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The Properties of Binomial Transforms for Modified (s,t)-Pell Matrix Sequence

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Abstract
In this study, we investigate a generalization of the modified Pell sequence, which is called (s,t)-modified Pell sequence. By considering this sequence, we define the matrix sequence whose elements are (s,t)-modified Pell numbers. Furthermore, we define various binomial transforms for modified (s,t)-Pell matrix sequence. Finally, we give some relationships for (s,t)-modified Pell matrix sequences such as Binet formulas, the generating functions, and some sum formulas.

Keywords: Binet formula, Binomial transforms, Generating function, Modified (s,t)-Pell sequence
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1. Introduction

Special integer sequences have been an important research subject for mathematicians for a very long time. These sequences are widely encontered in many scientific areas such as engineering, architecture, physics, and art. The Pell, Pell-Lucas, modified Pell sequence are very popular examples of second order recurrence sequences. The binomial transform is a discrete transformation of one sequence into another sequence with fascinating applications in combinatorics. The binomial transform give us interesting combinatorial properties. The binomial transforms of special integer sequence was firstly studied by Falcon and Plaza in [1]. Yilmaz and Taskara derived the binomial transforms of the Padovan and Perrin matrix sequences in [2]. Then, Bhadouria, Jhala, and Singh evaluated the binomial transforms of the k-Lucas sequence in [3]. The binomial transforms of the k-Jacobsthal and k-Jacobsthal-Lucas sequences were studied by Uygun and Erdoğdu in [4, 8]. In [5], binomial transform of quadrapell sequences and quadrapell matrix sequences are dealt with by Kızılates, Tuglu, and Çekim. Uygun computed the binomial transforms of the generalized (s,t)-Jacobsthal matrix sequence in [6]. Kwon gave the binomial transforms of the modified k-Fibonacci-like sequence in [7]. Yılmaz studied binomial transforms of the balancing and Lucas-balancing polynomials in [9]. Soykan studied binomial transforms of the generalized Tribonacci sequence, the generalized third order Pell sequence, the generalized fourth order Pell sequence, the generalized fifth order Pell sequence, the generalized Narayana sequence, the generalized Pentanacci sequence, the generalized Jacobsthal-Padovan numbers in [10]-[16]. Boyadzhiev presented new binomial identities for Bernoulli, Fibonacci, and harmonic numbers in [17]. A novel binomial transform based fragile watermarking technique has been proposed for color image authentication by Ghosal and Mandal in [18]. In[19], Wani, Catarino, and Halici studied generalized Pell sequence and its matrix sequence. Özkoç Öztürk and Gündüz obtained binomial transform for quadra Fibona-Pell sequence and quadra Fibona-Pell quaternion in [20].

The modified Pell numbers q_n are defined by the recurrence relation

$$q_n = 2q_{n-1} + q_{n-2},$$

for $n \geq 2$, beginning with the values $q_0 = 1$, $q_1 = 1$ in [21]. Because of the importance of special integer sequences, the mathematicians generalize these sequences. In this paper, we introduce a generalization of modified Pell numbers called (s,t)-Pell matrix sequence, depending on two real numbers s,t and $s^2+t > 0$. Modified (s,t)-Pell matrix sequence, $\{q_n\}_{n\in\mathbb{N}}$ is defined recurrently by

$$q_n(s,t) = 2sq_{n-1}(s,t) + tq_{n-2}(s,t), \ \ q_0(s,t) = 1 \ \text{ and } \ q_1(s,t) = s,$$

in [22]. Binet formula gives us the chance find any elements of the modified (s,t)-Pell matrix sequence easily. The Binet formula for (s,t)-Pell matrix sequence $q_n(s,t) = \frac{x_1^n + x_2^n}{2}$, where $x_1 = s + \sqrt{s^2+t}$, $x_2 = s - \sqrt{s^2+t}$.

The main goal of this paper is to apply different binomial transforms to the modified (s,t)-Pell matrix sequence and find some relations and properties of these new binomial transform sequences.

Definition 1.1. *The modified (s,t)-Pell matrix sequence is defined as in [22]*

$$\begin{aligned} \widehat{Q}_{n+1}(s,t) &= 2s\widehat{Q}_n(s,t) + t\widehat{Q}_{n-1}(s,t), \\ \widehat{Q}_0(s,t) &= \begin{pmatrix} s & 1 \\ t & -s/t \end{pmatrix} \text{ and } \widehat{Q}_1(s,t) = \begin{pmatrix} 2s^2+t & s \\ st & t \end{pmatrix}. \end{aligned} \tag{1.1}$$

2. Binomial Transform of Modified (s,t)-Pell Matrix Sequences

Definition 2.1. *The binomial transform of modified (s,t)-Pell matrix sequence is indicated as $\left\{\widetilde{B}_n(s,t)\right\}_{n\in\mathbb{N}}$ is defined by the following equation*

$$\widetilde{B}_n(s,t) = \sum_{i=0}^{n} \binom{n}{i} \widehat{Q}_i(s,t), \tag{2.1}$$

for any positive integer s, t and $s^2+t > 0$.

