Journal of New Theory
ISSN: 2149-1402

48 (2024) 61-77
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

Computation of the Golden Matrix Exponential Functions of Special Matrices

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Article Info
Received: 28 Jul 2024
Accepted: 25 Sep 2024
Published: 30 Sep 2024
doi:10.53570/jnt.1523798
Research Article

Abstract — Computation of the matrix exponential functions is important in solving various scientific and engineering problems due to their active role in solving differential equations. Accurate and effective computation of these functions determines the success of mathematical analysis and practical applications. Therefore, studying and understanding matrix exponential functions is the key to developing mathematical and engineering sciences. In the present paper, we aim to compute the values of the 1st and 2nd type Golden matrix exponential functions for some special matrices. We present the similarities and differences with the value of the well-known matrix exponential function for the same special matrices.

Keywords *Fibonacci sequence, Golden calculus, matrix exponential function*

Mathematics Subject Classification (2020) 15A24, 05A10

1. Introduction

Matrix functions play an important role in both theoretical and applied sciences. For example, they are essential in quantum mechanics, control theory, physics, mathematics, and engineering to solve optimization problems, compute eigenvalues, solve differential equations, and perform transformations that simplify complex systems. Among various matrix functions, matrix exponential and trigonometric functions are particularly noteworthy due to their wide applications and the rich mathematical properties they exhibit. The matrix exponential function is very important in solving linear differential equations and appears prominently in the study of linear dynamical systems. The study and application of these matrix functions are topics of intense research due to their theoretical importance and practical benefits. Researchers constantly explore new methods to compute these functions more efficiently and understand their behavior in different contexts. This ongoing research not only advances our mathematical knowledge but also leads to innovations in various fields of science and engineering. For some of the papers, which include the matrix exponential, trigonometric, and hyperbolic functions, see [1–13].

The matrix exponential function is defined by the Taylor series expansions as

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

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where A is the rth order matrix [9]. The matrix trigonometric and hyperbolic functions are defined by the rules

$$\cos(A) = \frac{e^{iA} + e^{-iA}}{2} \quad \text{and} \quad \sin(A) = \frac{e^{iA} - e^{-iA}}{2i}$$

and

$$\cosh(A) = \frac{e^{A} + e^{-A}}{2} \quad \text{and} \quad \sinh(A) = \frac{e^{A} - e^{-A}}{2}$$

where A is the rth order matrix [8,14].

Sastre et al. [5] provided an algorithm for computing matrix cosine function with the help of the Taylor series and cosine double angle formula. Some methods were presented by Defez and Jodar [15] to compute matrix exponential, sine, and cosine functions based on Hermite matrix polynomials. Besides, Defez et al. [16] introduced a method to compute hyperbolic matrix functions based on Hermite matrix polynomials.

Moreover, number sequences are indispensable tools in mathematical and scientific research. Investigating these sequences provides an in-depth look at scientific research and enables discoveries. For this reason, number sequences are considered the cornerstones of mathematical thinking and scientific progress. The most popular number sequence is undoubtedly the Fibonacci number sequence. The Fibonacci number sequence is defined by the recurrence relation, for $n \geq 1$,

$$F_{n+1} = F_n + F_{n-1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$ [17]. The Binet formula for the Fibonacci number sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ which are called the Golden and Silver ratios, respectively [17]. This number sequence finds extensive applications not only in theoretical mathematics but also across diverse scientific disciplines. In computer science, algorithms based on Fibonacci sequences are used for data structures and sorting problems. In economics, Fibonacci retracement is a popular tool in technical analysis to predict market movements. Additionally, the Fibonacci sequence appears in the study of population growth models. The pervasive presence of Fibonacci numbers in theoretical and applied sciences underscores their importance as a fundamental mathematical concept.

The Golden Fibonacci calculus is introduced by Pashaev and Nalci [18], which is an application of the Fibonacci number sequence. The authors defined the Golden derivative operator, Golden binomial expansion, Golden exponential functions, etc. We present some of the principle definitions of the Golden Fibonacci calculus.

The Fibonacci factorial $F_n!$ is defined by

$$F_n! = \prod_{i=1}^{n} F_i = F_n F_{n-1} F_{n-2} \cdots F_2 F_1$$

where F_n is the nth Fibonacci number [18]. The Golden binomial is defined as

$$(x+y)_F^n = \left(x + \alpha^{n-1} y\right)\left(x + \alpha^{n-2}\beta y\right) \cdots \left(x + \alpha\beta^{n-2} y\right)\left(x + \beta^{n-1} y\right)$$

where α and β are the Golden and Silver ratios, respectively [18]. The Golden binomial also holds the equality

$$(x+y)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k$$

where $\binom{n}{k}_F$ denotes the Fibonacci binomial coefficients which are defined by the rule

$$\binom{n}{k}_F = \frac{F_n!}{F_{(n-k)}!F_k!}$$

with $\binom{n}{0}_F = 1$ [19]. These coefficients are called as Fibonomial coefficients [19]. The 1st and 2nd type Golden exponential functions are defined as [18]

$$e_F^x = \sum_{n=0}^{\infty} \frac{(x)_F^n}{F_n!} \quad \text{and} \quad E_F^x = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{(x)_F^n}{F_n!}$$

Briefly,

$$e_F^x = \sum_{n=0}^{\infty} \frac{x^n}{F_n!} \quad \text{and} \quad E_F^x = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{x^n}{F_n!}$$

