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Binomial Expansion to 1-Tridiagonal Toeplitz Determinants

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Abstract. The determinants of 1-Tridiagonal Toeplitz matrices are computed using the Binomial Coefficient expansion considering two cases. Each expansion can be computed in parallel, which decreases algorithmic complexity and reduces the overall computation time.

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1. Introduction

The role of matrices in engineering and computational sciences is undeniable. Recently, it was proven that every $n \times n$ matrix can be expressed as a product of $\lfloor \frac{n}{2} \rfloor + 1$ Toeplitz matrices [8]. Ye et al. [8] furthermore, proved that any $n \times n$ matrix can be decomposed into a product of, at most, $2n + 5$ Toeplitz matrices.Their work delineated that $\lfloor \frac{n}{2} \rfloor + 1$ stands for the minimal count of r-Toeplitz matrices essential for expressing any generic $n \times n$ matrix. In an earlier publication, Mackey et al. [6] also examined the similarity between nxn matrices and Toeplitz matrices, demonstrating that any $n \times n$ matrix with $n \leq 4$ can be transformed into a Toeplitz matrix via a similarity transformation.

These discoveries place Toeplitz matrices at the center of matrix theory and its applications in engineering and the computational sciences. Moreover, Toeplitz matrices have found increasing use across a wide range of fields. To name a few, they play a pivotal role in spline function computation, parallel and distributed computing, signal and image processing, the solution of differential equations, boundary value problems, interpolation, physics, and polynomial and power series computations. One specific application concerns the spatial distribution of zeros of eigenpolynomials associated with Hermitian Toeplitz matrices, a topic particularly relevant to signal processing [7].

Thus, any work investigating the properties of Toeplitz matrices contributes to the advancement of their applications in engineering and the computational sciences, thereby furthering these fields. In this paper, we investigate one such property, namely, the determinant. We prove a theorem establishing a binomial expansion form for the determinants of 1-tridiagonal Toeplitz matrices.

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2. Definitions and Notations

Here, the notation W_n or ($W_n^{(1)}$) refers to the determinant of $T_n^{(1)}$ for $n \in \mathbb{N}$. The $n \times n$ Toeplitz Matrix is denoted by T_n. Throughout this paper, $T_n^{(k)}$ represents a $k-$ tridiagonal Toeplitz matrix, and $W_n^{(k)}$ stands for the determinants of $T_n^{(k)}, k > 1$.

Definition 2.1. A Toeplitz matrix is a matrix T_n with

$$t_{ij} = t_{i+1,j+1}, \quad \forall i, j = 1, 2, \cdots, n.$$

Or $T_n = [t_{i,j}]$, where $t_{i,j} = t_{i-j}$.

Definition 2.2. A k-tridiagonal Toeplitz matrix $T_n^{(k)}$ is defined as

$$T_{ij}^{(k)} = \begin{cases} a & ; \quad i = j \\ b & ; \quad j - i = k \\ c & ; \quad i - j = k \\ 0 & \text{otherwise} \end{cases},$$

for k $\in \mathbb{N}$, $i, j = 1, 2, \cdots, n$ and $a, b, c \in \mathbb{C}$.

Below is an illustration of a 1-tridiagonal Toeplitz matrix:

$$T_n^{(1)} = \begin{bmatrix} a & b & & & & & \\ c & a & b & & & & \\ & c & a & b & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & c & a \end{bmatrix}.$$

Note: 1−tridiagonal Toeplitz matrix is when $k = 1$.

3. Preliminaries

The literature provides works on the determinant of 1 and 2− tridiagonal Toeplitz matrices [2,3] and on tridiagonal 2-Toeplitz matrices [4].These references employ elementary row operations to derive recursive formulas. In contrast, our work uses binomial expansions to obtain a non-recursive expression for the determinant.

Bergum [1] elaborated on "a family of 1-Tridiagonal Matrices" and presented a recursive sequence of determinants given in Theorem 3.1.

Theorem 3.1 ([1]). *Determinants of $T_n^{(1)}$ are given recursively by:*

$$W_{n+2} = aW_{n+1} - bcW_n$$

for $n \geq 0$, with initial values, $W_0 = 1$ and $W_1 = a$.

Applying Theorem 3.1, Zhang [9] identified another expression for the determinants of 1-tridiagonal Toeplitz matrices reported in Theorem 3.2.

