymmetric dependence structures in the pricing of insurance products using the Archimedean and Khoudraji copula models. To assess the pricing implications of asymmetric dependence, actuarial calculations are performed on dependent real-life data from a Canadian insurance company. In the application part, we first estimate copula parameters using maximum likelihood techniques, followed by a goodness-of-fit procedure based on the Cramer-von Mises test statistic (S_n) [22] to compare candidate copulas. Finally, the best models are selected using the Akaike Information Criterion (AIC) and Bayesian Information Criteria (BIC) ensuring an optimal representation of dependency structures in insurance pricing. In recent years, the importance of relying on risk assessment and premium calculations in the insurance industry has increased. This study emphasizes that asymmetric dependence should not be ignored. For this purpose, studies in the literature are summarized.

Copulas, first introduced by Sklar [64], are frequently used in the literature to model dependency structures in various fields such as finance [29, 33, 34, 39, 57, 58], hydrology [7, 21, 32], medicine [42, 60] and environmental sciences [3, 35]. Our study focuses on the analysis of dependency structures and, like these studies, contributes to the understanding and modeling of dependencies between variables. Archimedean copulas are commonly used to model dependence in bivariate lifetimes and have been shown to be useful in various life insurance applications [2,6,15,17,48,49]. The recent literature has seen ongoing studies on the application of copulas in the field of insurance [25,28,45]. In addition, numerous studies have explored the role of asymmetry in insurance, particularly in automobile insurance [1, 11, 20, 43, 51, 53, 59, 61, 65]. Recently, the concept of asymmetry has been extensively examined in the field of life insurance [8, 9, 44]. However, it should be noted that only a limited number of studies link asymmetry with the concept of copula within the domain of life insurance [36, 38, 63]. Dufresne et al. [15] estimated lifetime by modeling joint lives with Archimedean copulas considering the age difference in life insurance using the Canadian insurance data. Kara [36, 38] theoretically examined the effect of asymmetric dependence on the pricing of joint life insurance and last survivor insurance policies with GFGM asymmetric copula models. The premium of a last survivor insurance policy is determined in [18] using the GFGM-Type II copula model. Frees et al. [19] conducted a study on life insurance related to dependent mortality.

This study differs from previous research by combining both symmetric and asymmetric copula models, offering a more flexible and innovative approach to modeling dependency structures in life insurance. While Dufresne et al. [15] analyzes under the assumption of symmetric dependence using Archimedean copulas, this study incorporates Khoudraji asymmetric copulas to model the joint lifetimes of spouses more realistically. The use of Khoudraji asymmetric copulas allows a more flexible treatment of dependency structures and extends the asymmetric dependency analyzed analytically by [36, 38] with GFGM copulas by testing it on real insurance data. Although Erawati and Subhan [18] use the GFGM-Type II copula only for policies that have died, this article applies the Khoudraji copula to mixed life insurance to illustrate the impact of asymmetric dependence on premium calculations. Moreover, although Frees et al. [19] analyzed dependent mortality, they did not evaluate asymmetric dependence structures from a financial perspective. This research fills these gaps and provides a comprehensive actuarial framework that takes asymmetric dependence into account, leading to more accurate premium calculations, improved risk management and stronger financial stability for insurers. This study on the impact of asymmetric dependence on premium calculations provides a new perspective to understand the importance of dependence in the insurance industry and to develop pricing strategies more effectively.

In the literature, elliptic and Vine copulas are also used for modeling dependent structures in the insurance and actuarial areas [12, 30, 55, 62, 67]. However, since the main focus of the study is to examine the effect of asymmetric dependence on premiums, copula functions that can directly model asymmetric dependence are preferred. Although various asymmetric models using Archimedean copulas are also available, only Khoudraji copulas were chosen as candidate copulas in our study. Khoudroji copulas, by definition, require the use of only independent and Archimedean copulas [41, 46]. As a result of the asymmetric tests performed in the study, two of the three populations were found to have asymmetric dependence, and therefore non-asymmetric elliptical copulas were not included. Moreover, Vine copulas are generally preferred in high-dimensional dependence modeling and offer a more complex structure to model the asymmetric structure of two populations. Since the estimation process of Vine copulas requires more computational complexity and does not directly include asymmetric structures, they are not included in the study.

The rest of the paper is organized as follows: Section 2 describes the data set and presents the rationale for symmetric and asymmetric dependence on life data that motivates the study. Section 3 describes marginal distributions, dependency models and insurance products. It also includes the analysis steps from the identification of dependence and its type to the selection of models. Section 4 includes actuarial applications to real insurance data, which constitutes the original part of the study. Here, it proposes model selection with goodness-of-fit tests using a real Canadian insurance dataset. Then, numerical applications are presented with real data to evaluate premium prices on various annuities and insurance products in the life insurance sector. Section 5 gives the results and discussion. Section 6 concludes the paper by evaluating the results.

2. Motivation

Previous studies such as [50] show that being married can significantly affect mortality. Moreover, the dependence between the lifetime of the male and the female has previously been analyzed by [6,15,19]. In these studies, the dependence of the data was first analyzed with common measures of dependence such as Kendall's τ and Spearman's ρ, and then modeled with Archimedean copulas. The data used in this study were received from a large Canadian life insurance company and include policies in force during the 5-year observation period from 29 December 1988 to 31 December 1993. The data set contains 14,947 contracts, of which 14,889 are male-female couples and 58 are same-sex couples (22 male-male and 36 female-female). Additionally, in this study, the coupling occurs at the time that an annuity is bought from the insurance company and the date of entry into the contract, date of birth and date of death of both insured persons are known.

In this study, our objective was to model the dependence between male and female lifetimes in married couples with symmetric-asymmetric copula functions for the reasons mentioned below. Thus, we aim to draw attention to the differences that arise in actuarial calculations by considering the types of dependence. For this purpose, we used the same data set from a large Canadian insurance company. In case the same couple had more than one policy, each couple was considered only once in the analysis and same-sex marriages were excluded. This reduced the number of observations to 12,856 couples. The age at the beginning of the observation is denoted as x_m for males and x_f for females.

Summary statistics for the dataset of 12,856 couples are presented in [15] (Table 1). According to these statistics, the average entry age is 66.39 for the whole population, 67.87 for males, and 64.91 for females. Among the 12,856 observed couples, only 193 experienced the death of both insured individuals during the 5-year observation period. To ensure methodological consistency, this study focuses solely on these 193 complete observations. Although the full dataset includes 12,856 couples, right-censored cases (where one or both individuals survived the period) were excluded from the modeling stage. This decision is consistent with prior studies such as [15,19], which also relied on complete data for dependence modeling and estimation. The observed differences in p-values and the parameter estimates compared to [15] are mainly attributable to the different sample selections, the complete cases in our study versus the full data set in theirs. Exchangeability tests were also applied only to the complete cases, as the exchTest() function cannot be used on censored data. Although it is theoretically possible to fit copula models, including Archimedean and Khoudraji-type copulas, to censored datasets, such applications require more advanced estimation methods that are beyond the scope of this study.

In this study, the data set is classified into three groups based on the ages at death of insured individuals: $x_m \geq x_f$ (i.e., the male died older than the female), $x_m < x_f$ (i.e., the female died older than the male) and the whole population including all couples. This classification is strictly based on the ages of death and may differ from the approach used in studies such as [15], where the grouping appears to have been performed according to the ages of entry. For each dataset, Spearman's correlation measure and asymmetric test results are calculated for the gender of the older partner and the results are given in Table 1. The values indicate that the ages at the death of spouses are positively correlated. The asymmetricity test was performed with the code "exchTest" in the R package. Although no asymmetry was observed in the entire population, a statistically significant asymmetry ($p < 0.05$) was observed when the grouping was based on the gender of the older partner.

Table 1. Spearman's ρ correlation measure and asymmetric test results

	Samples	ρ	Test value	p-value
$x_m \geq x_f$	144	0.8579	0.027344	0.02647
$x_m < x_f$	49	0.8777	0.034985	0.01948
whole	193	0.7955	0.016027	0.505

In actuarial applications, understanding the nature of the dependency between spouse lifetimes is essential to select appropriate models. The following section briefly reviews the relevant types of dependence and explains the rationale behind the modeling choices adopted in this study.

Dependence between the lifetimes of spouses has been widely studied in the actuarial literature and is most commonly associated with long-term factors, such as shared lifestyle, chronic health conditions, and sustained social interactions [6, 13–15, 31, 68], as summarized in [27]. In this context, long-term dependence refers to situations where the force of mortality of the surviving spouse remains constant or increases after the partner's death. This persistent association is shaped by the cumulative effects of cohabitation, mutual habits, and environmental similarities. As pointed out by [66], widely used Archimedean copula families (Clayton, Gumbel-Hougaard, Frank, and Joe) are suited to modeling long-term dependence and do not account for short-term or instantaneous effects. Although their study does not address Khoudraji-transformed versions, these asymmetric copulas preserve the core dependence structure of the base models and are therefore also suitable for modeling long-term dependence while allowing for asymmetry in the dependence relationship. Although the literature also considers short-term and instantaneous forms of dependence, these are not modeled in the present study and are mentioned solely for theoretical context. Short-term dependence generally refers to a temporary increase in mortality risk in the surviving spouse due to psychological effects, often described as the "broken heart" syndrome [47, 56]. Instantaneous dependence, resulting from shared exposure to external shocks (e.g., accidents or natural disasters), is typically addressed using common shock models [27]. However, it is necessary to determine whether the dependence in the lifetimes of spouses is symmetric or asymmetric. For this, exchangeability (symmetry) tests must first be applied. The rejection of exchangeability tests in certain cases supports the need for models that account for asymmetric dependence. Symmetric dependence occurs when the lifetimes of spouses are affected in similar ways, while asymmetric dependence is when the lifetimes of spouses are affected in different ways. The causes of asymmetric dependence between spouses in real life can be explained by certain factors that directly influence the dependency structure in the lifetimes of couples:

(1) Gender differences: Habits such as smoking, drinking alcohol, and participating in dangerous activities can increase the risk of death of men. Therefore, men's tendency to engage in higher-risk behaviors leads to differences in lifetimes between men and women. In other words, women tend to live longer than men, which can lead to asymmetric dependency.