Özvatan [19] obtained an estimation for the 1st type Golden exponential base number as

$$3.7041 < e_F < 3.7044$$

Using a similar method, an estimation for the 2nd type Golden exponential base number can be obtained as follows:

$$0.6958 < E_F < 0.6961$$

The Golden Taylor expansions

$$\cos_F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{F_{2n}!} \quad \text{and} \quad \sin_F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{F_{2n+1}!}$$

and

$$\cosh_F(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{F_{2n}!} \quad \text{and} \quad \sinh_F(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{F_{2n+1}!}$$

indicate some Golden trigonometric and Golden hyperbolic functions [18]. These functions also have the following representations [18]:

$$\sin_F(x) = \frac{e_F^{ix} - e_F^{-ix}}{2i} = \frac{E_F^x - E_F^{-x}}{2}$$

$$\cos_F(x) = \frac{e_F^{ix} + e_F^{-ix}}{2} = \frac{E_F^x + E_F^{-x}}{2}$$

$$\sinh_F(x) = \frac{e_F^{x} - e_F^{-x}}{2} = \frac{E_F^{ix} - E_F^{-ix}}{2i}$$

and

$$\cosh_F(x) = \frac{e_F^{x} + e_F^{-x}}{2} = \frac{E_F^{ix} + E_F^{-ix}}{2}$$

By starting from the divisibility problem for the Fibonacci numbers the Fibonacci divisors, and the corresponding hierarchy of Golden derivatives in powers of the Golden ratio are introduced by Pashaev [20]. The author also developed the corresponding quantum calculus. The concepts of the Golden Fibonacci calculus are extended to matrices in [21]. Here, we present some definitions of the Golden Fibonacci matrix calculus.

For the rth order commutable matrices A and B, the Golden binomial is defined as

$$(A+B)_F^n = \left(A + \alpha^{n-1}B\right)\left(A + \alpha^{n-2}\beta B\right)\dots\left(A + \alpha\beta^{n-2}B\right)\left(A + \beta^{n-1}B\right)$$

where α and β represent the Golden and Silver ratios, respectively [21]. The Golden binomial of the rth order commutable matrices A and B also holds the equality [21]:

$$(A+B)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F (-1)^{\frac{k(k-1)}{2}} A^{n-k} B^k$$

The 1st and 2nd type Golden matrix exponential functions have the following Golden Taylor series expansions:

$$e_F^A = \sum_{n=0}^{\infty} \frac{(A)_F^n}{F_n!} \quad \text{and} \quad E_F^A = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{(A)_F^n}{F_n!}$$

where A is the rth order matrix [21]. Briefly, we use the following notations throughout this paper:

$$e_F^A = \sum_{n=0}^{\infty} \frac{A^n}{F_n!} \quad \text{and} \quad E_F^A = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{A^n}{F_n!}$$

There are the following relations for the Golden matrix exponential functions

$$e_{-F}^A = E_F^A \quad \text{and} \quad e_F^A e_{-F}^B = e_F^{A+B}$$

where A and B are rth order commutable matrices [21]. Some Golden matrix trigonometric and hyperbolic functions are defined with the help of the Golden matrix exponential functions as follows:

$$\sin_F(A) = \frac{e_F^{iA} - e_F^{-iA}}{2i} = \frac{E_F^A - E_F^{-A}}{2}$$

$$\cos_F(A) = \frac{e_F^{iA} + e_F^{-iA}}{2} = \frac{E_F^A + E_F^{-A}}{2}$$

$$\sinh_F(A) = \frac{e_F^A - e_F^{-A}}{2} = \frac{E_F^{iA} - E_F^{-iA}}{2i}$$

and

$$\cosh_F(A) = \frac{e_F^A + e_F^{-A}}{2} = \frac{E_F^{iA} + E_F^{-iA}}{2}$$

where A is the rth order matrix [21].

In the present paper, we conduct a detailed examination of the 1st and 2nd type Golden matrix exponential functions. A thorough understanding of these functions may offer alternative approaches to solving differential equations, which play a significant role in various scientific fields. We explore what these functions represent for certain special matrices. In this process, we compare the findings of [6] and [13] regarding the matrix exponential function when similar matrices are used, with the findings we obtained for the 1st and 2nd type Golden matrix exponential functions. We note that to avoid similarity in proving our results, we provide proofs only for the 1st type Golden matrix exponential function, since the 2nd type can be derived in a similar manner.

2. Main Results

The matrix exponential function holds the equality $e^{0_r} = I_r$, for the rth order zero matrix 0_r, where I_r is the rth order identity matrix. We provide similar equalities for the Golden matrix exponential functions in the first proposition.

Proposition 2.1. For the Golden matrix exponential functions of the rth order zero matrix 0_r,

$$e_F^{0_r} = I_r \quad \text{and} \quad E_F^{0_r} = I_r$$

where I_r is the rth order identity matrix.

Proof. Considering the Golden Taylor series expansion of e_F^A, for $A = 0_r$,

$$e_F^{0_r} = \frac{I_r}{F_0!} + \sum_{n=1}^{\infty} \frac{0_r^n}{F_n!} = I_r$$

□

The matrix exponential function has the property $e^{(A^T)} = (e^A)^T$, where A^T is the transpose matrix of the rth order matrix A. The Golden matrix exponential functions have the following property similar to the matrix exponential function.

Proposition 2.2. Let A^T be the transpose matrix of the rth order matrix A. Then, for the Golden matrix exponential functions,

$$e_F^{(A^T)} = \left(e_F^A\right)^T \quad \text{and} \quad E_F^{(A^T)} = \left(E_F^A\right)^T$$

Proof. Using the Golden Taylor series expansion of the 1st type Golden matrix exponential function and the well known property $\left(A^T\right)^s = (A^s)^T$ of the matrix A, for $s \in \{1, 2, 3, \ldots\}$,

$$\begin{aligned} e_F^{(A^T)} &= \sum_{n=0}^{\infty} \frac{\left(A^T\right)^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \frac{(A^n)^T}{F_n!} \\ &= \left(\sum_{n=0}^{\infty} \frac{A^n}{F_n!}\right)^T \\ &= \left(e_F^A\right)^T \end{aligned}$$

□

For the matrix exponential function, $e^{mI_r} = (e^m)I_r$, where I_r is the rth order identity matrix and $m \in \mathbb{Z}$. The Golden matrix exponential functions have similar properties to the matrix exponential function, as follows:

Proposition 2.3. The Golden matrix exponential functions hold the following equalities

$$e_F^{mI_r} = e_F^m I_r \quad \text{and} \quad E_F^{mI_r} = E_F^m I_r$$

for the rth order identity matrix I_r and $m \in \mathbb{Z}$.

