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Some Algebraic Properties of Pythagorean Fuzzy Bi-ideals

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Research Article

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Abstract

Fuzzy sets have a significant place in solving problems involving uncertainty. There are many studies on fuzzy sets in decision-making, engineering, algebra, etc. In this study, we discuss the behavior of Pythagorean fuzzy sets, which are a kind of generalization of fuzzy sets in algebra. First, we define the Pythagorean fuzzy product and examine some of its properties. Then, we investigate the relationship between the Pythagorean fuzzy ideal and the Pythagorean fuzzy product. Then, we define Pythagorean fuzzy bi-ideal. We give the theorem that characterizes bi-ideals in terms of fuzzy bi-ideals. We examine the behavior of Pythagorean fuzzy bi-ideals under Pythagorean fuzzy product. We prove that the intersection, Cartesian product, and Pythagorean fuzzy product of Pythagorean fuzzy bi-ideals are also Pythagorean fuzzy bi-ideals. Furthermore, we investigate the image and the inverse image of the Pythagorean fuzzy bi-ideal.

Keywords: Pythagorean fuzzy sets, Pythagorean fuzzy ideals, Pythagorean fuzzy bi-ideals

Pythagorean Bulanık Bi-ideallerin Bazı Cebirsel Özellikleri

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Öz

Bulanık kümeler, belirsizlik içeren problemlerin çözümünde çok önemli bir yere sahiptir. Karar verme, mühendislik, cebir vb. alanlarda bulanık kümeler üzerine çok sayıda çalışma vardır. Bu çalışmada, bulanık kümelerin bir tür genelleştirilmesi olan Pythagorean bulanık kümelerin cebir de ki davranışlarını tartışıyoruz. İlk olarak, Pythagorean bulanık çarpımı tanımlıyoruz ve bazı özelliklerini inceliyoruz. Ardından, Pythagorean bulanık ideali ile Pythagorean bulanık çarpımı arasındaki ilişkiyi araştırıyoruz. Sonra, Pythagorean bulanık bi-ideali tanımlıyoruz. Bi-idealleri bulanık bi-idealler açısından karakterize eden teoremi veriyoruz. Pythagorean bulanık bi-ideallerin Pythagorean bulanık çarpım altındaki davranışlarını inceliyoruz. Pythagorean bulanık bi-ideallerin arakesitinin, kartezyen çarpımının ve Pythagorean bulanık çarpımının da Pythagorean bulanık bi-ideal olduğunu ispatlıyoruz. Dahası, Pythagorean bulanık bi-idealinin görüntüsünü ve ters görüntüsünü araştırıyoruz.

Anahtar Kelimeler: Pythagorean bulanık küme, Pythagorean bulanık ideal, Pythagorean bulanık bi-ideal

Introduction

In the universe, there are many unsolved problems with uncertainty. To solve these problems, scientists are producing new theories every day. Some of these theories are fuzzy set [1], rough set [2], and soft set [3]. In 1965, Zadeh [1] paved the way for many studies by defining a fuzzy set using the membership function. Thus, the fuzzy set theory began to be used in diverse fields such as science, medicine, engineering, etc. Many researchers have been working on combining fuzzy logic structures and functionalities with natural computing in recent years. The highly active development of new branches of fuzzy mathematics, such as fuzzy algebra, fuzzy decision-making, fuzzy rough sets, and fuzzy soft sets, to name just a few, has been greatly aided by fuzzy set theory in particular.

On the one hand, fuzzy sets were studied on different algebraic structures, and the other hand, generalizations of fuzzy sets were suggested. Intuitionistic fuzzy sets are one of the generalizations that may be defined with fuzzy sets. Atanassov [4] defined an intuitionistic fuzzy set and investigated various properties. After that, Yager [5] extended to Pythagorean fuzzy sets, a generalization of intuitionistic fuzzy sets. The idea was excellently suited to mathematically describe ambiguity and uncertainty and create a structured instrument to deal with imprecision in practical circumstances.

Yager [6] investigated features and application of Pythagorean fuzzy sets. Pythagorean fuzzy sets are used in Zhang [7]'s development of an extension of TOPSIS for multiple criteria decision-making. Some features of Pythagorean fuzzy sets were discussed in [8, 9]. Characterization of operations by Pythagorean and complex fuzzy set was studied in [10]. The continuous Pythagorean fuzzy information features were investigated by Gou et al. in [11]. The multiobjective optimization approach based on ratio analysis in [12] captures many features, including the criteria and alternatives for evaluating a multiple-criteria decision-making problem. Naz et al. [13] defined a new graph called the Pythagorean fuzzy graph. Some new distance measures of Pythagorean fuzzy sets were determined, and the application in medical diagnosis was studied by Xiao [14]. Wang et al. [15] suggested a multi-criteria decision-making approach that is entropy-based. Kirişçi and Şimşek [16] used Pythagorean fuzzy soft sets for the first time in an application related to infectious diseases. In [17], Kumar and Chen proposed a new entropy measure of Pythagorean fuzzy sets. In [18], Pythagorean fuzzy topological space was introduced, which expanded the concepts of intuitionistic fuzzy topological space and fuzzy topological space.

Moreover, some scientists started to study the behavior of Pythagorean fuzzy sets in algebra. Hussian et al. [19] researched rough Pythagorean fuzzy ideals in semigroups by establishing a relationship between rough sets and Pythagorean fuzzy sets. Bhunia et al. [20, 21] examined Pythagorean fuzzy subgroups, (α, β)-Pythagorean fuzzy subrings and ideals of rings. Razaq et al. [22, 23] defined Pythagorean fuzzy subgroup, Pythagorean fuzzy cosets, Pythagorean fuzzy normal subgroups, and Pythagorean fuzzy rings, which would pave the way for Pythagorean fuzzy sets in the field of algebra and analyzed theirs different algebraic properties. Adak et al. [24] studied Pythagorean fuzzy ideals of near rings.

The behavior of Pythagorean fuzzy sets in algebraic structures has become a matter of curiosity and has begun to be investigated. This study aims to generalize the Pythagorean fuzzy ideal and contribute to the literature by examining the algebraic properties of this new structure. To put it another way, we aimed to give this idea a more elevated identity and distinguish our study from others of a similar nature. This new Pythagorean fuzzy ideal type can also be used in future studies.