(2) Age difference: Generally, men are older than women and the risk of death increases with increasing age. This can create a natural asymmetry in survival probabilities between spouses, making the assumption of exchangeability unrealistic in dependency modeling.

(3) Health status: Each spouse may have different chronic diseases and genetic predispositions. In addition, women are often observed to use healthcare services more

frequently and attend regular check-ups, which can influence survival results. This may therefore lead to asymmetry in lifetimes.

(4) Social and economic factors: Differences in the social support networks of spouses and economic situation can lead to asymmetric dependence in lifetimes.

(5) Psychological factors: Spouses may have different psychological tolerance and stress coping mechanisms. This supports the need to model the dependency structure as asymmetric in lifetimes.

Such situations require modeling dependency using non-exchangeable asymmetric copula families such as Khoudraji. This study aims to examine the impact of asymmetric dependencies in spouse lifetimes on actuarial calculations. In this context, the Archimedian and asymmetric Khoudraji copula functions are used to calculate the required joint survival probabilities. Marginal distributions, dependent models and actuarial calculations for insurance products are briefly introduced in the next "Background" section of the study. The determination of dependency structures, parameter estimation, and model selection are discussed later.

3. Background

3.1. Marginal distributions

In actuarial sciences, survival models are constructed by defining the future lifetime of an individual as a random variable. For further details on the survival life models, please refer to the work of [4]. The lifetime of a newborn shall be modeled by a positive continuous random variable, say X with distribution function F and survival function S. The symbol (x) will be used to denote a life age x and $T_x = X - x|X > x$ is the remaining lifetime of (x). The actuarial symbols ${}_tp_x$ and ${}_tq_x$ are, respectively, the survival and distribution function of T_x. In fact, for a living (x), the probability that it survives for at least t years and the probability that it dies in t years are, respectively, given by

$$ {}_tp_x = P\left(T_x > t\right) = P\left(X > x + t \mid X > x\right) = \frac{P\left(X > x + t\right)}{P\left(X > x\right)} = \frac{S\left(x + t\right)}{S\left(x\right)} \tag{3.1} $$

$$ {}_tq_x = 1 - {}_tp_x = P\left(T_x \leq t\right) \tag{3.2} $$

When X has a probability density function f, then T_x has a probability density function given by

$$ {}_tf_x = {}_tp_x \mu(x + t) \tag{3.3} $$

where $\mu(.)$ is the hazard rate function, also called the force of mortality.

Life products were initially designed according to the life status of only one person; however, as the number of people in the same policy increased over time, multiple life statuses were taken into account [4]. As a result, the change in the number of people registered in the policy means that all kinds of calculations made regarding the products will change. Therefore, this transition from single-life products to multiple-life products requires an appropriate expansion of the actuarial structure. Actuarial calculations for multiple lives are common in insurance practices. In the following, (x) stands for the husband of the age (x) while (y) is the wife. Considering a couple (x, y), T_{xy} describes the remaining time until the first death between (x) and (y) and it is known as the joint life status. In contrast, $T_{\overline{xy}}$ is the time until death of the last survivor. In our study, only joint survivor status was analyzed. T_{xy} random variable can be written as

$$ T_{xy}{=}\min\left(T_x, T_y\right) . \tag{3.4} $$

The joint life function of two individuals with future lifetimes T_x and T_y is expressed as follows:

$$ {}_tp_{xy}{=}\ \mathrm{P}\left(T_{xy}{>}t\right) = P\left(T_x{>}t,\ T_y{>}t\right). \tag{3.5} $$

If the future lifetimes of individuals are independent, the probability that two individuals age x and y live together for at least t years and the probability that the first death occurs within t years are, respectively, given below:

$$ {}_tp_{xy} = P\left(T_x > t,\ T_y > t\right) = P\left(T_x > t\right)P\left(T_y > t\right) = \ {}_tp_x\ {}_tp_y \tag{3.6} $$

$$ {}_tq_{xy} = \ {}_tq_x + \ {}_tq_y - \ {}_tq_x\ {}_tq_y \tag{3.7} $$

Various parametric mortality laws are used in the pricing of actuarial products. Models such as the constant force of mortality, De Moivre, Gompertz, Makeham and Weibull are common in the literature. The choice of model depends on the characteristics of the data and the objective of the study. Frees et al. [19] and Carriere [6] showed in their studies that the dataset consisting of middle-aged, and elderly insured persons fits the Gompertz mortality law well. Therefore, in our study, like [15], focusing on the Gompertz model, we use the Gompertz law reparametrized by [5]. Gompertz's law of mortality, which states that mortality increases exponentially with age, is expressed as follows:

$$ S\left(x\right) = \exp\left(-\frac{B}{\ln c}\left(c^x - 1\right)\right) \quad ; B > 0, c > 1, x \geq 0 \tag{3.8} $$

$$ {}_tp_x = \exp\left(-\frac{Bc^x\left(c^t - 1\right)}{\ln c}\right) \tag{3.9} $$

The reparametrized Gompertz law is given as follows:

$$ e^{-\frac{m}{\sigma}} = \frac{B}{\ln c}\ , \quad e^{\frac{1}{\sigma}} = c \tag{3.10} $$

from which we obtain

$$ {}_tp_x = \exp\left(e^{\frac{x-m}{\sigma}}\left(1 - e^{\frac{t}{\sigma}}\right)\right) \tag{3.11} $$

where the mode m>0 and the dispersion parameter σ>0 are the new parameters of the distribution.

We estimate the Gompertz parameters with MLE for the whole populations, $x_m \geq x_f$ and $x_m < x_f$ and the results are given in Table 2.

Table 2. Gompertz parameter estimates

Parameters	whole	$x_m \geq x_f$	$x_m < x_f$
$\widehat{\boldsymbol{m}}_{\boldsymbol{m}}$	82.2435	83.09	79.1488
$\widehat{\boldsymbol{m}}_{\boldsymbol{f}}$	80.0767	79.3398	82.0524
$\widehat{\boldsymbol{\sigma}}_{\boldsymbol{m}}$	9.6568	9.6993	9.2545
$\widehat{\boldsymbol{\sigma}}_{\boldsymbol{f}}$	9.4047	9.3158	9.5551

3.2. Dependence models

Sklar [64] introduced the concept of copula to delineate the joint distribution function of a random vector by separating the behaviors of the marginals and the dependence structure. According to Sklar's theorem, for positively correlated and continuous variables X and Y, there exists a unique copula $\mathrm{C} : [0,1]^2 \to [0,1]$ that precisely defines the joint distribution function of the bivariate random vector (X, Y).

$$ F_{XY}\left(x, y\right) = P\left(X \leq x, Y \leq y\right) \ = \ C\left(F_X\left(x\right), F_Y\left(y\right)\right) = C\left(u, v\right) \tag{3.12} $$

where u and v show the continuous empirical marginal distribution functions of $F_X\ (x)$ and $F_Y\left(y\right)$, with uniform distribution $U(0,1)$, respectively. Originally, the most widely considered copula families satisfy the exchangeability, i.e. $C(u, v) = C(v, u)$. However, the asymmetric copula setting emerges from the unsatisfied exchangeability. According to the theorem defined by [16], for all $\alpha,\ \beta\ \in\ (0,1)$ and for all copulas C_1 and C_2, the function $C_{\alpha,\beta} : [0,1]^2 \rightarrow [0,1]$, defined by

$$C_{\alpha,\beta}(u,v) = C_1\left(u^{\overline{\alpha}},\ v^{\overline{\beta}}\right)\ C_2(u^{\alpha}, v^{\beta}) \tag{3.13}$$

is a copula, where $\overline{\alpha} = 1-\alpha$ and $\overline{\beta} = 1-\beta$. Here, an asymmetric copula is formed $\phi = (\alpha,\beta,\theta)$, $\alpha,\beta\ \epsilon(0,1)$, $\alpha\ \neq\ 1/2$, $\beta\ \neq\ 1/2$. If $\alpha=\beta$ then $C_{\alpha,\beta}$ is symmetric. For two symmetric copulas, C_1 and C_2, if $C_{\alpha,\beta}\,(u,v) = C_{\beta,\alpha}(v,u)$ then $C_{\alpha,\beta}$ is a symmetric copula. On the other side, if $C_{\alpha,\beta}\,(u,v) \neq C_{\beta,\alpha}(v,u)$ then $C_{\alpha,\beta}$ is defined as the asymmetric copula [39,40].