Proof. Considering the Golden Taylor series expansion of the 1st type Golden matrix exponential function and the property $I_r^s = I_r$ of identity matrix I_r, for $s \in \{1, 2, 3, \ldots\}$,

$$\begin{aligned} e_F^{mI_r} &= \sum_{n=0}^{\infty} \frac{(mI_r)^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \frac{m^n}{F_n!} I_r \\ &= e_F^m I_r \end{aligned}$$

□

The matrix exponential function satisfies the equation $e^A e^B = e^{A+B}$, for the rth order commutable matrices A and B. However, this property is not provided for the Golden matrix exponential functions. We investigate the Golden matrix exponential functions in terms of this property in the next proposition.

Proposition 2.4. For the rth order commutable matrices A and B,

$$e_F^A e_F^B \neq e_F^{A+B} \quad \text{and} \quad E_F^A E_F^B \neq E_F^{A+B}$$

Proof. Consider the matrix $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then,

$$e_F^{A+B} = e_F^{0_r} = I_r$$

where 0_r is the rth order zero matrix. On the other hand, for the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $e_F^A = \begin{bmatrix} e_F^1 & 0 \\ 0 & e_F^2 \end{bmatrix}$ and $e_F^B = \begin{bmatrix} e_F^{-1} & 0 \\ 0 & e_F^{-2} \end{bmatrix}$. Then,

$$e_F^A e_F^B = \begin{bmatrix} e_F^1 & 0 \\ 0 & e_F^2 \end{bmatrix} \begin{bmatrix} e_F^{-1} & 0 \\ 0 & e_F^{-2} \end{bmatrix}$$

Since $3.7041 < e_F < 3.7044$, it is clear that $e_F^a e_F^{-a} \neq 1$ for an arbitrary number a. Thus, $e_F^A e_F^B \neq I_r$. Therefore,

$$e_F^A e_F^B \neq e_F^{A+B}$$

□

The inverse of the matrix exponential e^A is $(e^A)^{-1} = e^{-A}$, for the rth order matrix A. However, this property is not held by the Golden matrix exponential functions. We provide this property by the next proposition.

Proposition 2.5. Let the matrices $\left(e_F^A\right)^{-1}$ and $\left(E_F^A\right)^{-1}$ be the inverses of e_F^A and E_F^A, respectively, for the rth order matrix A. Then,

$$\left(e_F^A\right)^{-1} \neq e_F^{-A} \quad \text{and} \quad \left(E_F^A\right)^{-1} \neq E_F^{-A}$$

Proof. If the inverse of e_F^A is equal to e_F^{-A}, the equality $e_F^A e_F^{-A} = I_r$ must be held, where I_r is the rth order identity matrix. It is clear from the example in the proof of Proposition 2.4 that this equality is unsatisfied. □

The matrix exponential function has the property $(e^A)^m = e^{mA}$, where A is the rth order matrix and $m \in \mathbb{Z}$. However, in the following proposition, we state that Golden matrix exponential functions do not have similar properties.

Proposition 2.6. The Golden matrix exponential functions satisfy the following inequalities

$$\left(e_F^A\right)^m \neq e_F^{mA} \quad \text{and} \quad \left(E_F^A\right)^m \neq E_F^{mA}$$

for the rth order matrix A and $m \in \mathbb{Z}$.

The proof is clear from Proposition 2.4.

Let A and B be rth order commutable matrices. Then, the matrix exponential function has the equality $e^A e^B = e^B e^A$. We investigate the same property for the Golden matrix exponential functions in the following proposition.

Proposition 2.7. For the Golden matrix exponential functions,

$$e_F^A e_F^B = e_F^B e_F^A \quad \text{and} \quad E_F^A E_F^B = E_F^B E_F^A$$

where A and B are rth order commutable matrices.

Proof. For the commutable rth order matrices A and B,

$$\begin{aligned} e_F^A e_F^B &= \sum_{n=0}^{\infty} \frac{A^n}{F_n!} \sum_{n=0}^{\infty} \frac{B^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \frac{B^n}{F_n!} \sum_{n=0}^{\infty} \frac{A^n}{F_n!} \\ &= e_F^B e_F^A \end{aligned}$$

□

The matrix exponential function has the property $e^{m\overline{I_r}} = \cosh(m) I_r + \sinh(m) \overline{I_r}$, for the rth order anti-identity matrix $\overline{I_r}$, where I_r is the rth order identity matrix. We investigate the Golden matrix exponential functions in terms of this property in the following proposition.

Proposition 2.8. Let $\overline{I_r} = \operatorname{adiag}[1, 1, \cdots, 1]$ be the rth order anti-identity matrix. Then,

$$e_F^{m\overline{I_r}} = \cosh_F(m) I_r + \sinh_F(m) \overline{I_r} \quad \text{and} \quad E_F^{m\overline{I_r}} = \cos_F(m) I_r + \sin_F(m) \overline{I_r}$$

where I_r is the rth order identity matrix.