The motivation for writing this paper is to propose a new approach to generalize Pythagorean fuzzy

ideals inspired by classical algebra. Secondly, a new framework can be presented by generating concepts such as Pythagorean fuzzy product and Pythagorean fuzzy bi-ideal obtained in this study.
In this study, basic definitions and concepts about Pythagorean fuzzy sets are first reminded. The Pythagorean fuzzy product is defined, and some basic properties are investigated. Moreover, the relation between the Pythagorean fuzzy product and the Pythagorean fuzzy ideal is examined. Then, the Pythagorean fuzzy bi-ideal is investigated, and its algebraic features are discussed. Additionally, it is shown that the image and preimage of Pythagorean fuzzy bi-ideal are Pythagorean fuzzy bi-ideal.

Preliminary

This section provided some basic definitions and concepts about Pythagorean fuzzy sets. Throughout this paper, S and T are two sets, and H is a ring unless indicated otherwise. From now on, PF will be used instead of "Pythagorean fuzzy" for brevity.

Definition 1. [1] Let $S \neq \emptyset$. A fuzzy set μ of S is described by a mapping $\mu : S \to [0, 1]$.

Definition 2. [5] $\Im = (f, g)$ is said to be a PF set of S and is symbolized by $f : S \to [0, 1]$ and $g : S \to [0, 1]$ such that $f^2(x) + g^2(x) \leq 1$ for all $x \in S$.

Definition 3. [22] Let $\Im = (f, g)$ be a PF set of S. The set $\Im_{(\alpha,\beta)} = \{x \in S | f^2(x) \geq \alpha \text{ and } g^2(x) \leq \beta\}$ for all $\alpha, \beta \in [0, 1]$ is said to be an (α, β)-level set of $\Im$.

Definition 4. Let $A \subseteq S$. $\chi_A = (\chi_A, \chi_{A^c})$ is said to be a PF characteristic function of S and is symbolized by
$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ and $\chi_{A^c}(x) = \begin{cases} 1 & \text{if } x \in A^c \\ 0 & \text{if } x \notin A^c \end{cases}$ for all $x \in S$.

Definition 5. [5] Let $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ be PF sets of S. Then, union $\Im_1 \cup \Im_2 = (f_1 \cup f_2, g_1 \cap g_2)$ is defined by $(f_1 \cup f_2)^2(x) = {f_1}^2(x) \vee {f_2}^2(x)$ and $(g_1 \cap g_2)^2(x) = {g_1}^2(x) \wedge {g_2}^2(x)$ for all $x \in S$. Then, intersection $\Im_1 \cap \Im_2 = (f_1 \cap f_2, g_1 \cup g_2)$ is defined by $(f_1 \cap f_2)^2(x) = {f_1}^2(x) \wedge {f_2}^2(x)$, $(g_1 \cup g_2)^2(x) = {g_1}^2(x) \vee {g_2}^2(x)$ for all $x \in S$.

Definition 6. Let $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ be PF sets of S and T, respectively. Then, cartesian product $\Im_1 \times \Im_2 = (f_1 \curlywedge f_2, g_1 \curlyvee g_2)$ is defined by $(f_1 \curlywedge f_2)^2(x, y) = {f_1}^2(x) \wedge {f_2}^2(y)$ and $(g_1 \curlyvee g_2)^2(x, y) = {g_1}^2(x) \vee {g_2}^2(y)$ for all $(x, y) \in S \times T$.

Definition 7. Let $\Im = (f, g)$ be a PF set of S and $\vartheta : S \to T$ be a function. Then, $\vartheta(\Im) = (\vartheta(f), \vartheta(g))$ image of $\Im$ under ϑ is symbolized by

$$\vartheta(f^2)(a) = \begin{cases} \vee\{f^2(x) : x \in S, \vartheta(x) = a\}, & \text{if } \vartheta^{-1}(a) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$\vartheta(g^2)(a) = \begin{cases} \wedge\{g^2(x) : x \in S, \vartheta(x) = a\}, & \text{if } \vartheta^{-1}(a) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for all $a \in T$. Then, $\vartheta^{-1}(\Im) = (\vartheta^{-1}(f), \vartheta^{-1}(g))$ pre-image of $\Im$ under ϑ is defined by $\vartheta^{-1}(f^2)(x) = f^2(\vartheta(x))$ and $\vartheta^{-1}(g^2)(x) = g^2(\vartheta(x))$ for all $x \in S$.

Definition 8. [22] Let $\Im = (f, g)$ be a PF set of H. $\Im = (f, g)$ is said to be a PF ring of H if

- $f^2(x - y) \geq f^2(x) \wedge f^2(y)$ and $g^2(x - y) \leq g^2(x) \vee g^2(y)$
- $f^2(xy) \geq f^2(x) \wedge f^2(y)$ and $g^2(xy) \leq g^2(x) \vee g^2(y)$

for all $x, y \in H$.

Definition 9. [22] Let $\Im = (f, g)$ be a PF set of H. $\Im = (f, g)$ is said to be a PF ideal of H if

- $f^2(x - y) \geq f^2(x) \wedge f^2(y)$ and $g^2(x - y) \leq g^2(x) \vee g^2(y)$
- $f^2(xy) \geq f^2(x) \vee f^2(y)$ and $g^2(xy) \leq g^2(x) \wedge g^2(y)$

for all $x, y \in H$.

Remark 1. Let $\Im = (f, g)$ be a PF ideal of H.

$$f^2(x) = f^2(\sum_{i=1}^{n} a_i b_i) \geq f^2(a_i) \text{ and } f^2(x) = f^2(\sum_{i=1}^{n} a_i b_i) \geq f^2(b_i)$$

and

$$g^2(x) = g^2(\sum_{i=1}^{n} a_i b_i) \leq g^2(a_i) \text{ and } g^2(x) = g^2(\sum_{i=1}^{n} a_i b_i) \leq g^2(b_i)$$

for all $x = \sum_{i=1}^{n} a_i b_i \in H$, where $1 \leq i \leq n$.

Theorem 1. [22] Let $\Im = (f, g)$ be a PF ideal of H.

- $\Im_{*+} = \{x \in H | f^2(x) = f^2(0)\}$ is an ideal of H.
- $\Im_{*-} = \{x \in H | g^2(x) = g^2(0)\}$ is an ideal of H.
- $\Im_{*} = \{x \in H | f^2(x) = f^2(0) \text{ and } g^2(x) = g^2(0)\}$ are ideals of H.