We denote by T_{x_f} and T_{x_m} the future lifetimes of women and men, respectively. The survival function of $\left(T_{x_f}, T_{x_m}\right)$is written in terms of copulas and marginal survival functions. It is given by

$$P\left(T_{x_f} > t_1, T_{x_m} > t_2\right) = \tilde{C}\left({}_{t_1}p_{x_f}, {}_{t_2}p_{x_m}\right) = {}_{t_1}p_{x_f} + {}_{t_2}p_{x_m} - 1 + C\left({}_{t_1}q_{x_f}, {}_{t_2}q_{x_m}\right) \tag{3.14}$$

For a review of existing copula families, see [10,54]. The Archimedean copula family is a popular statistical tool in life insurance applications, particularly due to its flexibility in modeling dependent random lifetimes. For further details, see [15,19,68]. This paper presents a discussion of four well-known Archimedean copulas as given below:

Clayton copula is defined as

$$C\,(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}; \theta > 0. \tag{3.15}$$

Frank copula is given by

$$C\,(u,v) = -\frac{1}{\theta}log\left\{1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{(e^{-\theta}-1)}\right\};\quad \theta \neq 0. \tag{3.16}$$

Gumbel copula is obtained by

$$C(u,v) = exp\{-\left[(-log\,(u)\ )^{\theta}\ + (-log\,(v)\ )^{\theta}\right]^{\frac{1}{\theta}}\};\ \ \theta \in [1,\infty)\,. \tag{3.17}$$

Joe copula is given by

$$C(u,v) = 1 - ((1-u)^{\theta} + (1-v)^{\theta} - (1-u)^{\theta}(1-v)^{\theta})^{\left(\frac{1}{\theta}\right)};\quad \theta \in [1,\infty)\,. \tag{3.18}$$

Khoudraji copulas are a class of asymmetrical copulas first introduced by [41]. Later, Liebscher [46] defined its general form. They may model both positive and negative dependency, and a wide variety of dependency constructs can be captured by varying their parameters.

In Eq (3.13) for $\alpha+\overline{\alpha}=1$, $\beta+\overline{\beta}=1$, if C_1 is independent copula $C(u,v)=uv$, and C_2 is a symmetrical Archimedean copula family with a dependency parameter θ given in (3.15)–(3.18). $C_{\alpha,\beta}$, $I^2 \rightarrow I$ is called Khoudraji copulas and the mathematical model is defined as

$$C_{KC}\,(u,v) = u^{1-\alpha}v^{1-\beta}C\left(u^{\alpha}, v^{\beta}\right),\quad \alpha\ \neq\ \beta \tag{3.19}$$

To select an appropriate model for the data, it is essential to first perform a symmetry test (or a test for the exchangeability of variables). The literature mentions measures such as R_n, S_n^* and T_n for this purpose [23]. Here, Cramér-von Mises statistic S_n^* is used for symmetry testing as

$$S_n^* = \int_0^1\int_0^1 \left\{\hat{C}_n\,(u,v) - \hat{C}_n\,(v,u)\right\}^2 d\hat{C}_n\,(v,u) \tag{3.20}$$

If the p-value for this statistic is less than 0.05, the null hypothesis $H_0:\ \hat{C}_n\,(u,v) = \hat{C}_n\,(v,u)$, which means the symmetry of the data, is rejected. This indicates that the data are asymmetrically dependent. In our study, the exchangeability test was conducted using the "exchTest" function in the "copula" package in the R software [26].

3.3. Multiple actuarial calculations for annuity and insurance contracts

Multiple life products are insurance policies designed based on the lifetime or health status of more than one person and are of major importance in terms of actuarial analyses and assessments. These products allow insurance companies to optimize their risk management strategies and perform more precise and accurate premium calculations. In addition, multiple life products allow insurance companies to make more comprehensive and detailed financial projections when determining their long-term liabilities and reserve requirements. These products, which play an important role in reinsurance agreements and diversification of insurance portfolios, also help to offer more flexible and appropriate solutions to customer needs. In summary, the premiums of multi-life insurance are used in reserve calculations, reinsurance agreements, financial reporting, risk management and marketing strategies. In this section, actuarial calculations are performed using Canadian insurance data. Using the equations given in Section 3.1 for joint lives, the necessary formulas to be able to perform actuarial calculations for annuities and insurance products are given below. More details can be obtained by [4, 24, 37, 52].

The net single premium of the term life annuity for two individuals at ages x and y is calculated by

$$\ddot{a}_{xy:\overline{n|}} = \sum_{t=0}^{n} v^t\, {}_tp_{xy} \tag{3.21}$$

In such annuities, the payments end with the first death. In actuarial science, these kinds of annuity are called joint life annuities. The net single premium of endowment life insurance arranged for two individuals at ages x and y is given by

$$A_{xy:\overline{n|}} = A^{1}_{xy\ :\overline{n|}} + {}_nE_{xy} \tag{3.22}$$

Here, $A^{1}_{xy\ :\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_{xy}$ and ${}_nE_{xy} = v^n\ {}_np_{xy}$ are net single premiums of a term and pure endowment insurances, respectively. In such insurance policies, the death benefit is paid at the end of the year in which the first death occurs.

The premium formula used in actuarial pricing for the endowment insurance product is given by

$$P_{xy:\overline{n|}} = \frac{A_{xy:\overline{n|}}}{\ddot{a}_{xy:\overline{n|}}}, \qquad {}_mP_{xy:\overline{n|}} = \frac{A_{xy:\overline{n|}}}{\ddot{a}_{xy:\overline{m|}}}\ (m < n) \tag{3.23}$$

3.4. Analysis process

The concept of asymmetric dependence, which constitutes the original contribution of this study, is analyzed on real data obtained from a Canadian insurance company (in Section 4). As indicated in the motivation section, these analyzes were performed on datasets divided into three populations (whole, $x_m \geq x_f$, $x_m < x_f$). The marginal parameters for these data sets were estimated using the maximum likelihood method under the assumption of a Gompertz distribution, and the parameter values provided in Section 1 were used in the analyzes performed in Section 4. Parameter estimation and model selection were applied only for analysis of real data in Section 4. This analysis process is generally summarized as follows:

(1) Determination of Dependency Structures
- Conduct Spearman correlation measurements for all populations in the data set.
- Apply the exchangeability test.

(2) Selection of candidate symmetric and asymmetric models

- Archimedean (ArchC) copula models: Clayton, Frank, Gumbel, Joe
- Khoudraji (KhoC) copula models: Kho-Cl, Kho-Fr, Kho-Gm, Kho-Joe

(3) Parameter Estimation
- Estimating parameters for ArchC and KhoC models using the Maximum Pseudo-Likelihood (MPL) method.

(4) Model Selection
- Calculating log-likelihood (LL), Cramer von Mises (S_n), p, AIC and BIC values for ArchC and KhoC models.
- Determining models that fit the data using the test statistic (p>0.05).
- Selecting symmetric and asymmetric models with the smallest AIC and BIC values.

(5) Calculation of Actuarial Premiums
- Obtaining joint survival probabilities for independent and dependent scenarios (Appendices B).
- Calculating single and annual net premiums for annuity and insurance products.
- Illustrating premium calculations comparing the most appropriate symmetric and asymmetric models.

4. Application

In this section, we analyze dependency structures and actuarial premiums between gender and age groups based on Canadian insurance data. In this dataset, three populations were created as a whole; ($x_m \geq x_f$); ($x_m < x_f$); (m: male, f: female). As described in the analysis process, the symmetric and asymmetric dependency structures were determined with the asymmetry test for each of these populations. The parameters were then estimated using the maximum pseudo-likelihood (MPL) method. Then, using the Cramer-von Mises (S_n) goodness-of-fit test, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) the most appropriate ArchC and KhoC models were determined and the results are given in Table 3. Here, the values in parentheses represent the standard errors of the parameter estimates.

According to the results of the asymmetry test presented in Table 1, the entire population is determined to be symmetric, while the $x_m \geq x_f$ and $x_m < x_f$ populations are identified as asymmetric. Based on the results of the goodness-of-fit test in Table 3, the Gumbel copula is the best-fitting model for the whole population, while Kho-Gm and Kho-Fr are the most suitable models for the $x_m \geq x_f$ and $x_m < x_f$ subpopulations, respectively. Here, among models with p values greater than 0.05* for the statistic S_n, the model with the smallest AIC and BIC values was selected as the model that was best adapted **. These results demonstrate that the model selections are consistent with the asymmetry test results.