Proof. For the rth order anti-identity matrix $\overline{I_r}$, $\overline{I_r}^{2s} = I_r$ and $\overline{I_r}^{2s-1} = \overline{I_r}$, for $s \in \{1, 2, 3, \cdots\}$, where I_r is rth order identity matrix. If we use these equalities in the Golden Taylor series expansion of the 1st type Golden matrix exponential function, then

$$\begin{aligned} e_F^{m\overline{I}} &= \sum_{n=0}^{\infty} \frac{\left(m\overline{I_r}\right)^n}{F_n!} \\ &= \frac{I_r}{F_0!} + \frac{m\overline{I_r}}{F_1!} + \frac{(mI_r)^2}{F_2!} + \frac{\left(m\overline{I_r}\right)^3}{F_3!} + \frac{(mI_r)^4}{F_4!} + \cdots \\ &= \left(\frac{1}{F_0!} + \frac{m^2}{F_2!} + \frac{m^4}{F_4!} + \cdots\right) I_r + \left(\frac{m}{F_1!} + \frac{m^3}{F_3!} + \frac{m^5}{F_5!} + \cdots\right) \overline{I_r} \\ &= \cosh_F(m) I_r + \sinh_F(m) \overline{I_r} \end{aligned}$$

□

The matrix exponential function provide the equality $e^{mD} = \operatorname{diag}\left[e^{md_1}, e^{md_2}, e^{md_3}, \cdots, e^{md_r}\right]$, for the rth order diagonal matrix D. We present the following proposition to indicate that the Golden matrix exponential functions have similar properties.

Proposition 2.9. Let $D = \operatorname{diag}[d_1, d_2, d_3, \cdots, d_r]$ be the rth order diagonal matrix. Then,

$$e_F^{mD} = \operatorname{diag}\left[e_F^{md_1}, e_F^{md_2}, e_F^{md_3}, \cdots, e_F^{md_r}\right]$$

and

$$E_F^{mD} = \operatorname{diag}\left[E_F^{md_1}, E_F^{md_2}, E_F^{md_3}, \cdots, E_F^{md_r}\right]$$

Proof. Considering the Golden Taylor series expansion of the 1st type Golden matrix exponential function $e_F^{mD} = \sum_{n=0}^{\infty} \frac{(mD)^n}{F_n!}$ and the sth power matrices $D^s = \operatorname{diag}[d_1^s, d_2^s, d_3^s, \cdots, d_r^s]$, for $s \in \{1, 2, 3, \ldots\}$, the proof is clear. □

The matrix exponential function holds $e^{m\overline{D}} = \cosh_F(m\xi) I_r + \frac{1}{\xi} \sinh(m\xi) \overline{D}$, for the rth order anti-diagonal matrix $\overline{D} = \operatorname{adiag}[d_1', d_2', d_3', \cdots, d_r']$ and identity matrix I_r, where $d_i' d_{r-i+1}' = \xi^2$ and $i \in \{1, 2, \ldots, r\}$. The Golden matrix exponential functions have similar properties to the matrix exponential functions, as follows:

Proposition 2.10. Let $\overline{D}$ be the rth order anti-diagonal matrix mentioned above. Then, the Golden matrix exponential functions hold the following equalities:

$$e_F^{m\overline{D}} = \cosh_F(m\xi) I_r + \frac{1}{\xi}\sinh_F(m\xi)\overline{D} \quad \text{and} \quad E_F^{m\overline{D}} = \cos_F(m\xi) I_r + \frac{1}{\xi}\sin_F(m\xi)\overline{D}$$

where $d'_i d'_{r-i+1} = \xi^2$ and I_r is the rth order identity matrix.

PROOF. Let $\overline{D} = \operatorname{adiag}[d'_1, d'_2, d'_3, \cdots, d'_r]$ be the rth order anti-diagonal matrix. Then,

$$\overline{D}^{2s} = \operatorname{adiag}\left[(d'_1 d'_r)^s, (d'_2 d'_{r-1})^s, (d'_3 d'_{r-2})^s, \cdots, (d'_r d'_1)^s\right] = \xi^{2s} I_r$$

and

$$\overline{D}^{2s-1} = \xi^{2s-1}\overline{D}$$

for $s \in \{1, 2, 3, \ldots\}$, where $d'_i d'_{r-i+1} = \xi^2$, $i \in \{1, 2, \cdots, r\}$, and I_r is the rth order identity matrix. If we substitute these equalities in the Golden Taylor series expansion of 1st type Golden matrix exponential function, then

$$\begin{aligned}
e_F^{m\overline{D}} &= \sum_{n=0}^{\infty} \frac{\left(m\overline{D}\right)^n}{F_n!} \\
&= \frac{I_r}{F_0!} + \frac{m\overline{D}}{F_1!} + \frac{\left(m\overline{D}\right)^2}{F_2!} + \frac{\left(m\overline{D}\right)^3}{F_3!} + \frac{\left(m\overline{D}\right)^4}{F_4!} + \cdots \\
&= \left(\frac{I_r}{F_0!} + \frac{\left(m\overline{D}\right)^2}{F_2!} + \frac{\left(m\overline{D}\right)^4}{F_4!} + \cdots\right) + \left(\frac{m\overline{D}}{F_1!} + \frac{\left(m\overline{D}\right)^3}{F_3!} + \frac{\left(m\overline{D}\right)^5}{F_5!} + \cdots\right) \\
&= \left(\frac{1}{F_0!} + \frac{(m\xi)^2}{F_2!} + \frac{(m\xi)^4}{F_4!} + \cdots\right) I_r + \left(\frac{m}{F_1!} + \frac{m^3\xi^2}{F_3!} + \frac{m^5\xi^4}{F_5!} + \cdots\right)\overline{D} \\
&= \cosh_F(m\xi) I_r + \frac{1}{\xi}\sinh_F(m\xi)\overline{D}
\end{aligned}$$

□

The matrix exponential function holds the equality $e^{mA} = \operatorname{diag}[e^m I_r, e^{-m} I_r]$, for the $2r$th order positive negative identity matrix $A = \begin{bmatrix} I_r & \\ & -I_r \end{bmatrix}$, where I_r is the rth order identity matrix. The Golden matrix exponential functions have properties similar to those of the matrix exponential function, as indicated in the next proposition.