Lemma 2.2. *Assume that n is any positive integer, then the binomial transform of modified (s,t)-Pell matrix sequence holds the following property*

$$\widetilde{B}_{n+1}(s,t) = \sum_{i=0}^{n} \binom{n}{i} \left[\widehat{Q}_i(s,t) + \widehat{Q}_{i+1}(s,t)\right].$$

Proof. By the property of combination as $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ and $\binom{n}{n+1} = 0$, we have

$$\begin{aligned} \widetilde{B}_{n+1}(s,t) &= \sum_{i=0}^{n+1} \binom{n+1}{i} \widehat{Q}_i(s,t) \\ &= \widehat{Q}_0(s,t) + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1}\right] \widehat{Q}_i(s,t) \\ &= \sum_{i=0}^{n} \binom{n}{i} \widehat{Q}_i(s,t) + \sum_{i=0}^{n} \binom{n}{i} \widehat{Q}_{i+1}(s,t). \end{aligned}$$

□

Theorem 2.3. *The following recurrence relation is valid for the binomial transform of modified (s,t)-Pell matrix sequence*

$$\widetilde{B}_{n+1}(s,t) = (2+2s)\widetilde{B}_n(s,t) + (t-2s-1)\widetilde{B}_{n-1}(s,t). \tag{2.2}$$

Proof. We see that by Lemma 2.2, $\widetilde{B}_{n+1}(s,t)=\sum\limits_{i=0}^{n}\binom{n}{i}\left[\widehat{Q}_i(s,t)+\widehat{Q}_{i+1}(s,t)\right]$. By (1.1) and (2.1), we have

$$\begin{aligned}\widetilde{B}_{n+1}(s,t) &= \sum_{i=0}^{n}\binom{n}{i}(\widehat{Q}_i(s,t)+\widehat{Q}_{i+1}(s,t))\\ &= \widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)+\sum_{i=1}^{n}\binom{n}{i}(\widehat{Q}_i(s,t)+\widehat{Q}_{i+1}(s,t))\\ &= \widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)+(1+2s)\sum_{i=1}^{n}\binom{n}{i}\widehat{Q}_i(s,t)+t\sum_{i=1}^{n}\binom{n}{i}\widehat{Q}_{i-1}(s,t).\end{aligned}$$

Therefore

$$\widetilde{B}_{n+1}(s,t)=-2s\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)+(1+2s)\widetilde{B}_n(s,t)+t\sum_{i=1}^{n}\binom{n}{i}\widehat{Q}_{i-1}(s,t). \tag{2.3}$$

If we replace n in place of $n+1$ in (2.3), we get

$$\begin{aligned}\widetilde{B}_n(s,t) &= -2s\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)+(1+2s)\widetilde{B}_{n-1}(s,t)+t\sum_{i=1}^{n-1}\binom{n-1}{i}\widehat{Q}_{i-1}(s,t)\\ &= -2s\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)+2s\widetilde{B}_{n-1}(s,t)+\sum_{i=1}^{n}\binom{n-1}{i-1}\widehat{Q}_{i-1}(s,t)+t\sum_{i=1}^{n-1}\binom{n-1}{i}\widehat{Q}_{i-1}(s,t).\end{aligned}$$

By $\binom{n-1}{n}=0$, we obtain

$$\begin{aligned}\widetilde{B}_n(s,t) &= -2s\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)+2s\widetilde{B}_{n-1}(s,t)+\sum_{i=1}^{n}\left[t\binom{n-1}{i}+\binom{n-1}{i-1}\right]\widehat{Q}_{i-1}(s,t)\\ &= -2s\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)+2s\widetilde{B}_{n-1}(s,t)+\sum_{i=1}^{n}\left[t\binom{n-1}{i}+\binom{n-1}{i-1}+t\binom{n-1}{i-1}-t\binom{n-1}{i-1}\right]\widehat{Q}_{i-1}(s,t)\\ &= -2s\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)+2s\widetilde{B}_{n-1}(s,t)+\sum_{i=1}^{n}\left[(1-t)\binom{n-1}{i-1}+t\binom{n}{i}\right]\widehat{Q}_{i-1}(s,t).\end{aligned}$$

We have

$$\widetilde{B}_n(s,t)=-2s\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)+(2s+1-t)\widetilde{B}_{n-1}(s,t)+t\sum_{i=1}^{n}\binom{n}{i}\widehat{Q}_i(s,t). \tag{2.4}$$

By substituting (2.3) into (2.4), we get the desired result as

$$\widetilde{B}_{n+1}(s,t)=(2+2s)\widetilde{B}_n(s,t)+(t-2s-1)\widetilde{B}_{n-1}(s,t).$$

□

Theorem 2.4. ***(Binet Formula)*** *The Binet formula of the binomial transform of modified (s,t)-Pell matrix sequence is given as*

$$\widetilde{B}_n(s,t)=\frac{\left[(1-b_2)\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)\right]b_1^n-\left[(1-b_1)\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)\right]b_2^n}{b_1-b_2}$$

where $b_1=s+1+\sqrt{s^2+t}$, $b_2=s+1-\sqrt{s^2+t}$.