Table 3. The parameter estimation and goodness-of-test results for each copula model

				whole				
Model	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\beta}$	LL	S_n	**p value**	**AIC**	**BIC**
Clayton	1.7175 (0.1803)	-	-	66.8452	0.2732	0.0026	-131.6904	-128.4277
Frank	8.4524 (0.7435)	-	-	98.5910	0.0372	0.0129	-195.1820	-191.9193
Gumbel**	2.5737 (0.1524)	-	-	107.5388	0.0118	0.6572*	-213.0776	-209.8149
Joe	3.2680 (0.2052)	-	-	96.9617	0.0630	0.0077	-191.9234	-188.6607
Kho-Cl	12.9120 (2.6977)	0.8210 (0.0544)	0.6399 (0.0496)	93.0434	0.1182	0.0026	-180.0868	-170.2987
Kho-Fr	12.6803 (1.8705)	0.9227 (0.0467)	0.8475 (0.0552)	101.9542	0.0313	0.0129	-197.9084	-188.1203
Kho-Gm**	2.6882 (0.2114)	0.9873 (0.0430)	0.9682 (0.0602)	107.6679	0.0120	0.6366*	-209.3358	-199.5477
Kho-Joe	3.6229 (0.2934)	0.9656 (0.0478)	0.9337 (0.0595)	97.7730	0.0555	0.0026	-189.5460	-179.7579
				$\boldsymbol{x_m \geq x_f}$				
Clayton	2.2615 (0.2562)	-	-	69.2077	0.2435	0.0034	-136.4154	-133.4456
Frank	10.5010 (1.1522)	-	-	95.5002	0.0457	0.0035	-189.0004	-186.0306
Gumbel**	3.2054 (0.2806)	-	-	109.5050	0.0167	0.2862*	-217.0100	-214.0402
Joe	4.2451 (0.4502)	-	-	100.3910	0.0467	0.0103	-198.7820	-195.8122
Kho-Cl	9.0243 (2.4914)	0.9969 (0.1033)	0.6931 (0.0866)	89.0878	0.1118	0.0724*	-172.1756	-163.2662
Kho-Fr	17.5281 (2.9835)	0.9972 (0.0515)	0.7815 (0.0674)	103.9150	0.0407	0.0241	-201.8300	-192.9206
Kho-Gm**	4.1327 (0.6130)	0.9997 (0.0606)	0.8433 (0.0658)	114.8375	0.0181	0.2931*	-223.6750	-214.7656
Kho-Joe	5.4445 (0.8553)	0.9923 (0.0894)	0.8388 (0.0777)	107.1481	0.0408	0.0103	-208.2962	-199.3868
				$\boldsymbol{x_m < x_f}$				
Clayton	2.6823 (0.4354)	-	-	25.9054	0.1184	0.0035	-49.8108	-47.9190
Frank	11.5220 (1.6796)	-	-	34.9102	0.0364	0.0310	-67.8204	-65.9286
Gumbel**	3.4914 (0.5374)	-	-	38.9184	0.0303	0.0862*	-75.8368	-73.9450
Joe	4.6290 (0.9404)	-	-	35.9466	0.0525	0.0172	-69.8932	-68.0014
Kho-Cl	12.8407 (7.7880)	0.7653 (0.1207)	0.9988 (0.1436)	42.2354	0.0395	0.2448*	-78.4708	-72.7953
Kho-Fr**	37.4053 (1.9693)	0.7133 (0.0778)	0.9820 (0.0714)	45.7855	0.0401	0.1414*	-85.5710	-79.8955
Kho-Gm	5.3987 (1.8039)	0.8168 (0.1022)	0.9984 (0.1166)	43.4638	0.0281	0.2034*	-80.9276	-75.2521
Kho-Joe	6.0364 (1.9875)	0.8587 (0.1227)	0.9994 (0.1348)	39.1878	0.0432	0.0310	-72.3756	-66.7001

The goodness-of-fit test results are further supported by the visualizations given in Figure 1 and Figure 2. Figure 1 presents scatter plots that compare the observed Canadian age-at-death data (black) with simulated data (red) from the best-fitting ArchC and KhoC models. Figure 2 shows contour plots of fitted (black, solid) and empirical (red, dashed) copulas. In both figures, results are shown for the entire population ("whole") and the subpopulations where the male age at death is greater than or equal to ($\geq$), or less than ($<$), that of the female.

Actuarial calculations were performed according to these selected models. Net single premiums and annual premiums were obtained for some annuities and insurance products using the Canadian insurance data set consisting of joint lives (x_f, x_m). In the calculations, the age pairs $(x = 50, y = 60)$, $(x = 55, y = 55)$ and $(x = 60, y = 50)$ were considered. This data set was analyzed for three populations: whole, $x_m \geq x_f$, $x_m < x_f$. The joint survival probabilities are given in Appendices B1 and B2, respectively, under symmetric (ArchC) and asymmetric (KhoC) dependence for the Canadian insurance dataset with population: whole, $x_m \geq x_f$, $x_m < x_f$. Using these probabilities, the actuarial calculations according to the ArchC and KhoC models for all populations are presented in Table 4.

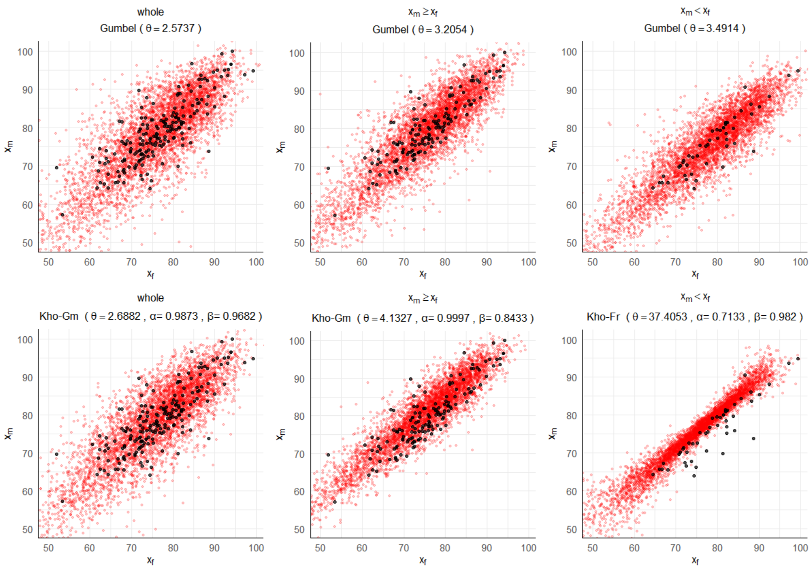

Figure 1. Scatter plots comparing observed Canadian data (black) with simulated data (red) from the best-fitting copula models presented in Table 3, across different populations (whole, $x_m \geq x_f$, $x_m < x_f$) Axes: x_f : age at death (female), x_m : age at death (male)

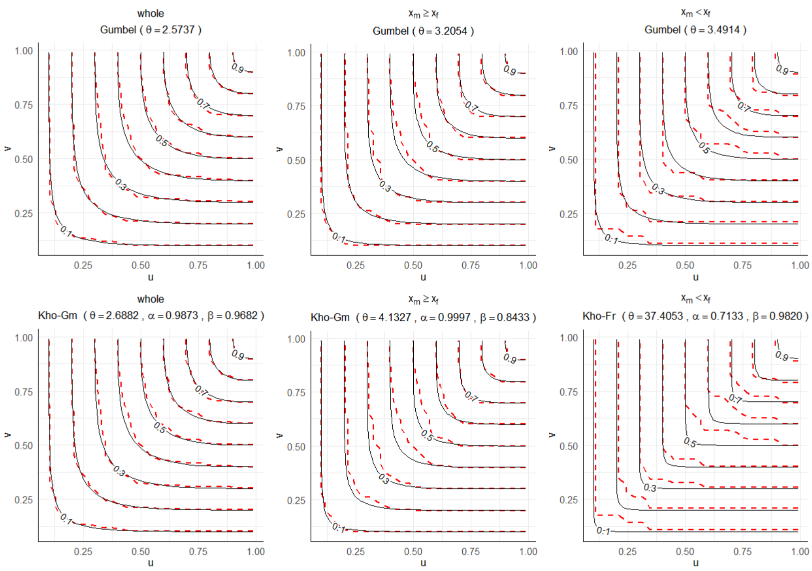

Figure 2. Contour plots comparing fitted (black, solid) and empirical (red, dashed) copulas for the best-fitting ArchC and KhoC models presented in Table 3, across different populations (whole, $x_m \geq x_f$, $x_m < x_f$)). Axes represent pseudo-observations u (female) and v (male)

Table 4 presents the net premiums, their relative ratios, and percentage changes across different populations (whole, $x_m \geq x_f, x_m < x_f$) for the independent, ArchC and KhoC models. The results show the differences between the independent and dependent models in terms of annuities ($\ddot{a}$), insurance value (A) and net annual premiums (P). Moreover, to visually assess the impact of dependency structures for different age groups (50-60, 55-55, 60-55), Figure 3 presents graphs showing the relative ratios of premiums for each population.

Table 4. Net premiums, relative ratios, and percentage changes according to the Independent, ArchC, and KhoC models