Proposition 2.11. Let A be the $2r$th order positive negative identity matrix mentioned above. Then,

$$e_F^{mA} = \operatorname{diag}\left[e_F^m I_r, e_F^{-m} I_r\right] \quad \text{and} \quad E_F^{mA} = \operatorname{diag}\left[E_F^m I_r, E_F^{-m} I_r\right]$$

where I_r is the rth order identity matrix.

PROOF. For the $2r$th order positive negative identity matrix $A = \begin{bmatrix} I_r & \\ & -I_r \end{bmatrix}$,

$$A^{2s} = I_{2r} \quad \text{and} \quad A^{2s+1} = A$$

where I_{2r} is the $2r$th order identity matrix and $s \in \{1, 2, 3, \ldots\}$. Then,

$$
\begin{aligned}
e_F^{mA} &= \sum_{n=0}^{\infty} \frac{(mA)^n}{F_n!} \\
&= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \left(\frac{I_{2r}}{F_0!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^4}{F_4!} + \cdots \right) + \left(\frac{mA}{F_1!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^5}{F_5!} + \cdots \right) \\
&= \left(\frac{1}{F_0!} + \frac{m^2}{F_2!} + \frac{m^4}{F_4!} + \cdots \right) I_{2r} + \left(\frac{m}{F_1!} + \frac{m^3}{F_3!} + \frac{m^5}{F_5!} + \cdots \right) A \\
&= \cosh_F (m)\, I_{2r} + \sinh_F (m)\, A \\
&= \operatorname{diag}\left[e_F^{m} I_r, e_F^{-m} I_r \right]
\end{aligned}
$$

□

The matrix exponential function satisfies $e^{mA} = \cos(m)\, I_{2r} + \sin(m)\, A$, for the $2r$th order positive negative anti-identity matrix $A = \begin{bmatrix} & I_r \\ -I_r & \end{bmatrix}$, where I_r is the rth order identity matrix. The following proposition gives the corresponding equalities for the Golden matrix exponential functions.

Proposition 2.12. Let A be the $2r$th order positive negative anti-identity matrix mentioned above. Then,

$$
e_F^{mA} = \cos_F (m)\, I_{2r} + \sin_F (m)\, A \quad \text{and} \quad E_F^{mA} = \cosh_F (m)\, I_{2r} + \sinh_F (m)\, A
$$

where I_{2r} is the $2r$th order identity matrix.

PROOF. For the $2r$th order positive negative anti-identity matrix A, $A^2 = -I_{2r}$, $A^3 = -A$, $A^4 = I_{2r}$, $A^5 = A$, $A^6 = -I_{2r}$, $A^7 = -A$, $A^8 = I_{2r}$, $A^9 = A$, ..., where I_{2r} is the $2r$th order identity matrix. Considering these equalities and the Golden Taylor series expansion of the Golden matrix trigonometric function,

$$
\begin{aligned}
e_F^{mA} &= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \left(\frac{I_{2r}}{F_0!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^4}{F_4!} + \cdots \right) + \left(\frac{mA}{F_1!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^5}{F_5!} + \cdots \right) \\
&= \left(\frac{1}{F_0!} - \frac{m^2}{F_2!} + \frac{m^4}{F_4!} - \cdots \right) I_{2r} + \left(\frac{m}{F_1!} - \frac{m^3}{F_3!} + \frac{m^5}{F_5!} - \cdots \right) A \\
&= \cos_F (m)\, I_{2r} + \sin_F (m)\, A
\end{aligned}
$$

□

For the rth order positive negative alternating identity matrix $A = \operatorname{diag}\left[1, -1, 1, -1, \cdots, (-1)^{r-1}\right]$, the matrix exponential function is equal to $e^{mA} = \operatorname{diag}\left[e^{m}, e^{-m}, e^{m}, e^{-m}, \cdots, e^{(-1)^{r-1}m}\right]$. In the following proposition, we investigate the Golden matrix exponential functions for this kind of matrix.

Proposition 2.13. For the rth order positive negative alternating identity matrix A mentioned above, the Golden matrix exponential are

$$
e_F^{mA} = \operatorname{diag}\left[e_F^{m}, e_F^{-m}, e_F^{m}, e_F^{-m}, \cdots, e_F^{(-1)^{r-1}m} \right]
$$

and

$$E_F^{mA} = \operatorname{diag}\left[E_F^{m}, E_F^{-m}, E_F^{m}, E_F^{-m}, \cdots, E_F^{(-1)^{r-1}m}\right]$$

PROOF. Let A be the rth order positive negative alternating identity matrix. Then, $A^{2s} = I_{2r}$ and $A^{2s+1} = A$, where I_{2r} is the $2r$th order identity matrix and $s \in \{1, 2, 3, \ldots\}$. Thus the proof is similar to the proof of Proposition 2.11.

□

The matrix exponential function holds $e^{mA} = \cos(m) I_r + \sin(m) A$, for the rth order positive negative alternating anti-identity matrix $A = \operatorname{adiag}\left[1, -1, 1, -1, \cdots, (-1)^{r-1}\right]$, where I_r is the rth order identity matrix. The Golden matrix exponential functions have similar properties to the matrix exponential function, as follows:

Proposition 2.14. Let A be the rth order positive negative alternating anti-identity matrix mentioned above. Then,

$$e_F^{mA} = \cos_F(m) I_r + \sin_F(m) A \quad \text{and} \quad E_F^{mA} = \cosh_F(m) I_r + \sinh_F(m) A$$

where I_r is the rth order identity matrix.