Proof. The characteristic equation of (2.2) is $x^2-(2+2s)x-(t-2s-1)=0$, whose roots are b_1 and b_2. For Binet formula, we assume that $\widetilde{B}_n(s,t)=c_1b_1^n+c_2b_2^n$. We know that $\widetilde{B}_0=\widehat{Q}_0(s,t)$ and $\widetilde{B}_1=\widehat{Q}_1(s,t)$. We substitute $n=0$ and $n=1$, it is obtained that $c_1=\frac{\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)-\widehat{Q}_0(s,t)b_2}{b_1-b_2}$, $c_2=\frac{\widehat{Q}_0(s,t)b_1-\widehat{Q}_1(s,t)-\widehat{Q}_0(s,t)}{b_1-b_2}$. □

Theorem 2.5. ***(Generating function)*** *The generating function of the binomial transform of the modified (s,t)-Pell matrix sequence is found as*

$$\widetilde{B}_n(s,t,x)=\frac{\widetilde{B}_0(s,t)+x\left(\widetilde{B}_1(s,t)-(2+2s)\widetilde{B}_0(s,t)\right)}{1-(2+2s)x-(t-2s-1)x^2}.$$

Proof. Assume the generating function of the binomial transform of the modified (s,t)-Pell matrix sequence as $\widetilde{B}_n(s,t,x)=\widetilde{B}_0(s,t)+\widetilde{B}_1(s,t)x+\widetilde{B}_2(s,t)x^2+\dots$ Let's multiply $\widetilde{B}_n(s,t,x)=\widetilde{B}_n$ by -$(2+2s)x$ and -$(t-2s-1)x^2$. it is obtained that

$$\begin{aligned}-(2+2s)x\widetilde{B}_n &= -(2+2s)x\widetilde{B}_0(s,t)-(2+2s)x^2\widetilde{B}_1(s,t)+\dots\\ -(t-2s-1)x^2\widetilde{B}_n &= -(t-2s-1)x^2\widetilde{B}_0(s,t)-(t-2s-1)x^3\widetilde{B}_1(s,t)+\dots\end{aligned}$$

By these three equalities and (2.2), we have

$$\begin{aligned}&\left[1-(2+2s)x-(t-2s-1)x^2\right]\widetilde{B}_n(s,t,x)\\ &\quad=\widetilde{B}_0(s,t)+x(\widetilde{B}_1(s,t)-(2+2s)\widetilde{B}_0(s,t))+x^2(\widetilde{B}_2(s,t)-(2+2s)\widetilde{B}_1(s,t)-(t-2s-1)\widetilde{B}_0(s,t))+\dots\\ &\quad=\widetilde{B}_0(s,t)+x(\widetilde{B}_1(s,t)-(2+2s)\widetilde{B}_0(s,t))+x^2(0)\end{aligned}$$

□

Theorem 2.6. *The sum of the binomial transform of (s,t)-Pell sequence is evaluated as*

$$\sum_{i=0}^{p-1}\widetilde{B}_{mi+n}(s,t)=\frac{\begin{array}{c}\widetilde{B}_n(s,t)-\widetilde{B}_{mp+n}(s,t)-(2s-t+1)^n\widetilde{B}_{m-n}(s,t)\\ +(2s-t+1)^m\widetilde{B}_{m(p-1)+n}(s,t)\end{array}}{1-(b_1^m+b_2^m)+(2s-t+1)^m}.$$

Proof. By the Binet formula of the binomial transform of modified (s,t)-Pell matrix sequence, we get

$$\begin{aligned}\sum_{i=0}^{p-1}\widetilde{B}_{mi+n}(s,t) &= c_1b_1^n\sum_{i=0}^{p-1}b_1^{mi}+c_2b_2^n\sum_{i=0}^{p-1}b_2^{mi}\\ &= c_1b_1^n\left(\frac{1-b_1^{mp}}{1-b_1^m}\right)+c_2b_2^n\left(\frac{1-b_2^{mp}}{1-b_2^m}\right)\\ &= \frac{\begin{array}{c}c_1b_1^n+c_2b_2^n-c_1b_2^mb_1^n-c_2b_1^mb_2^n\\ -c_1b_1^{mp+n}-c_2b_2^{mp+n}+c_2b_2^{mp+n}b_1^m+c_1b_1^{mp+n}b_2^m\end{array}}{1-(b_1^m+b_2^m)+(b_1b_2)^m}\\ &= \frac{\begin{array}{c}(c_1b_1^n+c_2b_2^n)-(c_1b_1^{mp+n}+c_2b_2^{mp+n})-(b_1b_2)^n[c_2b_2^{m-n}+c_1b_1^{m-n}]\\ -(b_1b_2)^m[c_2b_2^{m(p-1)+n}+c_1b_1^{m(p-1)+n}]\end{array}}{1-(b_1^m+b_2^m)+(b_1b_2)^m}\\ &= \frac{\begin{array}{c}\widetilde{B}_n(s,t)-\widetilde{B}_{mp+n}(s,t)-(2s-t+1)^n\widetilde{B}_{m-n}(s,t)\\ +(2s-t+1)^m\widetilde{B}_{m(p-1)+n}(s,t\end{array}}{1-(b_1^m+b_2^m)+(2s-t+1)^m}.\end{aligned}$$