Theorem 2. [22] Let $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ be two PF rings of H. Then, $\Im_1 \cap \Im_2$ is a PF ring of H.

On Pythagorean fuzzy ideals

In this section, we proposed the concept of the Pythagorean fuzzy product and investigated some properties of the Pythagorean fuzzy product.

Definition 10. Let $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ be PF sets of H. The PF product $\Im_1 \lessdot \gtrdot \Im_2 = (f_1 \lessdot f_2, g_1 \gtrdot g_2)$ is defined by

$$(f_1 \lessdot f_2)^2(x) = \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{f_1^2(a_i) \wedge f_2^2(b_i)\}$$

$$(g_1 \gtrdot g_2)^2(x) = \bigwedge_{x=\sum_{i=1}^{n} a_i b_i} \{g_1^2(a_i) \vee g_2^2(b_i)\}$$

for some $a_i, b_i \in H$, where every $a_i b_i \neq 0$ and $(f_1 \lessdot f_2)^2(x) = 0$, $(g_1 \gtrdot g_2)^2(x) = U$ if x is not be expressible as $x = \sum_{i=1}^{n} a_i b_i$ for all $x \in H$.

Example 1. Consider that the PF sets $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ of ring Z_3 as follows:
$f_1(\bar{0}) = 0.94$, $f_1(\bar{1}) = 0.23$, $f_1(\bar{2}) = 0.1$, $g_1(\bar{0}) = 0.11$, $g_1(\bar{1}) = 0.7$, $g_1(\bar{2}) = 0.78$
and
$f_2(\bar{0}) = 0.83$, $f_2(\bar{1}) = 1$, $f_2(\bar{2}) = 0.3$, $g_2(\bar{0}) = 0.14$, $g_2(\bar{1}) = 0$, $g_2(\bar{2}) = 0.55$

For example,
$\bar{1} = \bar{1}.\bar{1}$, $\bar{1} = \bar{1}.\bar{2} + \bar{1}.\bar{2}$, $\bar{1} = \bar{2}.\bar{1} + \bar{2}.\bar{1}$, $\bar{1} = \bar{1}.\bar{2} + \bar{2}.\bar{1}$, $\bar{1} = \bar{1}.\bar{1} + \bar{2}.\bar{1} + \bar{2}.\bar{2}$, $\bar{1} = \bar{1}.\bar{1} + \bar{1}.\bar{2} + \bar{2}.\bar{2}$, $\bar{1} = \bar{1}.\bar{1} + \bar{1}.\bar{1} + \bar{1}.\bar{1} + \bar{2}.\bar{2}$, $\bar{1} = \bar{1}.\bar{2} + \bar{1}.\bar{2} + \bar{1}.\bar{2} + \bar{2}.\bar{2}$, $\bar{1} = \bar{1}.\bar{2} + \bar{2}.\bar{1} + \bar{2}.\bar{1} + \bar{2}.\bar{2}$, $\bar{1} = \bar{2}.\bar{1} + \bar{2}.\bar{1} + \bar{2}.\bar{1} + \bar{2}.\bar{2}$, $\bar{1} = \bar{2}.\bar{2} + \bar{2}.\bar{2} + \bar{2}.\bar{2} + \bar{2}.\bar{2}$, $\bar{1} = \bar{1}.\bar{1} + \bar{1}.\bar{1} + \bar{1}.\bar{1} + \bar{1}.\bar{2} + \bar{1}.\bar{2}$, $\bar{1} = \bar{1}.\bar{1} + \bar{1}.\bar{1} + \bar{1}.\bar{1} + \bar{1}.\bar{2} + \bar{2}.\bar{1}$, $\bar{1} = \bar{1}.\bar{1} + \bar{1}.\bar{1} + \bar{1}.\bar{1} + \bar{2}.\bar{1} + \bar{2}.\bar{1}$, $\bar{1} = \bar{1}.\bar{1} + \bar{1}.\bar{1} + \bar{1}.\bar{2} + \bar{2}.\bar{1} + \bar{2}.\bar{2}$.
Similar equations are not included. Thus,

$$(f_1 \lessdot f_2)^2(\bar{1}) = \{f_1^2(\bar{1}) \cap f_2^2(\bar{1})\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{2}))\} \cup \{(f_1^2(\bar{2}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2}))\} \cup \{(f_1^2(\bar{2}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2}))\} \cup \{(f_1^2(\bar{2}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{2}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1}))\} \cup \{(f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{1}) \cap f_2^2(\bar{2})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{1})) \cap (f_1^2(\bar{2}) \cap f_2^2(\bar{2}))\}$$

and

$$(g_1 \gtrdot g_2)^2(\bar{1}) = \{g_1^2(\bar{1}) \cup g_2^2(\bar{1})\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{2}))\} \cap \{(g_1^2(\bar{2}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2}))\} \cap \{(g_1^2(\bar{2}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2}))\} \cap \{(g_1^2(\bar{2}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{2}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1}))\} \cap \{(g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{1}) \cup g_2^2(\bar{2})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{1})) \cup (g_1^2(\bar{2}) \cup g_2^2(\bar{2}))\}.$$

Then, it is obtained that $(f_1 \lessdot f_2)^2(\bar{1}) = 0,0529$ and $(g_1 \lessdot g_2)^2(\bar{1}) = 0.49$. Similarly, the others are obtained.

Theorem 3. Let $\Im_1 = (f_1, g_1)$, $F_2 = (f_2, g_2)$, $\Im_3 = (f_3, g_3)$ and $\Im_4 = (f_4, g_4)$ be PF sets of H. Then,

- $(\Im_1 \lessdot\gtrdot \Im_2) \lessdot\gtrdot \Im_3 = \Im_1 \lessdot\gtrdot (\Im_2 \lessdot\gtrdot \Im_3)$
- $(\Im_1 \lessdot\gtrdot \Im_2) = (\Im_2 \lessdot\gtrdot \Im_1)$ if H is a commutative ring
- $(\Im_1 \lessdot\gtrdot (\Im_2 \cap \Im_3) = (\Im_1 \lessdot\gtrdot \Im_2) \cap (\Im_1 \lessdot\gtrdot \Im_3)$
  $(\Im_1 \cap \Im_2) \lessdot\gtrdot \Im_3 = (\Im_1 \lessdot\gtrdot \Im_3) \cap (\Im_2 \lessdot\gtrdot \Im_3)$
- If $\Im_1 \subseteq \Im_2$, then $\Im_1 \lessdot\gtrdot \Im_3 \subseteq \Im_2 \lessdot\gtrdot \Im_3$ and $\Im_3 \lessdot\gtrdot \Im_1 \subseteq \Im_3 \lessdot\gtrdot \Im_2)$