Theorem 3.2. *[9, p.133] Given an $n \times n$ 1-tridiagonal Toeplitz matrix, $T_n^{(1)}$. Then, its determinant is:*

$$W_n = \begin{cases} a^n & if \quad bc = 0 \\ (n+1)(\frac{a}{2})^n & if \quad a^2 = 4bc \\ \frac{(\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta} & if \quad a^2 \neq 4bc, \end{cases}$$

where $\alpha = \frac{a+\sqrt{a^2-4bc}}{2}$ *and* $\beta = \frac{a-\sqrt{a^2-4bc}}{2}$

Recursive forms for the determinant of 2-tridiagonals came from two different sources [1] and [2]. Bergum identified a recursive formula via the application of the elementary operations. His result is given in Theorem 3.3.

Theorem 3.3 ([1])**.** *For* $k = 2$ *and* $n \geq 5$,

$$W_n^{(2)} = aW_{n-1}^{(2)} - abcW_{n-3}^{(2)} + b^2c^2W_{n-4}^{(2)}.$$

Borrowska's recursive form [2] made use of the gradual application of elementary operations on the determinant of sub-matrices of the same 2− tridiagonal Toeplitz matrix.
For instance, $W_{n-1}^{(2)}$ is the determinant of a sub-matrix, $T_{n-1}^{(2)}$, obtained from $T_n^{(2)}$ by deleting its last row and column. His result is reported in Theorem 3.4. Note that this theorem assumes nonzero determinants of sub-matrices.

Theorem 3.4 ([2])**.** *Let* $T_n^{(2)}$ *be an* $n \times n$ $2-$ *tridiagonal Toeplitz matrix. Then, its determinant is:*

$$W_n^{(2)} = \begin{cases} \left(W_{\frac{n}{2}}\right)^2, & n \text{ is even} \\ W_{\frac{n-1}{2}}W_{\frac{n+1}{2}}, & n \text{ is odd} \end{cases}$$

provided that the determinants of sub-matrices are nonzero.

4. Main Results

In this section, we share our main result regarding the determinants of 1−tridiagonal Toeplitz matrices for $n \geq 2$.

4.1. Binomial Coefficients.

Let us, first, look at two Lemmas used in the proof of our result.

Lemma 4.1. *[5, p.19]. Given* $\binom{n-1}{m-1}, \binom{n-1}{m}$ *for* $n, m \in \mathbb{N}$, $n \geq m$*. Then,*

$$\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}.$$

Proof. Its proof is given in [5, p.27]. □

Note: The equality,

$$\binom{(n+1-m)-1}{m} + \binom{(n+1-m)-1}{m-1} = \binom{n+1-m}{m},$$

is the result of Lemma 4.1 for $n^* = n + 1 - m$.

Lemma 4.2. *For* $n, m \in \mathbb{N}$ *with* $n = 2m + 1$,

$$\binom{n-m}{m+1} = \binom{n-m-1}{m}.$$

Proof. Let us consider the expression:

$$\binom{n-m}{m+1} - \binom{n-m-1}{m},$$

where $n = 2m + 1$.

Applying

$$\begin{aligned}\binom{n-m-1}{m} &= \frac{(n-m-1)!}{(n-2m-1)!m!} \\ &= \frac{(n-m-1)!(m+1)}{(n-2m-1)!m!(m+1)} \\ &= \frac{(n-m-1)!(m+1)}{(n-2m-1)!(m+1)!}\end{aligned}$$

and

$$\binom{n-m}{m+1} = \frac{(n-m)!}{(n-2m-1)!(m+1)!},$$

one obtains:

$$\begin{aligned}\binom{n-m}{m+1} - \binom{n-m-1}{m} &= \frac{(n-m-1)!\,[(m+1)-(n-m)]}{(m+1)!(n-2m-1)!}\\ &= \frac{(n-m-1)!(2m+1-n)}{(m+1)!(n-2m-1)!}\\ &= 0 \quad \text{for } \ n = 2m+1.\end{aligned}$$

□

Now, let us turn our attention to the main contribution, reported in Theorem 4.3.

Theorem 4.3. *Given $n \geq 2$ and the largest $m \in \mathbb{N}$ with $2m \leq n$. Then,*

$$W_n^{(1)} = \sum_{i=0}^{m} (-1)^i \binom{n-i}{i} a^{n-2i}(bc)^i \tag{4.1}$$

Proof. We prove the Theorem inductively in two cases.