□

3. The s-Binomial Transforms of the (s,t)-Pell Matrix Sequences

Definition 3.1. *The s-binomial transform of the modified (s,t)-Pell matrix sequence $\left\{\widetilde{O}_n(s,t)\right\}_{n\in N}$ is defined as*

$$\widetilde{O}_n(s,t)=\sum_{i=0}^{n}\binom{n}{i}s^n\hat{Q}_i(s,t). \tag{3.1}$$

We can easily see that $\widetilde{O}_n(s,t) = s^n \widetilde{B}_n(s,t)$.

Lemma 3.2. *The s-binomial transform of the modified (s,t)-Pell matrix sequence has the following property*

$$\widetilde{O}_{n+1}(s,t) = \sum_{i=0}^{n} \binom{n}{i} s^{n+1} \left[\hat{Q}_i(s,t) + \hat{Q}_{i+1}(s,t)\right]. \tag{3.2}$$

Proof. By (3.1), $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ and $\binom{n}{n+1} = 0$, it is computed that

$$\begin{aligned}
\widetilde{O}_{n+1}(s,t) &= \sum_{i=0}^{n+1} \binom{n+1}{i} s^{n+1} \hat{Q}_i(s,t) \\
&= s^{n+1}\hat{Q}_0(s,t) + \sum_{i=1}^{n+1} \binom{n}{i} s^{n+1} \hat{Q}_i(s,t) + \sum_{i=1}^{n+1} \binom{n}{i-1} s^{n+1} \hat{Q}_i(s,t) \\
&= \sum_{i=0}^{n} \binom{n}{i} s^{n+1} \hat{Q}_i(s,t) + \sum_{i=0}^{n} \binom{n}{i} s^{n+1} \hat{Q}_{i+1}(s,t).
\end{aligned}$$

□

Theorem 3.3. *The recurrence relation of the s-binomial transform of the modified (s,t)-Pell matrix sequence is given as*

$$\widetilde{O}_{n+1}(s,t) = s(2+2s)\widetilde{O}_n(s,t) + s^2(t-2s-1)\widetilde{O}_{n-1}(s,t). \tag{3.3}$$

Proof. The initial conditions for the s-binomial transform of the modified (s,t)-Pell matrix sequence are $\widetilde{O}_0(s,t) = \hat{Q}_0(s,t)$ and $\widetilde{O}_1(s,t) = s\hat{Q}_1(s,t)$. By Lemma 3.2 and (3.2), it is obtained that

$$\begin{aligned}
\widetilde{O}_{n+1} &= \sum_{i=0}^{n} \binom{n}{i} s^{n+1} (\hat{Q}_i(s,t) + \hat{Q}_{i+1}(s,t)) \\
&= s^{n+1}(\hat{Q}_0(s,t) + \hat{Q}_1(s,t)) + \sum_{i=1}^{n} \binom{n}{i} s^{n+1} (\hat{Q}_i(s,t) + \hat{Q}_{i+1}(s,t)) \\
&= s^{n+1}(\hat{Q}_0(s,t) + \hat{Q}_1(s,t)) + (1+2s)\sum_{i=1}^{n} \binom{n}{i} s^{n+1} \hat{Q}_i(s,t) + t\sum_{i=1}^{n} \binom{n}{i} s^{n+1} \hat{Q}_{i-1}(s,t).
\end{aligned}$$

Then we get

$$\widetilde{O}_{n+1}(s,t) = s^{n+1}(\hat{Q}_1(s,t) - 2s\hat{Q}_0(s,t)) + s(1+2s)\widetilde{O}_n(s,t) + t\sum_{i=1}^{n} \binom{n}{i} s^{n+1} \hat{Q}_{i-1}(s,t). \tag{3.4}$$