					Relative Ratios(%)			Percentage Changes(%)		
Population	Net Premiums	Ind	ArchC	KhoC	ArchC vs Ind	KhoC vs Ind	KhoC vs ArchC	ArchC vs Ind	KhoC vs Ind	KhoC vs ArchC
whole	$\ddot{a}_{50:60\ :\overline{10}\rceil}$	7.983783	8.088586	8.086820	1.013127	1.012906	0.999782	1.31	1.29	-0.02
	$\ddot{a}_{55:55\ :\overline{10}\rceil}$	8.109747	8.231620	8.231060	1.015028	1.014959	0.999932	1.50	1.50	-0.01
	$\ddot{a}_{60:50\ :\overline{10}\rceil}$	8.058702	8.170962	8.171078	1.013930	1.013945	1.000014	1.39	1.39	0.00
$x_m \geq x_f$	$\ddot{a}_{50:60\ :\overline{10}\rceil}$	7.953622	8.072883	8.050205	1.014995	1.012143	0.997191	1.50	1.21	-0.28
	$\ddot{a}_{55:55\ :\overline{10}\rceil}$	8.108063	8.257121	8.243131	1.018384	1.016658	0.998306	1.84	1.67	-0.17
	$\ddot{a}_{60:50\ :\overline{10}\rceil}$	8.083915	8.223776	8.225721	1.017301	1.017542	1.000237	1.73	1.75	0.02
$x_m < x_f$	$\ddot{a}_{50:60\ :\overline{10}\rceil}$	8.039206	8.193682	8.192043	1.019215	1.019011	0.999800	1.92	1.90	-0.02
	$\ddot{a}_{55:55\ :\overline{10}\rceil}$	8.079940	8.249342	8.200434	1.020966	1.014913	0.994071	2.10	1.49	-0.59
	$\ddot{a}_{60:50\ :\overline{10}\rceil}$	7.931432	8.065682	8.009783	1.016926	1.009879	0.993070	1.69	0.99	-0.69
whole	$A_{50:60\ :\overline{10}\rceil}$	0.767463	0.764410	0.764462	0.996022	0.996090	1.000068	-0.40	-0.39	0.01
	$A_{55:55\ :\overline{10}\rceil}$	0.763794	0.760244	0.760260	0.995352	0.995373	1.000021	-0.46	-0.46	0.00
	$A_{60:50\ :\overline{10}\rceil}$	0.765281	0.762011	0.762007	0.995727	0.995722	0.999995	-0.43	-0.43	0.00
$x_m \geq x_f$	$A_{50:60\ :\overline{10}\rceil}$	0.768341	0.764867	0.765528	0.995479	0.996339	1.000864	-0.45	-0.37	0.09
	$A_{55:55\ :\overline{10}\rceil}$	0.763843	0.759501	0.759909	0.994316	0.994850	1.000537	-0.57	-0.52	0.05
	$A_{60:50\ :\overline{10}\rceil}$	0.764546	0.760473	0.760416	0.994673	0.994598	0.999925	-0.53	-0.54	-0.01
$x_m < x_f$	$A_{50:60\ :\overline{10}\rceil}$	0.765848	0.761349	0.761397	0.994125	0.994188	1.000063	-0.59	-0.58	0.01
	$A_{55:55\ :\overline{10}\rceil}$	0.764662	0.759728	0.761152	0.993547	0.995410	1.001874	-0.65	-0.46	0.19
	$A_{60:50\ :\overline{10}\rceil}$	0.768987	0.765077	0.766705	0.994915	0.997032	1.002128	-0.51	-0.30	0.21
whole	$P_{50:60\ :\overline{10}\rceil}$	0.096128	0.094505	0.094532	0.983116	0.983397	1.000286	-1.69	-1.66	0.03
	$P_{55:55\ :\overline{10}\rceil}$	0.094182	0.092357	0.092365	0.980623	0.980708	1.000087	-1.94	-1.93	0.01
	$P_{60:50\ :\overline{10}\rceil}$	0.094963	0.093258	0.093257	0.982046	0.982035	0.999989	-1.80	-1.80	0.00
$x_m \geq x_f$	$P_{50:60\ :\overline{10}\rceil}$	0.096603	0.094745	0.095094	0.980767	0.984379	1.003684	-1.92	-1.56	0.37
	$P_{55:55\ :\overline{10}\rceil}$	0.094208	0.091981	0.092187	0.976361	0.978547	1.002240	-2.36	-2.15	0.22
	$P_{60:50\ :\overline{10}\rceil}$	0.094576	0.092472	0.092444	0.977753	0.977457	0.999697	-2.22	-2.25	-0.03
$x_m < x_f$	$P_{50:60\ :\overline{10}\rceil}$	0.095264	0.092919	0.092943	0.975384	0.975636	1.000258	-2.46	-2.44	0.03
	$P_{55:55\ :\overline{10}\rceil}$	0.094637	0.092096	0.092819	0.973150	0.980790	1.007851	-2.68	-1.92	0.79
	$P_{60:50\ :\overline{10}\rceil}$	0.096954	0.094856	0.095721	0.978361	0.987283	1.009119	-2.16	-1.27	0.91

The results in Table 4 and the graphs in Figure 3 show that the selected ArchC and KhoC models produce higher annuity ($\ddot{a}$) values than the independent model, but generally lower insurance (A) and annual premium (P) values. The relative differences (percentage changes) for annuity premiums range from 1.30% and 2.10%, while the differences for insurance values are less than 1%, and for annual premiums range between 1.27% and 2.68%. However, the relative ratios of the premium values calculated according to the ArchC and KhoC models are generally in the range of 1.000-1.009, indicating that there is no significant difference between the two models. However, it is expected that the premium differences between the dependent models and the independent model will increase as the benefit amounts increase.

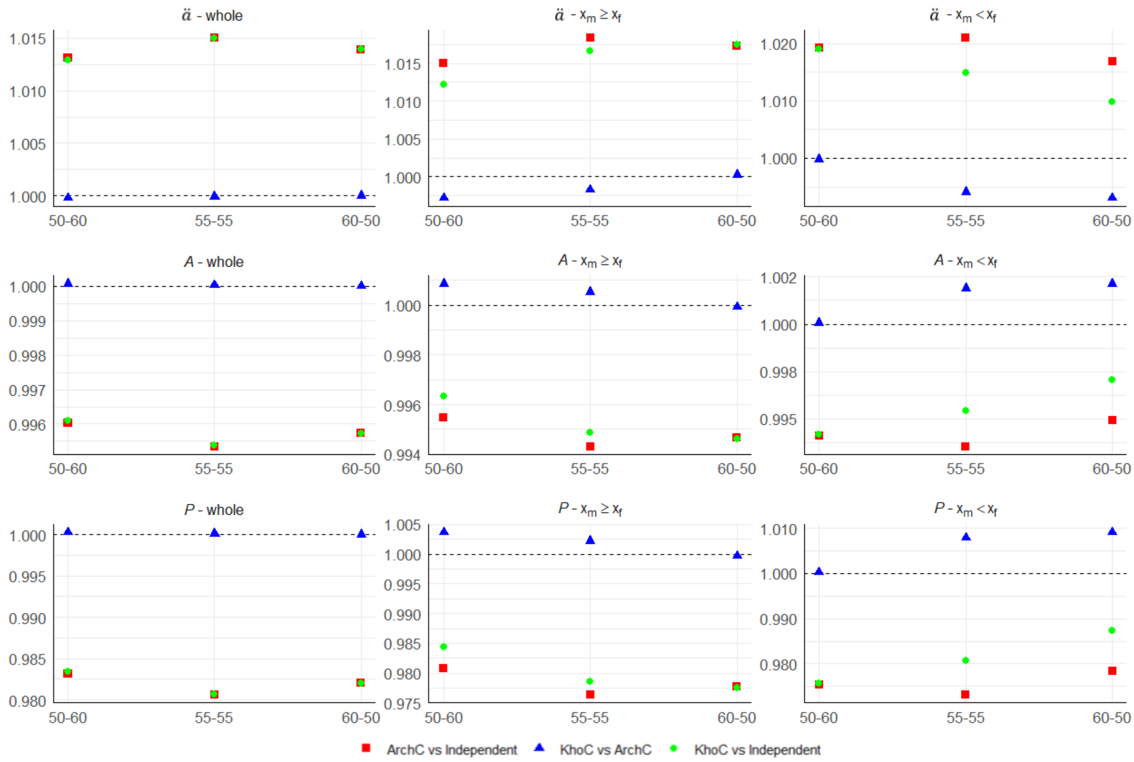

Figure 3. Relative premium ratios of ArchC, KhoC and Independent models by age groups and populations

The graphs in Figure 3 show the effect of dependency structures on premium values, in particular that the ArchC model produces higher premiums than the independent model, but the KhoC model generally produces lower or similar values compared to the ArchC model. It is also observed that premium rates vary between age groups, with particularly marked differences in $x_m \geq x_f$ ve $x_m < x_f$ populations. Although the ArchC model generally generates higher premiums than the independent model, the KhoC model is found to vary in different directions compared to both the independent and ArchC models.

Actuarial calculations are performed and interpreted in order to better explain the effect of the above-mentioned benefit levels on premium differences as follows: Net premium amounts are calculated for a 10-year annuity, which provides annual benefit payments of 10,000 units, and for a 10-year endowment life insurance policy, which provides a benefit payment of 10,000 units in the event of the policyholders death. Table 5 displays the premium differences ($DifPr$) and total premium differences ($Total\_DifPr$) based on the selected ArchC and KhoC models for different populations (whole $x_m \geq x_f, x_m < x_f$). Calculations were performed separately for the Canadian dataset (actual population size given in Table 1) and a sample size of N=1,000. Here, $DifPr$ represents the difference in premiums for a single individual, calculated as $DifPr = PrAsym - PrSym$. The total difference in premiums, denoted as $TotalDifPr = 10,000 \times DifPr \times Sample\ Size$, reflects the overall impact according to the sample size and the amount of benefit. The interest rate is set at %3, and the premium payment period is considered 10 years.