- **Case 1:** n=2m

Let us first verify $W_2^{(1)} = a^2 - bc$ and $W_3^{(1)} = a^3 - 2abc$. For both $n = 2$ and $n = 3$, we take $m = 1$.

$$\begin{aligned}W_2^{(1)} &= \sum_{i=0}^{1}(-1)^i \binom{2-i}{i} a^{2-2i}(bc)^i\\ &= a^2 - \binom{2-1}{1} a^{2-2}(bc)^1\\ &= a^2 - bc\end{aligned}$$

and

$$\begin{aligned}W_3^{(1)} &= \sum_{i=0}^{1}(-1)^i \binom{3-i}{i} a^{3-2i}(bc)^i\\ &= a^3 - \binom{3-1}{1} a^{3-2}(bc)^1\\ &= a^3 - 2abc\end{aligned}$$

Now, assume (4.1) for $k = n-1$ and $k = n$. By Theorem 3.1 and with the expressions:

$$W_n^{(1)} = \sum_{i=0}^{m} (-1)^i \binom{n-i}{i} a^{n-2i}(bc)^i$$

and

$$W_{n-1}^{(1)} = \sum_{i=0}^{m-1} (-1)^i \binom{n-1-i}{i} a^{n-1-2i}(bc)^i,$$

we write

$$\begin{aligned}W_{n+1} &= aW_n^{(1)} - bcW_{n-1}^{(1)}\\ &= a\sum_{i=0}^{m} (-1)^i \binom{n-i}{i} a^{n-2i}(bc)^i - bc\sum_{i=0}^{m-1} (-1)^i \binom{n-1-i}{i} a^{n-1-2i}(bc)^i.\end{aligned} \tag{4.2}$$

Let us now show that each Binomial term of $W_{n+1}^{(1)}$ comes from (4.2).

Taking $i = 0$ for $W_{n+1}^{(1)}$ gives

$$(-1)^0\binom{n+1-0}{0}a^{n+1-0}(bc)^0 = a^{n+1},$$

which is the first term ($i = 0$) of $W_n^{(1)}$ multiplied by "a".

That is,

$$\begin{aligned} a\left((-1)^0\binom{n-0}{0}a^{n-0}(bc)^0\right) &= a(a^n) \\ &= a^{n+1}. \end{aligned}$$

Now, let us consider any ith term of W_{n+1} for $0 < i < m$:

$$(-1)^i\binom{n+1-i}{i}a^{n+1-2i}(bc)^i,$$

the same as "a" times the ith term of W_n and "$-bc$" times $(i-1)th$ term of W_{n-1} That is:

$$\begin{aligned} & a\left((-1)^i\binom{n-i}{i}a^{n-2i}(bc)^i\right) - bc\left((-1)^{i-1}\binom{n-1-i+1}{i-1}a^{n-1-2(i-1)}(bc)^{i-1}\right) \\ &= \left((-1)^i\binom{n-i}{i}a^{n+1-2i}(bc)^i\right) + \left((-1)^i\binom{n-1-i+1}{i-1}a^{n+1-2i)}(bc)^i\right) \\ &= (-1)^i a^{n+1-2i}(bc)^i\left[\binom{n-i}{i} + \binom{n-i}{i-1}\right] \\ &= (-1)^i a^{n+1-2i}(bc)^i\left[\binom{n+1-i-1}{i} + \binom{n+1-i-1}{i-1}\right]. \end{aligned}$$

Hence, by Lemma 4.1, we get

$$(-1)^i a^{n+1-2i}(bc)^i\binom{n+1-i}{i}.$$

This completes this part of the proof.

For the last term $i = m$ of W_{n+1}, following a similar process gives:

$$\begin{aligned} & (-1)^m\binom{n+1-m}{m}a^{n+1-2m}(bc)^m \\ &= a\left((-1)^m\binom{n-m}{m}a^{n-2m}(bc)^m\right) \\ &- bc\left((-1)^{m-1}\binom{n-1-m+1}{m-1}a^{n-1-2(m-1)}(bc)^{m-1}\right). \end{aligned}$$

- **Case 2**: n=2m+1

This case follows steps similar to those applied in **Case 1**, differing in the verification of the $(m+1)^{th}$ term of W_{n+1}.

Let us verify this step. We consider $n + 1 = 2m + 2$ and $n - 1 = 2m$.

The $(m+1)^{th}$ term of W_{n+1}. is:

$$(-1)^{m+1}\binom{n+1-(m+1)}{m+1}a^{n+1-2(m+1)}(bc)^{m+1}. \tag{4.4}$$

The last term of W_{n-1} comes from $i = m$ for $n = 2m + 1$. i.e.

$$(-1)^m\binom{n-1-m}{m}a^{n-1-2m}(bc)^m. \tag{4.5}$$

Next, multiplying (4.5) by $-bc$ gives

$$(-1)^m\binom{n-1-m}{m}a^{n-1-2m}(bc)^m \times (-bc). \tag{4.6}$$

By Lemma4.2, we see that (4.6) is identical to (4.4).
This completes the proof. □

4.2. **Illustrations.** Now, we consider a few examples to demonstrate the Theorem.