PROOF. For the rth order positive negative alternating anti-identity matrix A, $A^2 = -I_r$, $A^3 = -A$, $A^4 = I_r$, $A^5 = A$, $A^6 = -I_r$, $A^7 = -A$, $A^8 = I_r$, $A^9 = A$, ..., where I_r is the rth order identity matrix. Using these equalities and the Golden Taylor series expansions of the Golden matrix trigonometric functions,

$$\begin{aligned} e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\ &= \left(\frac{I_r}{F_0!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^4}{F_4!} + \cdots\right) + \left(\frac{mA}{F_1!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^5}{F_5!} + \cdots\right) \\ &= \left(\frac{1}{F_0!} - \frac{m^2}{F_2!} + \frac{m^4}{F_4!} - \cdots\right) I_r + \left(\frac{m}{F_1!} - \frac{m^3}{F_3!} + \frac{m^5}{F_5!} - \cdots\right) A \\ &= \cos_F(m) I_r + \sin_F(m) A \end{aligned}$$

□

The matrix exponential function is equal to $e^{mA} = \cosh(m) I_r + \sinh(m) A$, for the rth order square identity matrix A with $A^2 = I_r$, where I_r is the rth order identity matrix. We provide the following proposition to give similar properties of the Golden matrix exponential functions.

Proposition 2.15. For the rth order square identity matrix A mentioned above,

$$e_F^{mA} = \cosh_F(m) I_r + \sinh_F(m) A \quad \text{and} \quad E_F^{mA} = \cos_F(m) I_r + \sin_F(m) A$$

where I_r is the rth order identity matrix.

PROOF. The matrix A with $A^2 = I_r$ has the equalities

$$A^{2s} = I_r \quad \text{and} \quad A^{2s-1} = A$$

for $s \in \{1, 2, 3, \ldots\}$, where I_r is the rth order identity matrix. Considering these equalities and the Golden Taylor series expansion of the Golden matrix hyperbolic function,

$$
\begin{aligned}
e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \left(\frac{I_r}{F_0!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^4}{F_4!} + \cdots\right) + \left(\frac{mA}{F_1!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^5}{F_5!} + \cdots\right) \\
&= \left(\frac{1}{F_0!} + \frac{m^2}{F_2!} + \frac{m^4}{F_4!} + \cdots\right) I_r + \left(\frac{m}{F_1!} + \frac{m^3}{F_3!} + \frac{m^5}{F_5!} + \cdots\right) A \\
&= \cosh_F(m)\, I_r + \sinh_F(m)\, A
\end{aligned}
$$

□

The matrix exponential function has the equality $e^{mA} = \cos(m)\, I_r + \sin(m)\, A$, where A is the rth order square anti-identity matrix and I_r is the rth order identity matrix. We investigate the Golden matrix exponential functions in terms of this property in the following proposition.

Proposition 2.16. Let A be the rth order square anti-identity matrix with $A^2 = -I_r$. Then,

$$e_F^{mA} = \cos_F(m)\, I_r + \sin_F(m)\, A \quad \text{and} \quad E_F^{mA} = \cosh_F(m)\, I_r + \sinh_F(m)\, A$$

Proof. For the rth order square anti-identity matrix A with $A^2 = -I_r$, $A^3 = -A$, $A^4 = I_r$, $A^5 = A$, $A^6 = -I_r$, $A^7 = -A$, $A^8 = I_r$, $A^9 = A$, ..., where I_r is the rth order identity matrix. Thus, the proof is similar to the proof of Proposition 2.14. □

The matrix exponential function has the property $e^{mA} = I_r + (e^m - 1)\, A$, for the rth order idempotent matrix A, where I_r is the rth order identity matrix. The Golden matrix exponential functions have similar properties to the matrix exponential function, as follows:

Proposition 2.17. Let A be the rth order idempotent matrix with $A^s = A$, for $s \in \{1, 2, 3, \ldots\}$. Then,

$$e_F^{mA} = I_r + (e_F^m - 1)\, A \quad \text{and} \quad E_F^{mA} = I_r + (E_F^m - 1)\, A$$

where I_r is the rth order identity matrix.

Proof. By the Golden Taylor series expansion of the 1st type Golden matrix exponential function,

$$
\begin{aligned}
e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \frac{I_r}{F_0!} + \left(\frac{m}{F_1!} + \frac{m^2}{F_2!} + \frac{m^3}{F_3!} + \frac{m^4}{F_4!} + \cdots\right) A \\
&= I_r + (e_F^m - 1)\, A
\end{aligned}
$$

□

The matrix exponential function has the equality $e^{mA} = I_r + (1 - e^{-m})\, A$, for the rth order anti-idempotent matrix A. In the following proposition, we research the Golden matrix exponential functions for this kind of matrix.

Proposition 2.18. For the rth order anti-idempotent matrix A with $A^s = (-1)^{s-1} A$, for $s \in \{1, 2, 3, \ldots\}$, the Golden matrix exponential functions are

$$e_F^{mA} = I_r + \left(1 - e_F^{-m}\right) A \quad \text{and} \quad E_F^{mA} = I_r + \left(1 + E_F^{-m}\right) A$$

where I_r is the rth order identity matrix.

Proof. By the Golden Taylor series expansions of the Golden hyperbolic functions,

$$\begin{aligned}
e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \frac{I_r}{F_0!} + \left(-\frac{m^2}{F_2!} - \frac{m^4}{F_4!} - \frac{m^6}{F_6!} - \cdots\right) A + \left(\frac{m}{F_1!} + \frac{m^3}{F_3!} + \frac{m^5}{F_5!} + \cdots\right) A \\
&= I_r + (1 - \cosh_F(m) + \sinh_F(m)) A \\
&= I_r + \left(1 - e_F^{-m}\right) A
\end{aligned}$$

□

Let $A = \begin{bmatrix} I_r & I_r \\ I_r & I_r \end{bmatrix}$ be the $2r$th order block identity matrix, where I_r is the rth order identity matrix. Then, the matrix exponential function has the equality $e^{mA} = I_{2r} + \frac{1}{2}\left(e^{2m} - 1\right) A$, where I_{2r} is the $2r$th order identity matrix. The following proposition indicates the Golden matrix exponential functions have similar equalities to the matrix exponential function for this kind of matrix.