- If $\Im_1 \subseteq \Im_2$ and $\Im_3 \subseteq \Im_4$, then $\Im_1 \lessdot \gtrdot \Im_3 \subseteq \Im_2 \lessdot \gtrdot \Im_4$

Note that

$$((f_1 \lessdot f_2) \lessdot f_3)^2(x) = \bigvee_{x=\sum_{i=1}^{n} a_i b_i c_i} \{f_1^2(a_i) \wedge f_2^2(b_i) \wedge f_3^2(c_i)\}$$

$$((g_1 \gtrdot g_2) \gtrdot g_3)^2(x) = \bigwedge_{x=\sum_{i=1}^{n} a_i b_i c_i} \{g_1^2(a_i) \vee g_2^2(b_i) \vee g_3^2(c_i)\}$$

for all $x \in H$.

Proposition 1. Let $\Im = (f, g)$ be a PF set of H. The following conditions are equivalent:

- $\Im \lessdot \gtrdot \Im \subseteq \Im$
- $$f^2(x) = f^2(\sum_{i=1}^{n} a_i b_i) \geq f^2(a_i) \wedge f^2(b_i)$$

  and

  $$g^2(x) = g^2(\sum_{i=1}^{n} a_i b_i) \leq g^2(a_i) \vee g^2(b_i)$$

  for all $x = \sum_{i=1}^{n} a_i b_i \in H$ where $1 \leq i \leq n$.

Theorem 4. Let $\Im = (f, g)$ be a PF set of H. $\Im = (f, g)$ be a PF ideal of H iff

- For all $x, y \in H$, $f^2(x - y) \geq f^2(x) \wedge f^2(y)$ and $g^2(x - y) \leq g^2(x) \vee g^2(y)$
- $\chi_H \lessdot \gtrdot \Im \subseteq \Im$ and $\Im \lessdot \gtrdot \chi_H \subseteq \Im$

Proof. Suppose $\Im = (f, g)$ is a PF ideal of H. Then, for all $x, y \in H$, $f^2(x - y) \geq f^2(x) \wedge f^2(y)$ and $g^2(x - y) \leq g^2(x) \vee g^2(y)$.

$$\begin{aligned}(\chi_H \lessdot f)^2(x) &= \textstyle\bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{{\chi_H}^2(a_i) \wedge f^2(b_i)\} \\ &\leq \textstyle\bigvee_{x=\sum_{i=1}^{n} a_i b_i} f^2(a_i b_i) \\ &\leq f^2(\textstyle\sum_{i=1}^{n} a_i b_i) \\ &= f^2(x)\end{aligned}$$

and

$$\begin{aligned}(\chi_H^c \lessdot g)^2(x) &= \textstyle\bigwedge_{x=\sum_{i=1}^{n} a_i b_i} \{{\chi_H^c}^2(a_i) \wedge g^2(b_i)\} \\ &\geq \textstyle\bigvee_{x=\sum_{i=1}^{n} a_i b_i} g^2(a_i b_i) \\ &\geq g^2(\textstyle\sum_{i=1}^{n} a_i b_i) \\ &= g^2(x)\end{aligned}$$

for all $x = \sum_{i=1}^{n} a_i b_i \in H$. Thus, $\chi_H \lessdot \gtrdot \Im \subseteq \Im$ followed. Similarly, $\Im \lessdot \gtrdot \chi_H \subseteq \Im$.
Conversely, let $f^2(x-y) \geq f^2(x) \wedge f^2(y)$, $g^2(x-y) \leq g^2(x) \vee g^2(y)$ for all $x, y \in H$ and $\chi_H \lessdot \gtrdot \Im \subseteq \Im$, $\Im \lessdot \gtrdot \chi_H \subseteq \Im$.

$$\begin{aligned}f^2(xy) &\geq (\chi_H \lessdot f)^2(xy) \\ &= \textstyle\bigvee_{xy=\sum_{i=1}^{n} a_i b_i} \{{\chi_H}^2(a_i) \wedge f^2(b_i)\} \\ &\geq {\chi_H}^2(x) \wedge f^2(y) \\ &= f^2(y)\end{aligned}$$

and

$$\begin{aligned} g^2(xy) &\leq (\chi_H^c \gtrdot g)^2(xy) \\ &= \bigwedge_{xy=\sum_{i=1}^n a_i b_i}\{{\chi_H^c}^2(a_i) \vee g^2(b_i)\} \\ &\leq {\chi_H^c}^2(x) \vee g^2(y) \\ &= g^2(y) \end{aligned}$$

for all $x, y \in H$. Similarly, $f^2(xy) \geq f^2(x)$ and $g^2(xy) \leq g^2(x)$. Then, it is followed that $\Im = (f, g)$ be a PF ideal of H.

Theorem 5. Let $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ be PF ideals of H. $\Im_1 \lessdot \gtrdot \Im_2$ is a PF ideal of H.

Proof.

$$\begin{aligned} & (f_1 \lessdot f_2)^2(x) \wedge (f_1 \lessdot f_2)^2(y) \\ = \; & \bigvee_{x=\sum_{i=1}^n a_i b_i}\{{f_1}^2(a_i) \wedge {f_2}^2(b_i)\} \wedge \bigvee_{y=\sum_{j=1}^m c_j d_j}\{{f_1}^2(c_j) \wedge {f_2}^2(d_j)\} \\ = \; & \bigvee_{x=\sum_{i=1}^n a_i b_i} \bigvee_{y=\sum_{j=1}^m c_j d_j}\{{f_1}^2(a_i) \wedge f^2(b_i) \wedge {f_1}^2(c_j) \wedge {f_2}^2(d_j)\} \\ \leq \; & \bigvee_{x+y=\sum_{k=1}^t x_k y_k}\{{f_1}^2(x_k) \wedge {f_2}^2(y_k)\} \\ = \; & (f_1 \lessdot f_2)^2(x+y) \end{aligned}$$