We substitute n in place of $n+1$ in (3.4) as

$$\begin{aligned}
\widetilde{O}_n(s,t) &= s^n(\hat{Q}_1(s,t) - 2s\hat{Q}_0(s,t)) + s(1+2s)\widetilde{O}_{n-1}(s,t) + t\sum_{i=1}^{n-1} \binom{n-1}{i} s^n \hat{Q}_{i-1}(s,t) \\
&= s^n(\hat{Q}_1(s,t) - 2s\hat{Q}_0(s,t)) + 2s^2\widetilde{O}_{n-1}(s,t) + \sum_{i=1}^{n} \binom{n-1}{i-1} s^n \hat{Q}_i(s,t) + t\sum_{i=1}^{n-1} \binom{n-1}{i} s^n \hat{Q}_{i-1}(s,t).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\widetilde{O}_n(s,t) &= s^n(\hat{Q}_1(s,t) - 2s\hat{Q}_0(s,t)) + 2s^2\widetilde{O}_{n-1}(s,t) + \sum_{i=1}^{n} \left[t\binom{n-1}{i} + \binom{n-1}{i-1} + t\binom{n-1}{i-1} - t\binom{n-1}{i-1}\right] s^n \hat{Q}_{i-1}(s,t) \\
&= s^n(\hat{Q}_1(s,t) - 2s\hat{Q}_0(s,t)) + 2s^2\widetilde{O}_{n-1}(s,t) + \sum_{i=1}^{n} \left[(1-t)\binom{n-1}{i-1} + t\binom{n}{i}\right] s^n \hat{Q}_{i-1}(s,t).
\end{aligned}$$

$$\widetilde{O}_n(s,t) = s^n(\hat{Q}_1(s,t) - 2s\hat{Q}_0(s,t)) + 2s^2\widetilde{O}_{n-1}(s,t) + t\sum_{i=1}^{n} \binom{n}{i} s^n \hat{Q}_{i-1}(s,t) + (1-t)s\widetilde{O}_{n-1}(s,t). \tag{3.5}$$

By substituting the equality (3.4) in (3.5), the result is obtained that as

$$\begin{aligned} s\widetilde{O}_n(s,t) &= s^{n+1}(\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t))+2s^3\widetilde{O}_{n-1}(s,t)+\widetilde{O}_{n+1}(s,t) \\ &\quad -s^{n+1}(\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t))-(1+2s)s\widetilde{O}_n(s,t)+(1-t)s^2\widetilde{O}_{n-1}(s,t) \\ &= (2s^3-ts^2+s^2)\widetilde{O}_{n-1}(s,t)-(1+2s)\widetilde{O}_n(s,t)+\widetilde{O}_{n+1}(s,t). \end{aligned}$$

□

Theorem 3.4. *The Binet formula for the binomial transform of modified (s,t)-Pell matrix sequence is given as*

$$\widetilde{O}_n(s,t)=\frac{\left[(s-o_2)\hat{Q}_0(s,t)+s\hat{Q}_1(s,t)\right]o_1^n-\left[(s-o_1)\hat{Q}_0(s,t)+s\hat{Q}_1(s,t)\right]o_2^n}{o_1-o_2}.$$

Proof. The characteristic equation of the recurrence formula (3.3) is $x^2-s(2+2s)x+s^2(2s+t-1)=0$, whose roots are $o_1=s(s+1)+s\sqrt{s^2+t}$ and $o_2=s(s+1)-s\sqrt{s^2+t}$. The Binet formula of the s-binomial transform of modified (s,t)-Pell matrix sequence is demonstrated by $\widetilde{O}_n(s,t)=c_1^n o_1^n+c_2 o_2^n$. We know that $\widetilde{B}_0=\widehat{Q}_0(s,t)$ and $\widetilde{B}_1=\widehat{Q}_1(s,t)$. We substitute $n=0$ and $n=1$ in this equality, it is obtained that $c_1=\frac{\widehat{Q}_0(s,t)+\widehat{Q}_1(s,t)-\widehat{Q}_0(s,t)b_2}{b_1-b_2}$ and $c_2=\frac{\widehat{Q}_0(s,t)b_1-\widehat{Q}_1(s,t)-\widehat{Q}_0(s,t)}{b_1-b_2}$. □

Theorem 3.5. *The generating function for the binomial transform of modified (s,t)-Pell matrix sequence is obtained as*

$$\widetilde{O}_n(s,t,x)=\widetilde{O}_n=\frac{\widetilde{O}_0(s,t)+x\left[\widetilde{O}_1(s,t)-s(2+2s)\widetilde{O}_0(s,t)\right]}{1-s(2+2s)x+s^2(2s-t+1)x^2}.$$

Proof. Let's product $\widetilde{O}_n(s,t,x)$ with -$s(2+2s)x$ and $s^2(2s-t+1)x^2$, we obtain the following equalities:

$$\begin{aligned} -s(2+2s)x\widetilde{O}_n &= -s(2+2s)x\widetilde{O}_0(s,t)-s(2+2s)x^2\widetilde{O}_1(s,t)+\ldots \\ s^2(2s-t+1)x^2\widetilde{O}_n &= s^2(2s-t+1)x^2\widetilde{O}_0(s,t)+s^2(2s-t+1)x^3\widetilde{O}_1(s,t)+\ldots \end{aligned}$$