Proposition 2.19. For the $2r$th order block identity matrix A mentioned above,

$$e_F^{mA} = I_{2r} + \frac{1}{2}\left(e_F^{2m} - 1\right) A \quad \text{and} \quad E_F^{mA} = I_{2r} + \frac{1}{2}\left(E_F^{2m} - 1\right) A$$

where I_{2r} is the $2r$th order identity matrix.

Proof. For the $2r$th order block identity matrix A, $A^s = 2^{s-1}A$, for $s \in \{2, 3, \ldots\}$. Substituting these equalities in the Golden Taylor series expansion of the 1st type Golden matrix exponential function,

$$\begin{aligned}
e_F^{mA} &= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(m)^2 2A}{F_2!} + \frac{(m)^3 2^2 A}{F_3!} + \frac{(m)^4 2^3 A}{F_4!} + \cdots \\
&= \frac{I_{2r}}{F_0!} + \frac{1}{2}\left(\frac{2m}{F_1!} + \frac{(2m)^2}{F_2!} + \frac{(2m)^3}{F_3!} + \frac{(2m)^4}{F_4!} + \cdots\right) A \\
&= I_{2r} + \frac{1}{2}\left(e_F^{2m} - 1\right) A
\end{aligned}$$

□

Let $A = \begin{bmatrix} -I_r & I_r \\ I_r & -I_r \end{bmatrix}$, where I_r is the rth order identity matrix. Then, the matrix exponential function for A is equal to $e^{mA} = I_{2r} - \frac{1}{2}\left(e^{2m} - 1\right) A$. In the next proposition, we investigate the Golden matrix exponential functions for this kind of matrix.

Proposition 2.20. Let A be the $2r$th order block identity matrix mentioned above. Then,

$$e_F^{mA} = I_{2r} - \frac{1}{2}\left(e_F^{2m} - 1\right) A \quad \text{and} \quad E_F^{mA} = I_{2r} - \frac{1}{2}\left(E_F^{2m} - 1\right) A$$

where I_{2r} is the $2r$th order identity matrix.

Proof. Using the Golden Taylor series expansion of the 1st type Golden matrix exponential function and considering $A^s = (-2)^{s-1} A$, for $s \in \{2, 3, \ldots\}$,

$$
\begin{aligned}
e_F^{mA} &= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(m)^2\,2A}{F_2!} + \frac{(m)^3\,2^2A}{F_3!} + \frac{(m)^4\,2^3A}{F_4!} + \cdots \\
&= \frac{I_{2r}}{F_0!} + \frac{1}{2}\left(\frac{2m}{F_1!} - \frac{(2m)^2}{F_2!} + \frac{(2m)^3}{F_3!} - \frac{(2m)^4}{F_4!} + \cdots\right) A \\
&= \frac{I_{2r}}{F_0!} - \frac{1}{2}\left(\frac{(2m)^2}{F_2!} + \frac{(2m)^4}{F_4!} + \frac{(2m)^6}{F_6!} + \cdots\right) A + \frac{1}{2}\left(\frac{2m}{F_1!} + \frac{(2m)^3}{F_3!} + \frac{(2m)^5}{F_5!} + \cdots\right) A \\
&= I_{2r} - \frac{1}{2}\left(\cosh_F(2m) - 1 + \sinh_F(2m)\right) A \\
&= I_{2r} - \frac{1}{2}\left(e_F^{2m} - 1\right) A
\end{aligned}
$$

□

For the $2r$th order matrix $A = \begin{bmatrix} I_r & -I_r \\ -I_r & I_r \end{bmatrix}$, where I_r is the rth order identity matrix, the matrix exponential function for A is equal to $e^{mA} = I_{2r} - \frac{1}{2}\left(e^{2m} - 1\right) A$, where I_{2r} is the $2r$th order identity matrix. The Golden matrix exponential functions behave similarly to matrix exponential function, as stated in the proposition below.

Proposition 2.21. Let the matrix A be as mentioned above. Then,

$$
e_F^{mA} = I_{2r} - \frac{1}{2}\left(e_F^{2m} - 1\right) A \quad \text{and} \quad E_F^{mA} = I_{2r} - \frac{1}{2}\left(E_F^{2m} - 1\right) A
$$

where I_{2r} is the $2r$th order identity matrix.

The proof is similar to the proof of Proposition 2.20, considering $A^s = (2)^{s-1} A$, for $s \in \{2, 3, \ldots\}$.

The matrix exponential function is equal to $e^{mA} = I_r - \frac{1}{r}\left(e^{rm} - 1\right) A$, for the rth order unity matrix A, i.e., all entries of A equal to 1, where I_r is the rth order identity matrix. We investigate the Golden matrix exponential functions for the unity matrix.

Proposition 2.22. Let A be the rth order unity matrix. Then,

$$
e_F^{mA} = I_r - \frac{1}{r}\left(e_F^{rm} - 1\right) A \quad \text{and} \quad E_F^{mA} = I_r - \frac{1}{r}\left(E_F^{rm} - 1\right) A
$$

where I_r is the rth order identity matrix.

Proof. For the unity matrix A, $A^s = r^{s-1}A$ for $s \in \{2, 3, \ldots\}$. Considering these equalities and the Golden Taylor series expansion of the 1st type Golden matrix exponential function,

$$
\begin{aligned}
e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(m)^2\,rA}{F_2!} + \frac{(m)^3\,r^2A}{F_3!} + \frac{(m)^4\,r^3A}{F_4!} + \cdots \\
&= \frac{I_r}{F_0!} + \frac{1}{r}\left(\frac{rm}{F_1!} + \frac{(rm)^2}{F_2!} + \frac{(rm)^3}{F_3!} + \frac{(rm)^4}{F_4!} + \cdots\right) A \\
&= I_r - \frac{1}{r}\left(e_F^{rm} - 1\right) A
\end{aligned}
$$

□

The matrix exponential function is $e^{mA} = \sum_{n=0}^{s-1} \frac{(mA)^n}{n!}$, for the nilponent matrix A with $A^s = 0$, where $s \in \{2, 3, \ldots\}$. The next proposition includes the values of the Golden matrix exponential functions for the nilpotent matrix.