and

$$\begin{aligned} & (g_1 \gtrdot g_2)^2(x) \vee (g_1 \gtrdot g_2)^2(y) \\ = \; & \bigwedge_{x=\sum_{i=1}^n a_i b_i}\{{g_1}^2(a_i) \vee {g_2}^2(b_i)\} \vee \bigwedge_{y=\sum_{j=1}^m c_j d_j}\{{g_1}^2(c_j) \vee {g_2}^2(d_j)\} \\ = \; & \bigwedge_{x=\sum_{i=1}^n a_i b_i} \bigwedge_{y=\sum_{j=1}^m c_j d_j}\{{g_1}^2(a_i) \vee g^2(b_i) \wedge {g_1}^2(c_j) \wedge {g_2}^2(d_j)\} \\ \geq \; & \bigwedge_{x+y=\sum_{k=1}^t x_k y_k}\{{g_1}^2(x_k) \vee {g_2}^2(y_k)\} \\ = \; & (g_1 \gtrdot g_2)^2(x+y) \end{aligned}$$

and

$$\begin{aligned} (f_1 \lessdot f_2)^2(x) &= \bigvee_{x=\sum_{i=1}^n a_i b_i}\{{f_1}^2(a_i) \wedge {f_2}^2(b_i)\} \\ &= \bigvee_{-x=\sum_{i=1}^n (-a_i) b_i}\{{f_1}^2(a_i) \wedge f^2(b_i)\} \\ &= \bigvee_{-x=\sum_{i=1}^n (-a_i) b_i}\{{f_1}^2(-a_i) \wedge f^2(b_i)\} \\ &= (f_1 \lessdot f_2)^2(-x) \end{aligned}$$

and

$$\begin{aligned} (g_1 \gtrdot g_2)^2(x) &= \bigwedge_{x=\sum_{i=1}^n a_i b_i}\{{g_1}^2(a_i) \vee {g_2}^2(b_i)\} \\ &= \bigwedge_{-x=\sum_{i=1}^n (-a_i) b_i}\{{g_1}^2(a_i) \vee g^2(b_i)\} \\ &= \bigwedge_{-x=\sum_{i=1}^n (-a_i) b_i}\{{g_1}^2(-a_i) \vee {g_2}^2(b_i)\} \\ &= (g_1 \gtrdot g_2)^2(-x) \end{aligned}$$

for all $x, y \in H$. Similarly, $(f_1 \lessdot f_2)^2(xy) \geq (f_1 \lessdot f_2)^2(x) \vee (f_1 \lessdot f_2)^2(y)$ and $(g_1 \gtrdot g_2)^2(xy) \leq (g_1 \gtrdot g_2)^2(x) \wedge (g_1 \gtrdot g_2)^2(y)$. Consequently, $\Im_1 \lessdot \gtrdot \Im_2$ is a PF ideal of H.

Theorem 6. Let $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ be PF ideals of H. Then, $\Im_1 \lessdot \gtrdot \Im_2 \subseteq \Im_1 \cap \Im_2$.

Proof. Since $\Im_1 = (f_1, g_1)$ is a PF ideal of H, we have $\chi_H \lessdot\gtrdot \Im_1 \subseteq \Im_1$ and $\Im_1 \lessdot\gtrdot \chi_H \subseteq \Im_1$. Moreover, since $\Im_1 \subseteq \chi_H$ and $\Im_2 \subseteq \chi_H$, we have $\Im_1 \lessdot\gtrdot \Im_2 \subseteq \chi_H \lessdot\gtrdot \Im_2 \subseteq \Im_2$ and $\Im_1 \lessdot\gtrdot \Im_2 \subseteq \Im_1 \lessdot\gtrdot \chi_H \subseteq \Im_1$. It follows that $\Im_1 \lessdot \gtrdot \Im_2 \subseteq \Im_1 \cap \Im_2$.

On Pythagorean fuzzy bi-ideals

In this section, we proposed the concept of Pythagorean fuzzy bi-ideal. Then, we proved some results on the Pythagorean fuzzy bi-ideal. Also, we researched some properties of Pythagorean bi-ideal by Pythagorean fuzzy product.

Definition 11. Let $\Im = (f, g)$ be a PF ring of H. $\Im = (f, g)$ is said to be a PF bi-ideal of H if $f^2(xyz) \geq f^2(x) \wedge f^2(z)$ and $g^2(xyz) \leq g^2(x) \vee g^2(z)$ for all $x, y, z \in H$.

Example 2. Let $H = \left\{ \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{0} & \bar{0} \end{pmatrix} : \bar{x} \in \mathbb{Z}_2 \right\}$ be a ring. Consider that the PF set $\Im = (f, g)$ of ring H as follows:
$f\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.87$, $f\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.32$, $f\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.32$, $f\begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.78$,
and
$g\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.02$, $g\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.65$, $g\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.65$, $g\begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.48$
Let's take $A = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$, $B = \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$, $C = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$, $D = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$.
Then, it is easily shown that
$f^2(AAB) \geq f^2(A) \wedge f^2(B)$, $g^2(AAB) \leq g^2(A) \vee g^2(B)$
$f^2(CCB) \geq f^2(C) \wedge f^2(B)$, $g^2(CCB) \leq g^2(C) \vee g^2(B)$
$f^2(DCB) \geq f^2(D) \wedge f^2(B)$, $g^2(DCB) \leq g^2(D) \vee g^2(B)$
. . .
All variations are proven to be displayed this way. Hereby, $\Im = (f, g)$ is a PF bi-ideal of H.

Remark 2. Let $\Im = (f, g)$ be a PF bi-ideal of H.

$$f^2(x) = f^2(\sum_{i=1}^{n} a_i b_i c_i) \geq f^2(a_i) \wedge f^2(c_i)$$

and

$$g^2(x) = g^2(\sum_{i=1}^{n} a_i b_i c_i) \leq g^2(a_i) \vee g^2(c_i)$$

for all $x = \sum_{i=1}^{n} a_i b_i c_i \in H$ where $1 \leq i \leq n$.

Theorem 7. Let $\Im = (f, g)$ be a PF set of H. $\Im = (f, g)$ be a PF bi-ideal of H iff

- For all $x, y \in H$, $f^2(x - y) \geq f^2(x) \wedge f^2(y)$ and $g^2(x - y) \leq g^2(x) \vee g^2(y)$
- $\Im \lessdot \gtrdot \Im \subseteq \Im$
- $\Im \lessdot \gtrdot \chi_H \lessdot \gtrdot \Im \subseteq \Im$

Proof. It can be made by using Definition 11.