By these equalities and (3.3), we get the generating function. □

4. The Rising Binomial Transform of the (s,t)-Pell Matrix Sequence

Definition 4.1. *The rising binomial transform of the modified (s,t)-Pell matrix sequence $\left\{\widetilde{I}_n(s,t)\right\}_{n\in N}$ is defined by the following formula*

$$\widetilde{I}_n(s,t)=\sum_{i=0}^{n}\binom{n}{i}s^i\hat{Q}_i(s,t), \tag{4.1}$$

Lemma 4.2. *The Binet formula of the modified (s,t)-Pell matrix sequence is*

$$\hat{Q}_n=Mx_1^n-Nx_2^n$$

where $M=\frac{\hat{Q}_1-x_2\hat{Q}_0}{x_1-x_2}$, $N=\frac{\hat{Q}_1-x_1\hat{Q}_0}{x_1-x_2}$, $x_1=s+\sqrt{s^2+t}$ *and* $x_2=s-\sqrt{s^2+t}$ *[22].*

Theorem 4.3. *The Binet formula of the rising binomial transform of the modified (s,t)-Pell matrix sequence is*

$$\widetilde{I}_n(s,t)=M(sx_1+1)^n-N(sx_2+1)^n.$$

Proof. By (4.1), we have

$$\begin{aligned}\widetilde{I}_n(s,t) &= \sum_{i=0}^{n}\binom{n}{i}s^i\hat{Q}_i(s,t)\\ &= M\sum_{i=0}^{n}\binom{n}{i}s^ix_1^i - N\sum_{i=0}^{n}\binom{n}{i}s^ix_2^i\\ &= M(sx_1+1)^n - N(sx_2+1)^n.\end{aligned}$$

□

Theorem 4.4. *For $n \geqslant 1$, the rising binomial transform of the modified (s,t)-Pell matrix sequence has the following recurrence relation*

$$\widetilde{I}_{n+1}(s,t) = (2s^2+2)\widetilde{I}_n(s,t) - (1-s^2t+2s^2)\widetilde{I}_{n-1}(s,t).$$

Proof. By Binet formula of the rising binomial transform of the modified (s,t)-Pell matrix sequence, we get

$$\begin{aligned}&(2s^2+2)\widetilde{I_n}(s,t) - (1-s^2t+2s^2)\widetilde{I_{n-1}}(s,t)\\ &\quad= (2s^2+2)[M(sx_1+1)^n - N(sx_2+1)^n] - (1-s^2t+2s^2)[M(sx_1+1)^{n-1} - N(sx_2+1)^{n-1}]\\ &\quad= M(sx_1+1)^{n-1}\left[(2s^2+2)(sx_1+1)-(1-s^2t+2s^2)\right] - N(sx_2+1)^{n-1}\left[(2s^2+2)(sx_2+1)-(1-s^2t+2s^2)\right]\\ &\quad= M(sx_1+1)^{n-1}\left[(sx_1+1)^2\right] - N(sx_2+1)^{n-1}\left[(sx_2+1)^2\right]\\ &\quad= \widetilde{I_{n+1}}(s,t).\end{aligned}$$

□

Theorem 4.5. *The generating function of rising binomial transform of the modified (s,t)-Pell matrix sequence is denoted by $\widetilde{I}_n(s,t,x)$ and given as*

$$\widetilde{I}_n(s,t,x) = \widetilde{I}_0(s,t)+\widetilde{I}_1(s,t)x+\widetilde{I}_2(s,t)x^2+\ldots = \widetilde{I}_n = \frac{\widetilde{I}_0(s,t)+x\left[\widetilde{I}_1(s,t)-(2s^2+2)\widetilde{I}_0(s,t)\right]}{1-(2s^2+2)x+(1-s^2t+2s^2)x^2}.$$

Proof. By following same computations in Theorem 3.5, we obtain the generating function as

$$\begin{aligned}-(2s^2+2)x\widetilde{I}_n &= -(2s^2+2)x\widetilde{I}_0(s,t)-(2s^2+2)x^2\widetilde{I}_1(s,t)+\ldots\\ (1-s^2t+2s^2)x^2\widetilde{I}_n &= (1-s^2t+2s^2)x^2\widetilde{I}_0(s,t)+(1-s^2t+2s^2)x^3\widetilde{I}_1(s,t)+\ldots\\ \left[1-(2s^2+2)x+(1-s^2t+2s^2)x^2\right]\widetilde{I}_n &= \widetilde{I}_0(s,t)-(2s^2+2)x\widetilde{I}_0(s,t)+x\widetilde{I}_1(s,t)+x^2(0).\end{aligned}$$