Example 4.4. Consider the following 1−tridiagonal Toeplitz matrix.

$$T_4^{(1)} = \begin{bmatrix} \mathrm{e} & -5 & 0 & 0 \\ \mathrm{i} & \mathrm{e} & -5 & 0 \\ 0 & \mathrm{i} & \mathrm{e} & -5 \\ 0 & 0 & \mathrm{i} & \mathrm{e}. \end{bmatrix}$$

Its determinant is $e^4 + 15ie^2 - 25$. Here, i is an imaginary unit.
Let us now verify the determinant applying Theorem 4.3. Here, $n = 4$, $m = 2$, $a = e$, $b = -5$, and $c = i$, hence the following expression:

$$\begin{aligned} W_4^{(1)} &= \sum_{j=0}^{2}(-1)^j\binom{4-j}{j}(\mathrm{e})^{4-2j}((-5)(i))^j \\ &= \binom{4-0}{0}(\mathrm{e})^{4-2(0)}[(-5)(\mathrm{i})]^0 - \binom{4-1}{1}(\mathrm{e})^{4-2(1)}((-5)(\mathrm{i}))^1 \\ &\quad + \binom{4-2}{2}(\mathrm{e})^{4-2(2)}[(-5)(\mathrm{i})]^2. \end{aligned}$$

We get:

$$\begin{aligned} W_4^{(1)} &= 1\mathrm{e}^4 - 3\mathrm{e}^2(-5\mathrm{i}) + 1(-5\mathrm{i})^2 \\ &= \mathrm{e}^4 + 15ie^2 - 25. \end{aligned}$$

Now, we turn our attention to an odd size 1−tridiagonal Toeplitz matrix.

Example 4.5. Take the following 1−tridiagonal Toeplitz matrix with determinant 23400.

$$T_5^{(1)} = \begin{bmatrix} 6 & 2 & 0 & 0 & 0 \\ -7 & 6 & 2 & 0 & 0 \\ 0 & -7 & 6 & 2 & 0 \\ 0 & 0 & -7 & 6 & 2 \\ 0 & 0 & 0 & -7 & 6 \end{bmatrix}.$$

Here, $a = 6, b = 2, c = -7, n = 5$, and $m = 2$. Applying Theorem 4.3, we get the following:

$$\begin{aligned} W_5^{(1)} &= \sum_{j=0}^{2}(-1)^j\binom{5-j}{j}(6)^{5-2j}[(-7)(2)]^j \\ &= \binom{5-0}{0}(6)^5 - \binom{5-1}{1}(6)^{5-2}(-7(2))^1 \\ &\quad + \binom{5-2}{2}(6)^{5-4}(-7(2))^2 \\ &= (6)^5 - 4((6)^3(-14)^1 + 3(6)^1(-14)^2 \\ &= 7776 + 12096 + 3528 \\ &= 23400. \end{aligned}$$

In the next example, we consider a complex-valued 1−tridiagonal toeplitz matrix.

Example 4.6. Given the 1−tridiagonal Toeplitz matrix with determinant $20\pi\mathrm{i} - 8\mathrm{i}$. Here, i is an imaginary unit.

$$T_3^{(1)} = \begin{bmatrix} 2\mathrm{i} & -5 & 0 \\ \pi & 2\mathrm{i} & -5 \\ 0 & \pi & 2\mathrm{i} \end{bmatrix}.$$

Here, $a = 2\mathrm{i}, c = \pi, b = -5, n = 3$, and $m = 1$. By Theorem 4.3, we have:

$$\begin{aligned} W_3^{(1)} &= \sum_{j=0}^{1} (-1)^j \binom{3-j}{j} (2i)^{3-2j} [(\pi)(-5)]^j \\ &= \binom{3-0}{0}(2\mathrm{i})^3 - \binom{3-1}{1}(2\mathrm{i})^{3-2}(\pi(-5))^1 \\ &= -8\mathrm{i} - 2(2\mathrm{i}(-5\pi)) \\ &= -8\mathrm{i} + 20\pi\mathrm{i}. \end{aligned}$$

5. Conclusion

We presented our main contribution in Theorem 4.3 and provided a complete proof. The contribution is a method for computing the determinants of 1-tridiagonal Toeplitz matrices using binomial expansions. We also verified our results via examples of both real- and complex-valued 1-tridiagonal Toeplitz matrices of odd and even sizes.

Given the widespread use of Toeplitz matrices and their determinants in the computational sciences, and in light of recent developments showing that any matrix can be decomposed into a product of Toeplitz matrices [8], our result offers new opportunities for innovation and advancement, particularly in reducing algorithmic complexity. One area where we anticipate a significant impact is the parallel computation of determinants. Many existing algorithms rely on the basic definition of the determinant, which is computationally expensive. In contrast, the binomial expansion in Theorem 4.3 allows each term to be computed in parallel, leading to faster determinant computation.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Both authors made equal contributions to this work.

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