Theorem 8. Let $\emptyset \neq A \subseteq H$. A is a bi-ideal of H iff χ_A is a PF bi-ideal of H.

Proof. Let A be a bi-ideal of H and $x, y, z \in H$.

Let $x, y \in A$. We have $x - y \in A$ and $xy \in A$. Then, $\chi_A^2(x-y) = \chi_A^2(xy) = 1$ and $\chi_{A^c}^2(x-y) = \chi_{A^c}^2(xy) = 0$. H is easy to see that $\chi_A^2(x-y) \geq \chi_A^2(x) \wedge \chi_A^2(y)$, $\chi_{A^c}^2(x-y) \leq \chi_{A^c}^2(x) \vee \chi_{A^c}^2(y)$ and $\chi_A^2(xy) \geq \chi_A^2(x) \wedge \chi_A^2(y)$, $\chi_{A^c}^2(xy) \leq \chi_{A^c}^2(x) \vee \chi_{A^c}^2(y)$.

Let $x, y \notin A$. We have $x-y \in A$ or $x-y \notin A$, $xy \in A$ or $xy \notin A$. In this situation, $\chi_A^2(x-y) \geq \chi_A^2(x) \wedge \chi_A^2(y)$, $\chi_{A^c}^2(x-y) \leq \chi_{A^c}^2(x) \vee \chi_{A^c}^2(y)$ and $\chi_A^2(xy) \geq \chi_A^2(x) \wedge \chi_A^2(y)$, $\chi_{A^c}^2(xy) \leq \chi_{A^c}^2(x) \vee \chi_{A^c}^2(y)$.

Therefore, χ_A is a PF ring of H. Similarly, it is shown that $\chi_A^2(x-y) \geq \chi_A^2(x) \wedge \chi_A^2(y)$, $\chi_{A^c}^2(x-y) \leq \chi_{A^c}^2(x) \vee \chi_{A^c}^2(y)$ and $\chi_A^2(xy) \geq \chi_A^2(x) \wedge \chi_A^2(y)$, $\chi_{A^c}^2(xy) \leq \chi_{A^c}^2(x) \vee \chi_{A^c}^2(y)$. Finally, χ_A is a PF bi-ideal of H.

Conversely, let χ_A be a PF bi-ideal of H and $x, y \in A$. It is followed that $\chi_A^2(x) = 1$, $\chi_A^2(y) = 1$ and $\chi_{A^c}^2(x) = 0$, $\chi_{A^c}^2(y) = 0$. Since χ_A is a PF bi-ideal of H, $\chi_A^2(x-y) \geq \chi_A^2(x) \wedge \chi_A^2(y)$, $\chi_{A^c}^2(x-y) \leq \chi_{A^c}^2(x) \vee \chi_{A^c}^2(y)$ and $\chi_A^2(xy) \geq \chi_A^2(x) \wedge \chi_A^2(y)$, $\chi_{A^c}^2(xy) \leq \chi_{A^c}^2(x) \vee \chi_{A^c}^2(y)$ for all $x, y \in A$. From here, we get $\chi_A^2(x-y) \geq 1$, $\chi_{A^c}^2(x-y) \leq 0$ and $\chi_A^2(xy) \geq 1$, $\chi_{A^c}^2(xy) \leq 0$. Thus, $\chi_A^2(x-y) = 1$, $\chi_{A^c}^2(x-y) = 0$ and $\chi_A^2(xy) = 1$, $\chi_{A^c}^2(xy) = 0$. It is followed that $x - y \in A$ and $xy \in A$. Therefore, A is a subring of H. Similarly, it is shown that $xyz \in A$ for all $x, z \in A$ and $y \in H$. Finally, A is a bi-ideal of H.

Example 3. Let $H = \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{0} & \bar{0} \end{pmatrix} : \bar{a}, \bar{b} \in \mathbb{Z}_3 \right\}$ be a ring, and $A = \left\{ \begin{pmatrix} \bar{0} & \bar{b} \\ \bar{0} & \bar{0} \end{pmatrix} : \bar{b} \in \mathbb{Z}_3 \right\} \subseteq H$.
The PF characteristic function

$$\chi_A = (\chi_A, \chi_{A^c})$$

as follows:

$\chi_A \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 1$, $\chi_A \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 1$, $\chi_A \begin{pmatrix} \bar{0} & \bar{2} \\ \bar{0} & \bar{0} \end{pmatrix} = 1$, $\chi_A \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0$, $\chi_A \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0$,
$\chi_A \begin{pmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{0} \end{pmatrix} = 0$, $\chi_A \begin{pmatrix} \bar{2} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0$, $\chi_A \begin{pmatrix} \bar{2} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0$, $\chi_A \begin{pmatrix} \bar{2} & \bar{2} \\ \bar{0} & \bar{0} \end{pmatrix} = 0$

and

$\chi_A^c \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0$, $\chi_A^c \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0$, $\chi_A^c \begin{pmatrix} \bar{0} & \bar{2} \\ \bar{0} & \bar{0} \end{pmatrix} = 0$, $\chi_A^c \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 1$, $\chi_A^c \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 1$,
$\chi_A^c \begin{pmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{0} \end{pmatrix} = 1$, $\chi_A^c \begin{pmatrix} \bar{2} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 1$, $\chi_A^c \begin{pmatrix} \bar{2} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 1$, $\chi_A^c \begin{pmatrix} \bar{2} & \bar{2} \\ \bar{0} & \bar{0} \end{pmatrix} = 1$

Since χ_A is a PF bi-ideal of H, A is a bi-ideal of H. Also, since A is a bi-ideal of H, χ_A is a PF bi-ideal of H.

Theorem 9. Every PF ideal of a ring is a PF bi-ideal of a ring.

Proof. Let $\Im = (f, g)$ be a PF ideal of a ring H. $\Im = (f, g)$ is a PF ring of H by definition of a PF ideal. Moreover, for all $x, y, z \in H$,

$$f^2(xyz) = f^2((xy)z) \geq f^2(z) \geq f^2(x) \wedge f^2(z)$$

and

$$g^2(xyz) = g^2((xy)z) \leq g^2(z) \leq g^2(x) \vee g^2(z)$$

Consequently, $\Im = (f, g)$ is a PF bi-ideal of H.