Table 5. Differences Based on Selected Copula Models for Sample Sizes in the Canadian Dataset and N=1,000

Populations	Premiums	Compared Models	DifPr (single person)	TotalDifPr (N:Canadian dataset)	TotalDifPr (N:1,000)
whole	$\ddot{a}_{55:55:\overline{10}\rceil}$	ArchC-Independent	0.121873	235,214.9	1,218,730
	$\ddot{a}_{55:55:\overline{10}\rceil}$	KhoC-Independent	0.121313	234,134.1	1,213,130
	$\ddot{a}_{60:50:\overline{10}\rceil}$	KhoC-ArchC	0.000116	223.88	1,160
	$A_{50:60:\overline{10}\rceil}$	KhoC-ArchC	0.000052	100.36	520
	$P_{50:60:\overline{10}\rceil}$	KhoC-ArchC	0.000027	52.11	270
$x_m \geq x_f$	$\ddot{a}_{55:55:\overline{10}\rceil}$	ArchC-Independent	0.149058	214,643.5	1,490,580
	$\ddot{a}_{60:50:\overline{10}\rceil}$	KhoC-Independent	0.141806	204,200.6	1,418,060
	$\ddot{a}_{60:50:\overline{10}\rceil}$	KhoC-ArchC	0.001945	2,800.8	19,450
	$A_{50:60:\overline{10}\rceil}$	KhoC-ArchC	0.000661	951.84	6,610
	$P_{50:60:\overline{10}\rceil}$	KhoC-ArchC	0.000349	502.56	3,490
$x_m < x_f$	$\ddot{a}_{55:55:\overline{10}\rceil}$	ArchC-Independent	0.169402	83,006.98	1,694,020
	$\ddot{a}_{50:60:\overline{10}\rceil}$	KhoC-Independent	0.152837	74,890.13	1,528,370
	$A_{60:50:\overline{10}\rceil}$	KhoC-ArchC	0.001628	797.72	16,280
	$P_{60:50:\overline{10}\rceil}$	KhoC-ArchC	0.000865	423.85	8,650

Table 5 shows that when 10-year endowment insurance policies with a 10,000 benefit are sold to 144 people in the $x_m \geq x_f$ population, the total annual premium difference $P_{50:60:\overline{10}\rceil}$ for the KhoC-ArchC model is 502.56 units for the Canadian population, while for N=1,000, this difference increases to 3,490 units. Similarly, for the $x_m < x_f$ population, the annual premium $P_{60:50:\overline{10}\rceil}$ for 49 individuals is 423.85 units for the Canadian population, while for N = 1,000, this difference reaches 8,650 units. Similar calculations and interpretations can be made for annuity premiums: for the entire population, the premium difference for the ArchC independent model is 0.121873, and the total premium difference is 235,214.9 units for the Canadian population and 1,218,730 units for N=1,000. The difference for the KhoC-independent model is slightly lower. In the $x_m \geq x_f$ population, the premium difference for the ArchC-independent model is 0.149058 and the total premium difference is 214,643.5 units for the Canadian population and 1,490,580 units for N=1,000. The strongest dependence effect is observed in the population $x_m < x_f$, where the spread of the annuity premium reaches 0.169402 and the total spread of the premium is 83,006,98 units for the Canadian population and 1,694,020 units for N=1,000. According to these results, in all populations, the impact of dependency structures on premium calculations increases with increasing benefit levels and the highest sensitivity is observed for annuity premiums. Although the differences between the ArchC and KhoC models are small, both models produce significant differences compared to the independent model, especially in annuity premiums. Moreover, as the population size increases, total premium differences scale linearly and the effect of dependency becomes more pronounced.

In conclusion, the impact of dependency structures on insurance premium calculations varies according to population type and benefit levels. Therefore, insurance companies should consider the dependency effect in their premium estimates, especially for long-term contracts and high-benefit policies. In particular, this suggests that insurance companies should adopt more cautious reserve policies compared to the independent model. However, these differences are less pronounced in the whole population, indicating that dependency structures have a more significant impact in cases where the lifetime of the females exceeds the lifetime of the males. These findings emphasize the importance of considering dependency structures in premium calculations, particularly when modeling asymmetric survival patterns. Since the dependency measure of spouses in a population with such an asymmetric data structure is unpredictable, it will create riskier portfolios. This implies that premiums should be overestimated. Otherwise, this deficit calculation will adversely affect the economy due to the loss of insurance companies and thus the extra liability to the state.

These findings reveal the concrete real-world implications of including dependence as well as asymmetric dependence structures in actuarial calculations. Traditional models that may not consider dependence and the type of dependence can lead to financial instability by under- or overestimating premiums. For example, ignoring asymmetric dependence in different populations may lead to mispricing, while setting high premiums in populations with weak asymmetry may affect the competitiveness of insurance products. The use of asymmetric copula models allows insurers to refine their pricing strategies so that premiums accurately reflect the true dependency structure of joint lifetimes. This not only improves risk management, but also reduces insolvency risks by enabling more efficient capital allocation. In addition, actuaries can use these findings to improve capital adequacy to ensure that insurers maintain adequate reserves. Therefore, the integration of asymmetric dependence into premium calculations is critical for both financial sustainability and policyholder protection. This clearly demonstrates the need for actuaries to adopt more flexible dependency modeling approaches.

5. Results and discussion

The premiums in the dependent case are lower than in the independent case. However, the premiums in the asymmetric case are not always higher than in the symmetric case; they vary depending on the selected model and the dependence structure. In some cases, asymmetric models yield higher premiums, while in others they result in lower premiums compared to symmetric models. Therefore, no general rule can be established regarding the relationship between symmetric and asymmetric premium calculations.

There is no significant difference between the symmetric and asymmetric premium calculations for the entire population. This finding is consistent with the results of the symmetry test, which confirms that the whole population does not exhibit an asymmetric structure. However, when the population is divided by age, asymmetry emerges in the data structures, leading to more pronounced differences in premium calculations. Thus, when age-based subpopulations are considered, the impact of asymmetry on premium calculations becomes more significant. This highlights the importance of working with different age groups rather than treating the entire population as a homogeneous group in the model selection process.

Actuarial calculations differ on the basis of the independence-dependence and symmetric-asymmetric cases. Although Gumbel is the best fit model for the whole population, the Kho-Gm model is the best-fitting model for the $x_m \geq x_f$ population, and the Kho-Fr model is the best-fitting model for the $x_m < x_f$ population. For the $x_m < x_f$ population, which exhibits an asymmetric dependence, the selected Kho-Fr model generally results in higher premiums than the Gumbel model under the age intervals considered in this study. However, this is not a universal trend in all cases. In some cases, the premium differences between symmetric and asymmetric models are more pronounced than the differences between models selected for different populations. However, the results show that the relative ratios of the premium values calculated using the ArchC and KhoC models are low, indicating that there is no significant difference between the two models. This indicates that the choice between the ArchC and KhoC models does not have a significant effect on the premium amounts under current conditions. However, as coverage levels increase, the premium differences between the dependent models (ArchC and KhoC) and the independent model become more significant. This result shows that the impact of dependency assumptions on premium calculations increases especially in high-coverage insurance policies and emphasizes that insurance companies should carefully evaluate the dependency structure in long-term and high-coverage contracts.

Our study presents results consistent with the literature on modeling joint lifetimes using copulas. For example, Frees et al. [19] showed that premiums calculated according to dependent lifetimes are lower than under the independence assumption; this result is consistent across all subpopulations in our study. However, unlike Frees et al.[19], we focused on both symmetric and asymmetric dependence structures and found that premiums calculated according to asymmetric models (e.g., Kho-Fr) are not always higher or lower than those calculated according to symmetric models (e.g., Gumbel) and vary depending on the model and subpopulation chosen. On the other hand, Dufresne et al. [15] found asymmetric structures in dependent lifetimes due to age differences and emphasized that this should be considered in model selection. Our findings also show that asymmetric dependency structures occur, especially in subpopulations separated according to the age of spouses. Moreover, while previous studies such as Carriere [6] and Denuit et al. [14] are limited to Archimedean copulas, we model asymmetric dependence on real data by including Khoudraji copulas and show the impact of this structure on premium calculations. In this respect, our study emphasizes the importance of asymmetric dependence in insurance pricing based on joint lifetime and provides a more realistic and sensitive assessment for heterogeneous populations. Therefore, this study, which comparatively addresses both symmetric and asymmetric dependence structures, makes a meaningful contribution to the existing literature.

6. Conclusion

This paper proposes symmetric and asymmetric copula models to model the bivariate lifetimes commonly observed in joint life insurance applications. Our study aims to examine the impact of asymmetric dependence, particularly on the pricing of various annuities and insurance products across different age groups. For this purpose, we perform actuarial applications on real data sets. Initially, using real insurance data, survival probabilities for specific ages in single-life scenarios have been computed under the assumption that the marginal distributions of spouses follow the Gompertz distribution. Subsequently, in the context of multiple life scenarios, the joint survival probabilities of spouses have been obtained under the assumptions of independent and dependent future lifetimes. Here, these probabilities have been derived using Archimedean and Khoudraji copula models, which allow for both symmetric and asymmetric dependence modeling. To assess the effect of asymmetric dependence on actuarial calculations, we compute net single and annual premiums based on selected copula models. Using the Canadian insurance data set, we categorize populations as a whole, $x_m \geq x_f$ and $x_m < x_f$. Through a goodness-of-fit analysis among candidate models, we identify optimal copula models and compute premiums for various annuity and insurance products. The results of empirical data indicate differences in net annual premiums under the dependence versus independence assumptions. In particular, the findings of the real data highlight significant discrepancies in the premiums calculated under symmetric and asymmetric models, especially for the population $x_m < x_f$.

The results of our applications focusing on the dependence of joint lifetimes suggest that lifetime dependence and asymmetry should be considered when evaluating the prices of annuities and insurance products. These results provide valuable insights for the insurance industry, particularly in improving pricing accuracy and risk assessment, while also leading to a more realistic assessment of other actuarial quantities such as lifetime and reserves. Furthermore, recognizing the significance of asymmetric dependence can enhance the design of life insurance products tailored to different demographic groups. For example, insurers may consider differentiated pricing strategies based on age differences between spouses, as the findings suggest that certain populations exhibit higher premiums under asymmetric dependency models. This approach can contribute to more equitable pricing structures, ensuring that policyholders pay premiums that reflect their real joint survival risk.