□

5. The Falling Binomial Transform of The (s,t)-Pell Matrix Sequence

Definition 5.1. *For any positive integer n, the falling binomial transform of the modified (s,t)-Pell matrix sequence $\left\{\widetilde{D}_n(s,t)\right\}_{n\in N}$ is defined as*

$$\widetilde{D}_n(s,t) = \sum_{i=0}^{n}\binom{n}{i}s^{n-i}\hat{Q}_i(s,t). \tag{5.1}$$

Lemma 5.2. *The falling binomial transform of the modified (s,t)-Pell matrix sequence verifies the relation*

$$\widetilde{D}_{n+1}(s,t) = \sum_{i=0}^{n}\binom{n}{i}s^{n-i}\left[s\hat{Q}_i(s,t)+\hat{Q}_{i+1}(s,t)\right].$$

Proof. By (5.1), we have the initial conditions $\widetilde{D}_0(s,t)=\hat{Q}_0(s,t)$ and $\widetilde{D}_1(s,t)=s\hat{Q}_0(s,t)+\hat{Q}_1(s,t)$. Then, we get

$$\begin{aligned}\widetilde{D}_{n+1}(s,t) &= \sum_{i=0}^{n+1}\binom{n+1}{i}s^{n+1-i}\hat{Q}_i(s,t)\\ &= s^{n+1}\hat{Q}_0(s,t)+\sum_{i=1}^{n}s^{n+1-i}\binom{n}{i}\hat{Q}_i(s,t)+\sum_{i=0}^{n+1}s^{n-i}\binom{n}{i}\hat{Q}_{i+1}(s,t)\\ &= \sum_{i=0}^{n}s^{n+1-i}\binom{n}{i}\hat{Q}_i(s,t)+\sum_{i=0}^{n}s^{n-i}\binom{n}{i}\hat{Q}_{i+1}(s,t).\end{aligned}$$

□

Theorem 5.3. *The recurrence relation of the falling binomial transform of the modified (s,t)-Pell matrix sequence is given as*

$$\widetilde{D}_{n+1}(s,t)=4s\widetilde{D}_n(s,t)+(3s^2-t)\widetilde{D}_{n-1}(s,t).$$

Proof. By Lemma 5.2 and (1.1), we obtain

$$\begin{aligned}\widetilde{D}_{n+1}(s,t) &= \sum_{i=0}^{n}\binom{n}{i}s^{n-i}\left[s\hat{Q}_i(s,t)+\hat{Q}_{i+1}(s,t)\right]\\ &= s^n\left[s\hat{Q}_0(s,t)+\hat{Q}_1(s,t)\right]+\sum_{i=1}^{n}\binom{n}{i}s^{n-i}\left[s\hat{Q}_i(s,t)+\hat{Q}_{i+1}(s,t)\right]\\ &= s^n\left[s\hat{Q}_0(s,t)+\hat{Q}_1(s,t)\right]+3s\sum_{i=1}^{n}\binom{n}{i}s^{n-i}\hat{Q}_i(s,t)+t\sum_{i=1}^{n}\binom{n}{i}s^{n-i}\hat{Q}_{i-1}(s,t).\end{aligned}$$

Then we get

$$\widetilde{D}_{n+1}(s,t)=s^n\left[\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t)\right]+3s\widetilde{D}_n(s,t)+t\sum_{i=1}^{n}\binom{n}{i}s^{n-i}\hat{Q}_{i-1}(s,t). \tag{5.2}$$

If we substitute for n in place of $n+1$ in this equality (5.2), we get

$$\begin{aligned}\widetilde{D}_n(s,t) &= s^{n-1}\left[\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t)\right]+3s\widetilde{D}_{n-1}(s,t)+t\sum_{i=1}^{n-1}\binom{n-1}{i}s^{n-1-i}\hat{Q}_{i-1}(s,t)\\ &= s^{n-1}\left[\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t)\right]+3s\sum_{i=0}^{n-1}\binom{n-1}{i}s^{n-1-i}\hat{Q}_i(s,t)+t\sum_{i=1}^{n-1}\binom{n-1}{i}s^{n-1-i}\hat{Q}_{i-1}(s,t)\\ &= s^{n-1}\left[\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t)\right]+3s\sum_{i=1}^{n-1}\binom{n-1}{i-1}s^{n-i}\hat{Q}_{i-1}(s,t)+\frac{t}{s}\sum_{i=1}^{n-1}\binom{n-1}{i}s^{n-i}\hat{Q}_{i-1}(s,t).\end{aligned}$$