Proposition 2.23. For the rth order nilpotent matrix A with $A^s = 0$, for $s \in \{2, 3, \ldots\}$,

$$e_F^{mA} = \sum_{n=0}^{s-1} \frac{(mA)^n}{F_n!}$$

and

$$E_F^{mA} = \sum_{n=0}^{s-1} (-1)^{\frac{n(n-1)}{2}} \frac{(mA)^n}{F_n!}$$

PROOF. Since $A^s = A^{s+1} = A^{s+2} = \cdots = 0_r$, where 0_r is the rth order zero matrix, the result is clear. □

Corollary 2.24. A strictly triangular matrix which is a triangular matrix with zero diagonal entries has similar results to the matrix in Proposition 2.23, because it is also a nilpotent matrix.

The matrix exponential function of the rth order matrix A can be obtained via its similar matrix B, that is, $e^{mA} = Pe^{mB}P^{-1}$, where P is a non-singular matrix. We investigate this property for the Golden matrix exponential functions.

Proposition 2.25. Let the rth order matrices A and B be similar. Then, the Golden matrix exponential functions of A can be calculated by the rules

$$e_F^{mA} = Pe_F^{mB}P^{-1} \quad \text{and} \quad E_F^{mA} = PE_F^{mB}P^{-1}$$

where P is a non singular matrix such that $A = PBP^{-1}$.

PROOF. By applying the Golden Taylor series expansion of the 1st type Golden matrix exponential function,

$$\begin{aligned}
e_F^{mA} &= e_F^{mPBP^{-1}} \\
&= \frac{I_r}{F_0!} + \frac{mPBP^{-1}}{F_1!} + \frac{(mPBP^{-1})^2}{F_2!} + \frac{(mPBP^{-1})^3}{F_3!} + \frac{(mPBP^{-1})^4}{F_4!} + \cdots \\
&= I_r + mPBP^{-1} + \frac{m^2PB^2P^{-1}}{F_2!} + \frac{m^3PB^3P^{-1}}{F_3!} + \frac{m^4PB^4P^{-1}}{F_4!} + \cdots \\
&= P\left(I_r + mB + \frac{(mB)^2}{F_2!} + \frac{(mB)^3}{F_3!} + \frac{(mB)^4}{F_4!} + \cdots\right)P^{-1} \\
&= Pe_F^{mB}P^{-1}
\end{aligned}$$

where I_r is the rth order identity matrix. □

For the rth order Jordan matrix A, the matrix exponential function is equal to $e^{mA} = [c_{ij}]_{i,j=1}^r$ with

$$c_{ij} = \begin{cases} \frac{m^k}{k!}e^{m\lambda}, & i = j - k \text{ and } k \in \{0, 1, 2, \cdots, r-1\} \\ 0, & \text{otherwise} \end{cases}$$

Finally, we explore the Golden matrix exponential functions for the Jordan matrix A.

Proposition 2.26. For the rth order Jordan matrix $A = [a_{ij}]_{i,j=1}^{r}$ with $a_{ij} = \begin{cases} \lambda, & i=j \\ 1, & i=j-1 \\ 0, & \text{otherwise} \end{cases}$, the Golden matrix exponential functions are

$$e_F^{mA} = \left[c_{ij}^{*}\right]_{i,j=1}^{r} \quad \text{with} \quad c_{ij}^{*} = \begin{cases} \dfrac{m^k}{F_k!} e_F^{m\lambda}, & i=j-k \\ 0, & \text{otherwise} \end{cases}$$

and

$$E_F^{mA} = \left[c_{ij}^{**}\right]_{i,j=1}^{r} \quad \text{with} \quad c_{ij}^{**} = \begin{cases} \dfrac{m^k}{F_k!} E_F^{m\lambda}, & i=j-k \\ 0, & \text{otherwise} \end{cases}$$

where $k \in \{0, 1, 2, \dots, r-1\}$.

Proof. The sth power of matrix A is

$$A^s = \begin{bmatrix} \lambda^s & s\lambda^{s-1} & \frac{s(s-1)}{2!}\lambda^{s-2} & \frac{s(s-1)(s-2)}{3!}\lambda^{s-3} & \cdots & \frac{s(s-1)(s-2)\dots(s-r+1)}{(r-1)!}\lambda^{s-r+1} \\ 0 & \lambda^s & s\lambda^{s-1} & \frac{s(s-1)}{2!}\lambda^{s-2} & \cdots & \vdots \\ 0 & 0 & \lambda^s & s\lambda^{s-1} & \cdots & \frac{s(s-1)(s-2)}{3!}\lambda^{s-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^s & s\lambda^{s-1} \\ 0 & 0 & 0 & \cdots & 0 & \lambda^s \end{bmatrix}$$

and the Golden Taylor series expansion of the 1st type Golden matrix exponential function

$$e_F^{mA} = \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots$$

This completes the proof. □

3. Conclusion

In the present paper, we have computed the values of the 1st and 2nd type Golden matrix exponential functions for some special matrices. We have presented a comparative analysis of these values with the well-known matrix exponential function value for the same special matrices. We believe that a good understanding of these functions will enable the development of new alternative approaches to solving differential equations and optimization problems in various sciences and engineering. For future research, the solution of the linear Golden 1st order autonomous system $D_F^t f = Af$, where D_F is the Golden time derivative, can be written as Golden matrix exponential $f = e_F(At) f_0$, where A is an rth order matrix. Additionally, by selecting different forms for the matrix A, various dynamical systems with Golden evolution can be defined.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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