In future studies, considering other dependency factors, such as age differences, and conducting similar studies with asymmetric models across different demographic and geographic regions will further enhance the applicability of these methods. Expanding the scope of research to include policyholder behavior, health status, and socioeconomic factors could further refine actuarial models and support the development of more robust insurance products.

Acknowledgements

We would like to thank the editor and the referees for their constructive comments and suggestions, which significantly improved this article. We also wish to thank the Society of Actuaries, through the courtesy of Edward (Jed) Frees and Emiliano A. Valdez, for allowing use of the data in this paper.

Author contributions. All authors have contributed equally to the preparation of this manuscript.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. This work was supported by the Scientific Research Projects Coordination Unit of Kırıkkale University under project number 2024/030.

Data availability. The dataset used in this study was provided by the Society of Actuaries and is not publicly available. Access was granted through the courtesy of Edward (Jed) Frees and Emiliano A. Valdez.

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APPENDIX

Appendix A: Marginal and joint survival probabilities

Table A1. Marginal survival probabilities (whole)

	whole					
$_tp_{xy}$	t=1		t=5		t=10	
Age	**Male**	**Female**	**Male**	**Female**	**Male**	**Female**
40	0.998627	0.998419	0.991494	0.990152	0.97738	0.973616
45	0.997696	0.997311	0.985765	0.983299	0.962329	0.955518
50	0.996137	0.995428	0.976226	0.971747	0.937589	0.92549
55	0.993525	0.992233	0.960422	0.952398	0.897489	0.876543
60	0.989158	0.986819	0.934473	0.920354	0.834006	0.799126
65	0.98187	0.977673	0.892489	0.868282	0.737395	0.682774
70	0.969761	0.962303	0.826223	0.786351	0.599741	0.522376
75	0.949772	0.936702	0.725882	0.664304	0.423995	0.331194
80	0.917148	0.894691	0.58411	0.498557	0.236923	0.152513
85	0.864893	0.827485	0.405613	0.305908	0.0892103	0.040757
90	0.7838	0.724518	0.219939	0.133233	0.0173181	0.0043145
95	0.664426	0.577884	0.0787406	0.0323832	0.00110589	0.0000944662
100	0.503519	0.393292	0.0140447	0.00291714	0.0000109274	1.41573×10^{-7}
105	0.316154	0.204321	0.000778046	0.0000485316	4.71237×10^{-9}	2.21196×10^{-12}
110	0.144774	0.0670399	6.05668×10^{-6}	4.55777×10^{-8}	1.06013×10^{-14}	1.46516×10^{-20}

Table A2. Marginal survival probabilities ($x_m \geq x_f$)

	$x_m \geq x_f$					
$_tp_{xy}$	t=1		t=5		t=10	
Age	**Male**	**Female**	**Male**	**Female**	**Male**	**Female**
40	0.998723	0.998341	0.992096	0.989643	0.979	0.972175
45	0.997863	0.997163	0.9868	0.98235	0.965086	0.952881
50	0.996424	0.995153	0.977995	0.970001	0.942228	0.920762
55	0.994019	0.991724	0.963428	0.949238	0.905159	0.868317
60	0.990005	0.985887	0.939519	0.914751	0.846325	0.785445
65	0.98332	0.975983	0.900806	0.858644	0.756245	0.661618
70	0.972227	0.959273	0.83952	0.770538	0.626358	0.493364
75	0.953932	0.931352	0.746091	0.640285	0.456862	0.298673
80	0.924064	0.885467	0.61234	0.466469	0.26935	0.126584
85	0.876131	0.812165	0.43987	0.271367	0.111193	0.0291554
90	0.80137	0.700578	0.252787	0.107439	0.0252744	0.00236628
95	0.690193	0.544089	0.0999829	0.0220245	0.00211507	0.0000322571
100	0.537477	0.353095	0.0211543	0.0014646	0.0000332136	2.07964×10^{-8}
105	0.353585	0.168545	0.00157006	0.0000141994	3.16625×10^{-8}	7.25686×10^{-14}
110	0.175373	0.0475755	0.0000201664	5.11069×10^{-9}	2.76906×10^{-13}	3.36303×10^{-23}

Table A3. Marginal survival probabilities (x_m<x_f)

	$x_m < x_f$					
$_tp_x$	t=1		t=5		t=10	
Age	**Male**	**Female**	**Male**	**Female**	**Male**	**Female**
40	0.998341	0.998648	0.98963	0.991603	0.972081	0.977592
45	0.997154	0.997719	0.982267	0.98587	0.952557	0.962477
50	0.995121	0.996154	0.969754	0.976271	0.919956	0.937498
55	0.991639	0.993518	0.948648	0.960284	0.866576	0.896806
60	0.985692	0.989086	0.913485	0.933896	0.782071	0.832101
65	0.975566	0.981651	0.85614	0.890999	0.655781	0.733319
70	0.958428	0.969231	0.765974	0.82303	0.484696	0.592481
75	0.929709	0.948626	0.632783	0.719878	0.288478	0.413405
80	0.882406	0.914843	0.455888	0.574271	0.118384	0.225223
85	0.806754	0.860538	0.259678	0.392189	0.0256639	0.0808161
90	0.691707	0.776106	0.0988296	0.206064	0.00186049	0.0143332
95	0.531165	0.651982	0.0188252	0.069557	0.000020576	0.000773975
100	0.337567	0.485862	0.001093	0.0111272	9.02531×10^{-9}	5.61766×10^{-6}
105	0.155038	0.295783	8.25736×10^{-6}	0.000504858	1.55483×10^{-14}	1.37907×10^{-9}
110	0.0407757	0.128008	1.88297×10^{-9}	2.7316×10^{-6}	1.98725×10^{-24}	1.11574×10^{-15}

Table A4. The joint survival probabilities under independence

	whole			$x_m \geq x_f$			$x_m < x_f$		
t	$p(x$=50, y=60)	$p(x$=55, y=55)	$p(x$=60, y=50)	$p(x$=50, y=60)	$p(x$=55, y=55)	$p(x$=60, y=50)	$p(x$=50, y=60)	$p(x$=55, y=55)	$p(x$=60, y=50)
1	0.983007	0.985809	0.984636	0.982361	0.985793	0.985207	0.984260	0.985211	0.981901
2	0.964457	0.970281	0.967857	0.963106	0.970241	0.969040	0.967055	0.969017	0.962136
3	0.944250	0.953320	0.949567	0.942130	0.953244	0.951404	0.948285	0.951313	0.940599
4	0.922286	0.934826	0.929670	0.919334	0.934704	0.932200	0.927849	0.931999	0.917188
5	0.898473	0.914705	0.908071	0.894622	0.914523	0.911335	0.905650	0.910972	0.891809
6	0.872725	0.892861	0.884683	0.867914	0.892605	0.888718	0.881599	0.888137	0.864380
7	0.844971	0.869207	0.859426	0.839139	0.868862	0.864266	0.855615	0.863406	0.834833
8	0.815157	0.843667	0.832234	0.808251	0.843214	0.837908	0.827632	0.836702	0.803125
9	0.783252	0.816176	0.803056	0.775224	0.815598	0.809586	0.797600	0.807963	0.769238
10	0.749252	0.786688	0.771864	0.740069	0.785965	0.779264	0.765496	0.777150	0.733190

Appendix B: The joint survival probabilities under dependence for Canadian insurance dataset with population: whole, $x_m \geq x_f$, $x_m < x_f$

Table B1. The joint survival probabilities for Archimedean copula models (ArchC)