Let's take care of $\binom{n-1}{n}=0$, then we get

$$\begin{aligned}\widetilde{D}_n(s,t) &= s^{n-1}\left[\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t)\right]+\sum_{i=1}^{n}\left[\frac{t}{s}\binom{n-1}{i}+3s\binom{n-1}{i-1}\right]s^{n-i}\hat{Q}_{i-1}(s,t)\\ &= s^{n-1}\left[\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t)\right]+\sum_{i=1}^{n}\left[\frac{t}{s}\binom{n-1}{i}+3s\binom{n-1}{i-1}+\frac{t}{s}\binom{n-1}{i-1}-\frac{t}{s}\binom{n-1}{i-1}\right]s^{n-i}\hat{Q}_{i-1}(s,t)\\ &= s^{n-1}\left[\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t)\right]+\sum_{i=1}^{n}\left[(3s-\frac{t}{s})\binom{n-1}{i-1}+\frac{t}{s}\binom{n}{i}\right]s^{n-i}\hat{Q}_{i-1}(s,t).\end{aligned}$$

Therefore, it is obtained that

$$\widetilde{D}_n(s,t)=s^{n-1}\left[\hat{Q}_1(s,t)-2s\hat{Q}_0(s,t)\right]+\frac{t}{s}\sum_{i=1}^{n}\binom{n}{i}s^{n-i}\hat{Q}_{i-1}(s,t)+(3s-\frac{t}{s})\widetilde{D}_{n-1}(s,t). \tag{5.3}$$

By substituting the above equality (5.2) into (5.3), we get

$$\widetilde{D}_{n+1}(s,t) = 4s\widetilde{D}_n(s,t) + (3s^2 - t)\widetilde{D}_{n-1}(s,t). \tag{5.4}$$

□

Theorem 5.4. *The Binet formula of the falling binomial transform of the modified (s,t)-Pell matrix sequence is as follows*

$$\widetilde{D}_n(s,t) = \frac{\left[(s-d_2)\hat{Q}_0(s,t) + s\hat{Q}_1(s,t)\right] d_1^n - \left[(s-d_1)\hat{Q}_0(s,t) + s\hat{Q}_1(s,t)\right] d_2^n}{d_1 - d_2}.$$

Proof. The characteristic polynomial equation of recurrence formula (5.4) is $x^2 - 4sx + 3s^2 - t = 0$, whose solutions are $d_1 = 2s + \sqrt{s^2+t}$ and $d_2 = 2s - \sqrt{s^2+t}$. Let's show that $\widetilde{D}_n(s,t) = c_1 d_1^n + c_2 d_2^n$. We substitute $n = 0$ and $n = 1$ in this equality, it is obtained that $c_1 = \frac{(s-d_2)\hat{Q}_0(s,t)+s\hat{Q}_1(s,t)}{d_1-d_2}$ and $c_2 = \frac{-(s-d_1)\hat{Q}_0(s,t)-s\hat{Q}_1(s,t)}{d_1-d_2}$. □

Theorem 5.5. *The generating function of the falling binomial transform of the modified (s,t)-Pell matrix sequence is as follows*

$$\widetilde{D}_n(s,t.x) = \frac{\widetilde{D}_0(s,t) + x\left[\widetilde{D}_1(s,t) - 4s\widetilde{D}_0(s,t)\right]}{1 - 4sx + (3s^2-t)x^2}.$$

Proof. If we product $\widetilde{D}_n(s,t,x)$ by $-4sx$ and $(3s^2-t)x^2$, it is obtained that

$$\begin{aligned} -4sx\widetilde{D}_n(s,t.x) &= -4sx\widetilde{D}_0(s,t) - 4sx^2\widetilde{D}_1(s,t) + \ldots \\ (3s^2-t)x^2\widetilde{D}_n(s,t.x) &= (3s^2-t)x^2\widetilde{D}_0(s,t) + (3s^2-t)x^3\widetilde{D}_1(s,t) + \ldots \end{aligned}$$

By these equalities and (5.4), it is computed that

$$\widetilde{D}_n(s,t,x) = \frac{\widetilde{D}_0(s,t) + x\left[\widetilde{D}_1(s,t) - 4s\widetilde{D}_0(s,t)\right]}{1 - 4sx + (3s^2-t)x^2}.$$

□

Theorem 5.6. *Assume that m, n are any positive integers. Then the sum of the s-binomial transform of (s,t)-Pell matrix sequence is given as*

$$\sum_{i=0}^{p-1} D_{mi+n}(s,t) = \frac{\begin{array}{c} D_n(s,t) - (3s^2-t)^n D_{m-n} - D_{mp+n}(s,t) \\ +(3s^2-t)^m D_{m(p-1)+n}(s,t) \end{array}}{1 - (d_1^m + d_2^m) + (3s^2-t)^m}.$$

6. Conclusion

Depending on two real parameters s and t, we get a generalization of modified Pell sequence which is called (s,t)-modified Pell sequence. By the elements of the (s,t)-modified Pell numbers, we obtain (s,t)-modified Pell matrix sequence. Then, we define different binomial transforms of the modified (s,t)-Pell matrix sequence such as rising, falling and s-binomial transforms. We also denote some properties of the binomial transforms of (s,t)-modified Pell matrix sequences such as Binet formulas, the generating functions and sum formulas. In the future, the binomial transforms of the generalizations of the other sequences will be studied.

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