Example 4. Consider the PF bi-ideal in Example 2.
Since $f^2\left(\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}\cdot\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}\right) \neq f^2\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} \vee f^2\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}$, $\Im = (f, g)$ is not a PF ideal of H. However, $\Im = (f, g)$ is a PF bi-ideal of H.

Theorem 10. Let $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ be two PF bi-ideals of H. $\Im_1 \cap \Im_2$ is a PF bi-ideal of H.

Proof. It is known that $\Im_1 \cap \Im_2$ is a PF ring of H by Theorem 2.

$$\begin{aligned}(f_1 \cap f_2)^2(xyz) &= (f_1)^2(xyz) \wedge (f_2)^2(xyz)\\ &\geq [(f_1)^2(x) \wedge (f_1)^2(z)] \wedge [(f_2)^2(x) \wedge (f_2)^2(z)]\\ &= (f_1)^2(x) \wedge (f_2)^2(x) \wedge (f_1)^2(z) \wedge (f_1)^2(z)\\ &= (f_1 \cap f_2)^2(x) \wedge (f_1 \cap f_2)^2(z)\end{aligned}$$

and

$$\begin{aligned}(g_1 \cap g_2)^2(xyz) &= (g_1)^2(xyz) \vee (g_2)^2(xyz)\\ &\leq [(g_1)^2(x) \vee (g_1)^2(z)] \wedge [(g_2)^2(x) \vee (g_2)^2(z)]\\ &= (g_1)^2(x) \vee (g_2)^2(x) \vee (g_1)^2(z) \vee (g_1)^2(z)\\ &= (g_1 \cap g_2)^2(x) \vee (g_1 \cap g_2)^2(z)\end{aligned}$$

for all $x, y, z \in H$. Thereby, $\Im_1 \cap \Im_2$ is a PF bi-ideal of H.

Example 5. Let $H = \left\{\begin{pmatrix}\bar{x} & \bar{y}\\ \bar{0} & \bar{0}\end{pmatrix} : \bar{x} \in \mathbb{Z}_2\right\}$ be a ring. Consider that the PF sets $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ of ring H as follows:

$f_1\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.9$, $f_1\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.4$, $f_1\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.4$, $f_1\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.6$,
$g_1\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.2$, $g_1\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.8$, $g_1\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.8$, $g_1\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.5$

and

$f_2\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.6$, $f_2\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.1$, $f_2\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.1$, $f_2\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.5$,
$g_2\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.2$, $g_2\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.7$, $g_2\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.7$, $g_2\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.4$

Then, $\Im_1 = (f_1, g_2)$ and $\Im_2 = (f_2, g_2)$ are PF bi-ideals of H. Also,
$(f_1 \cap f_2)^2\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.36$, $(f_1 \cap f_2)^2\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.01$, $(f_1 \cap f_2)^2\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.01$, $(f_1 \cap f_2)^2\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.25$,
$(g_1 \cap g_2)^2\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.04$, $(g_1 \cap g_2)^2\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.64$, $(g_1 \cap g_2)^2\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.64$, $(g_1 \cap g_2)^2\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix} = 0.25$.
Herefrom, it is easily shown by Definition 11 that $\Im_1 \cap \Im_2$ is a PF bi-ideal of H.

Theorem 11. Let $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ be two PF bi-ideals of H. Then, $\Im_1 \times \Im_2 = (f_1 \curlywedge f_2, g_1 \curlyvee g_2)$ is a PF bi-ideal of $H \times H$.

Proof. For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in H \times H$,

$$\begin{aligned}
& (f_1 \leftthreetimes f_2)^2((x_1, x_2) - (y_1, y_2))(f_1 \leftthreetimes f_2)^2((x_1, x_2) - (y_1, y_2)) \\
= \; & (f_1 \leftthreetimes f_2)^2(x_1 - y_1, x_2, y_2) \\
= \; & (f_1)^2(x_1 - y_1) \wedge (f_2)^2(x_2 - y_2) \\
\geq \; & [(f_1)^2(x_1) \wedge (f_1)^2(y_1)] \wedge [(f_2)^2(x_2) \wedge (f_2)^2(y_2)] \\
= \; & (f_1)^2(x_1) \wedge (f_2)^2(x_2) \wedge (f_1)^2(y_1) \wedge (f_2)^2(y_2) \\
= \; & (f_1 \leftthreetimes f_2)^2(x_1, x_2) \wedge (f_1 \leftthreetimes f_2)^2(y_1, y_2)
\end{aligned}$$

Similarly, it is shown that $(f_1 \leftthreetimes f_2)^2((x_1, x_2)(y_1, y_2)) \geq (f_1 \leftthreetimes f_2)^2(x_1, x_2) \wedge (f_1 \leftthreetimes f_2)^2(y_1, y_2)$. Moreover,

$$\begin{aligned}
(g_1 \rightthreetimes g_2)^2((x_1, x_2) - (y_1, y_2)) &= (g_1 \rightthreetimes g_2)^2(x_1 - y_1, x_2, y_2) \\
&= (g_1)^2(x_1 - y_1) \vee (g_2)^2(x_2 - y_2) \\
&\leq [(g_1)^2(x_1) \vee (g_1)^2(y_1)] \vee [(g_2)^2(x_2) \vee (g_2)^2(y_2)] \\
&= (g_1)^2(x_1) \wedge (g_2)^2(x_2) \vee (g_1)^2(y_1) \vee (g_2)^2(y_2) \\
&= (g_1 \rightthreetimes g_2)^2(x_1, x_2) \vee (g_1 \rightthreetimes g_2)^2(y_1, y_2)
\end{aligned}$$