	whole											
	$p(x=50, y=60)$				$p(x=55, y=55)$				$p(x=60, y=50)$			
t	**Clayton**	**Frank**	**Gumbel**	**Joe**	**Clayton**	**Frank**	**Gumbel**	**Joe**	**Clayton**	**Frank**	**Gumbel**	**Joe**
1	0.986569	0.983357	0.984389	0.983119	0.990462	0.986159	0.987292	0.985920	0.988645	0.984979	0.986050	0.984745
2	0.971835	0.965881	0.967971	0.964938	0.979948	0.971728	0.974114	0.970761	0.976182	0.969267	0.971485	0.968328
3	0.955697	0.947495	0.950406	0.945406	0.968369	0.956673	0.960129	0.954483	0.962521	0.952810	0.955974	0.950705
4	0.938050	0.928111	0.931543	0.924477	0.955633	0.940954	0.945203	0.937049	0.947567	0.935552	0.939376	0.931836
5	0.918792	0.907634	0.911266	0.902111	0.941639	0.924526	0.929235	0.918430	0.931222	0.917425	0.921587	0.911689
6	0.897820	0.885959	0.889473	0.878278	0.926286	0.907338	0.912140	0.898605	0.913387	0.898355	0.902515	0.890239
7	0.875037	0.862972	0.866073	0.852957	0.909466	0.889333	0.893842	0.877561	0.893962	0.878261	0.882080	0.867470
8	0.850350	0.838557	0.840987	0.826134	0.891071	0.870446	0.874271	0.855295	0.872851	0.857053	0.860207	0.843375
9	0.823681	0.812596	0.814149	0.797809	0.870993	0.850605	0.853363	0.831813	0.849959	0.834638	0.836831	0.817958
10	0.794966	0.784974	0.785508	0.767995	0.849126	0.829735	0.831063	0.807135	0.825201	0.810916	0.811898	0.791233
	$x_m \geq x_f$											
	$p(x=50, y=60)$				$p(x=55, y=55)$				$p(x=60, y=50)$			
t	**Clayton**	**Frank**	**Gumbel**	**Joe**	**Clayton**	**Frank**	**Gumbel**	**Joe**	**Clayton**	**Frank**	**Gumbel**	**Joe**
1	0.985819	0.982796	0.984206	0.982519	0.990773	0.986227	0.987854	0.985949	0.989638	0.985630	0.987182	0.985359
2	0.970264	0.964836	0.967548	0.963775	0.980595	0.972010	0.975325	0.970908	0.978261	0.970759	0.973889	0.969691
3	0.953231	0.945998	0.949653	0.943720	0.969379	0.957294	0.962030	0.954848	0.965781	0.955323	0.959748	0.952966
4	0.934616	0.926147	0.930359	0.922311	0.957032	0.942021	0.947824	0.937746	0.952109	0.939249	0.944616	0.935156
5	0.914313	0.905140	0.909537	0.899507	0.943456	0.926122	0.932601	0.919581	0.937151	0.922459	0.928386	0.916241
6	0.892220	0.882826	0.887075	0.875272	0.928547	0.909524	0.916268	0.900339	0.920809	0.904865	0.910963	0.896203
7	0.868242	0.859049	0.862874	0.849572	0.912200	0.892144	0.898740	0.880012	0.902985	0.886372	0.892263	0.875030
8	0.842294	0.833655	0.836844	0.822379	0.894305	0.873892	0.879940	0.858596	0.883580	0.866876	0.872207	0.852715
9	0.814302	0.806496	0.808912	0.793666	0.874753	0.854670	0.859792	0.836093	0.862498	0.846268	0.850722	0.829257
10	0.784215	0.777444	0.779024	0.763414	0.853433	0.834372	0.838228	0.812510	0.839646	0.824432	0.827742	0.804657
	$x_m < x_f$											
	$p(x=50, y=60)$				$p(x=55, y=55)$				$p(x=60, y=50)$			
t	**Clayton**	**Frank**	**Gumbel**	**Joe**	**Clayton**	**Frank**	**Gumbel**	**Joe**	**Clayton**	**Frank**	**Gumbel**	**Joe**
1	0.988891	0.984769	0.986564	0.984446	0.990723	0.985733	0.987648	0.985402	0.985650	0.982419	0.984064	0.982092
2	0.976691	0.969098	0.972610	0.967848	0.980482	0.971123	0.974941	0.969828	0.969904	0.964183	0.967267	0.962945
3	0.963310	0.952893	0.957738	0.950178	0.969189	0.956097	0.961462	0.953256	0.952654	0.945137	0.949212	0.942514
4	0.948651	0.936050	0.941792	0.931412	0.956749	0.940576	0.947059	0.935669	0.933793	0.925118	0.929729	0.920761
5	0.932619	0.918460	0.924657	0.911532	0.943061	0.924478	0.931622	0.917053	0.913216	0.903957	0.908685	0.897646
6	0.915113	0.900005	0.906232	0.890519	0.928020	0.907712	0.915055	0.897398	0.890818	0.881477	0.885964	0.873133
7	0.896033	0.880558	0.886423	0.868361	0.911516	0.890181	0.897270	0.876702	0.866505	0.857502	0.861463	0.847185
8	0.875281	0.859988	0.865146	0.845047	0.893438	0.871780	0.878188	0.854963	0.840189	0.831857	0.835091	0.819769
9	0.852761	0.838156	0.842322	0.820569	0.873671	0.852396	0.857731	0.832184	0.811800	0.804383	0.806774	0.790850
10	0.828386	0.814922	0.817883	0.794919	0.852104	0.831913	0.835831	0.808372	0.781286	0.774941	0.776455	0.760397

Table B2. The joint survival probabilities for Khoudraji copula models (KhoC)

	whole											
	$p(x{=}50, y{=}60)$				$p(x{=}55, y{=}55)$				$p(x{=}60, y{=}50)$			
t	Kho-Cl	Kho-Fr	Kho-Gm	Kho-Joe	Kho-Cl	Kho-Fr	Kho-Gm	Kho-Joe	Kho-Cl	Kho-Fr	Kho-Gm	Kho-Joe
1	0.983768	0.983492	0.984367	0.983135	0.986884	0.986307	0.987283	0.985936	0.986140	0.985127	0.986051	0.984761
2	0.966465	0.966283	0.967915	0.965001	0.973151	0.972210	0.974093	0.970827	0.971856	0.969753	0.971488	0.968393
3	0.947843	0.948158	0.950308	0.945546	0.958513	0.957569	0.960095	0.954631	0.956709	0.953725	0.955979	0.950852
4	0.927770	0.928942	0.931394	0.924721	0.942842	0.942271	0.945152	0.937315	0.940470	0.936914	0.939385	0.932098
5	0.906138	0.908473	0.911057	0.902484	0.926040	0.926217	0.929166	0.918848	0.922978	0.919204	0.921601	0.912100
6	0.882852	0.886601	0.889194	0.878798	0.908020	0.909311	0.912052	0.899207	0.904113	0.900480	0.902534	0.890831
7	0.857829	0.863183	0.865716	0.853634	0.888702	0.891461	0.893731	0.878377	0.883778	0.880628	0.882104	0.868270
8	0.831000	0.838093	0.840543	0.826970	0.868016	0.872573	0.874135	0.856349	0.861889	0.859535	0.860237	0.844407
9	0.802311	0.811218	0.813608	0.798795	0.845897	0.852559	0.853200	0.833125	0.838374	0.837088	0.836867	0.819239
10	0.771728	0.782468	0.784859	0.769105	0.822289	0.831326	0.830870	0.808715	0.813171	0.813179	0.811938	0.792772
	$x_m \geq x_f$											
	$p(x{=}50, y{=}60)$				$p(x{=}55, y{=}55)$				$p(x{=}60, y{=}50)$			
t	Kho-Cl	Kho-Fr	Kho-Gm	Kho-Joe	Kho-Cl	Kho-Fr	Kho-Gm	Kho-Joe	Kho-Cl	Kho-Fr	Kho-Gm	Kho-Joe
1	0.983278	0.982957	0.983811	0.982567	0.987117	0.986429	0.987611	0.985999	0.987106	0.985853	0.987152	0.985410
2	0.965439	0.965180	0.966638	0.963950	0.973648	0.972588	0.974776	0.971102	0.973959	0.971469	0.973868	0.969891
3	0.946214	0.946315	0.948152	0.944076	0.959279	0.958224	0.961129	0.955268	0.960161	0.956630	0.959767	0.953409
4	0.925467	0.926119	0.928201	0.922876	0.943871	0.943155	0.946529	0.938464	0.945563	0.941182	0.944702	0.935931
5	0.903083	0.904402	0.906662	0.900282	0.927320	0.927230	0.930866	0.920654	0.930059	0.924996	0.928566	0.917430
6	0.878966	0.881008	0.883428	0.876222	0.909531	0.910314	0.914048	0.901808	0.913547	0.907954	0.911264	0.897878
7	0.853029	0.855812	0.858405	0.850628	0.890420	0.892278	0.895989	0.881893	0.895926	0.889942	0.892713	0.877253
8	0.825203	0.828717	0.831514	0.823433	0.869906	0.873001	0.876607	0.860883	0.877083	0.870850	0.872832	0.855534
9	0.795435	0.799652	0.802688	0.794572	0.847917	0.852371	0.855828	0.838749	0.856898	0.850565	0.851549	0.832702
10	0.763697	0.768576	0.771886	0.763991	0.824386	0.830285	0.833582	0.815465	0.835248	0.828973	0.828797	0.808742
	$x_m < x_f$											
	$p(x{=}50, y{=}60)$				$p(x{=}55, y{=}55)$				$p(x{=}60, y{=}50)$			
t	Kho-Cl	Kho-Fr	Kho-Gm	Kho-Joe	Kho-Cl	Kho-Fr	Kho-Gm	Kho-Joe	Kho-Cl	Kho-Fr	Kho-Gm	Kho-Joe
1	0.987335	0.985424	0.986631	0.984512	0.987266	0.986276	0.987329	0.985467	0.983265	0.982785	0.983478	0.982154
2	0.974621	0.970918	0.972832	0.968105	0.974058	0.972385	0.974227	0.970078	0.965450	0.964695	0.965933	0.963170
3	0.961287	0.955952	0.958172	0.950745	0.959991	0.957768	0.960299	0.953799	0.946247	0.945228	0.947029	0.942974
4	0.947050	0.940269	0.942487	0.932400	0.944916	0.942172	0.945394	0.936596	0.925509	0.924200	0.926612	0.921493
5	0.931648	0.923696	0.925656	0.913040	0.928718	0.925439	0.929399	0.918440	0.903117	0.901492	0.904559	0.898652
6	0.914855	0.906088	0.907574	0.892635	0.911301	0.907449	0.912219	0.899299	0.878969	0.877009	0.880763	0.874374
7	0.896505	0.887313	0.888144	0.871155	0.892572	0.888107	0.893762	0.879143	0.852975	0.850671	0.855131	0.848579
8	0.876482	0.867242	0.867276	0.848570	0.872447	0.867331	0.873947	0.857943	0.825062	0.822411	0.827583	0.821187
9	0.854700	0.845742	0.844884	0.824851	0.850848	0.845049	0.852695	0.835668	0.795175	0.792180	0.798060	0.792122
10	0.831089	0.822675	0.820891	0.799964	0.827704	0.821200	0.829938	0.812286	0.763282	0.759952	0.766524	0.761312