Similarly, it is shown that $(g_1 \rightthreetimes g_2)^2((x_1, x_2)(y_1, y_2)) \leq (g_1 \rightthreetimes g_2)^2(x_1, x_2) \vee (g_1 \rightthreetimes g_2)^2(y_1, y_2)$. Thus, $\Im_1 \times \Im_2$ is a PF ring of $H \times H$. Now,

$$\begin{aligned}
(f_1 \leftthreetimes f_2)^2((x_1, x_2)(y_1, y_2)(z_1, z_2)) &= (f_1 \leftthreetimes f_2)^2((x_1 y_1 z_1) - (x_2 y_2 z_2)) \\
&= (f_1)^2(x_1 y_1 z_1) \wedge (f_2)^2(x_2 y_2 z_2) \\
&\geq [(f_1)^2(x_1) \wedge (f_1)^2(z_1)] \wedge [(f_2)^2(x_2) \wedge (f_2)^2(z_2)] \\
&= (f_1)^2(x_1) \wedge (f_2)^2(x_2) \wedge (f_1)^2(z_1) \wedge (f_2)^2(z_2) \\
&= (f_1 \leftthreetimes f_2)^2(x_1, x_2) \wedge (f_1 \leftthreetimes f_2)^2(z_1, z_2)
\end{aligned}$$

Similarly, it is shown that $(g_1 \rightthreetimes g_2)^2((x_1, x_2)(y_1, y_2)(z_1, z_2)) \leq (g_1 \rightthreetimes g_2)^2(x_1, x_2) \wedge (g_1 \rightthreetimes g_2)^2(z_1, z_2)$.

Example 6. Consider the PF bi-ideal in Example 5. Then,
$(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.36$, $(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.01$,
$(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.01$, $(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.25$,
$(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.16$, $(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.01$,
$(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.01$, $(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.16$,
$(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.16$, $(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.01$,
$(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.01$, $(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.16$,
$(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.36$, $(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}, \begin{pmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{pmatrix}\right) = 0.01$,

$(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.01$, $(f_1 \leftthreetimes f_2)^2\left(\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.25$

and

$(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{0}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{0}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.04$, $(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{0}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{0}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.49$,

$(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{0}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.49$, $(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{0}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.16$,

$(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{0}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{0}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.64$, $(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{0}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{0}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.64$,

$(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{0}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.64$, $(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{0}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.64$,

$(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{1}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{0}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.64$, $(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{1}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{0}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.64$,

$(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{1}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.64$, $(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{1}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.64$,

$(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{0}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.25$, $(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{0}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.49$,

$(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{0}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.49$, $(g_1 \rightthreetimes g_2)^2\left(\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix},\begin{pmatrix}\bar{1}&\bar{1}\\\bar{0}&\bar{0}\end{pmatrix}\right)=0.25$.

Herefrom, it is easily shown by Definition 11 that $\Im_1 \times \Im_2$ is a PF bi-ideal of H.

Theorem 12. Let $\Im_1 = (f_1, g_1)$ and $\Im_2 = (f_2, g_2)$ be two PF bi-ideals of H. $\Im_1 \lessdot\gtrdot \Im_2$ is a PF bi-ideal of H.

Proof.

$$\begin{aligned}
& (f_1 \lessdot f_2)^2(x) \wedge (f_1 \lessdot f_2)^2(y) \\
= \; & \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{f_1^2(a_i) \wedge f_2^2(b_i)\} \wedge \bigvee_{y=\sum_{i=1}^{n} c_i d_i} \{f_1^2(c_i) \wedge f_2^2(d_i)\} \\
= \; & \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \{f_1^2(a_i) \wedge f_2^2(b_i)\} \wedge \bigvee_{-y=\sum_{i=1}^{n} -c_i d_i} \{f_1^2(-c_i) \wedge f_2^2(d_i)\} \\
= \; & \bigvee_{x=\sum_{i=1}^{n} a_i b_i} \bigvee_{-y=\sum_{i=1}^{n} -c_i d_i} \{f_1^2(a_i) \wedge f_1^2(-c_i) \wedge f_2^2(b_i)\} \wedge f_2^2(d_i)\} \\
\leq \; & \bigvee_{x-y=\sum_{i=1}^{k} x_i y_i} \{f_1^2(x_i) \wedge f_2^2(y_i)\} \\
= \; & (f_1 \lessdot f_2)^2(x-y)
\end{aligned}$$

$$\begin{aligned}
& (g_1 \gtrdot g_2)^2(x) \vee (g_1 \gtrdot g_2)^2(y) \\
= \; & \bigwedge_{x=\sum_{i=1}^{n} a_i b_i} \{g_1^2(a_i) \vee g_2^2(b_i)\} \vee \bigwedge_{y=\sum_{i=1}^{n} c_i d_i} \{g_1^2(c_i) \vee g_2^2(d_i)\} \\
= \; & \bigwedge_{x=\sum_{i=1}^{n} a_i b_i} \{g_1^2(a_i) \vee g_2^2(b_i)\} \vee \bigwedge_{-y=\sum_{i=1}^{n} -c_i d_i} \{g_1^2(-c_i) \vee g_2^2(d_i)\} \\
= \; & \bigwedge_{x=\sum_{i=1}^{n} a_i b_i} \bigwedge_{-y=\sum_{i=1}^{n} -c_i d_i} \{g_1^2(a_i) \vee g_1^2(-c_i) \vee g_2^2(b_i)\} \vee g_2^2(d_i)\} \\
\geq \; & \bigwedge_{x-y=\sum_{i=1}^{k} x_i y_i} \{g_1^2(x_i) \vee g_2^2(y_i)\} \\
= \; & (g_1 \gtrdot g_2)^2(x-y)
\end{aligned}$$

and

$$\begin{aligned}
& (f_1 \lessdot f_2)^2(x) \wedge (f_1 \lessdot f_2)^2(z) \\
= & \bigvee_{x=\sum_{i=1}^m a_ib_i}\{f_1^2(a_i) \wedge f_2^2(b_i)\} \wedge \bigvee_{z=\sum_{i=1}^n c_id_i}\{f_1^2(c_i) \wedge f_2^2(d_i)\} \\
= & \bigvee_{x=\sum_{i=1}^n a_ib_i}\bigvee_{z=\sum_{i=1}^n c_id_i}\{f_1^2(a_i) \wedge f_1^2(c_i) \wedge f_2^2(b_i)\} \wedge f_2^2(d_i)\} \\
\leq & \bigvee_{xyz=\sum_{i=1}^n x_iy_iz_i}\{f_1^2(x_i) \wedge f_1^2(y_i) \wedge f_1^2(e_i) \wedge f_2^2(b_i)\} \wedge f_2^2(d_i) \wedge f_2^2(h_i)\}\} \\
& \bigvee_{x-y=\sum_{i=1}^k x_iy_i}\{f_1^2(x_i) \wedge f_2^2(y_i)\} \\
= & (f_1 \lessdot f_2)^2(x-y)(f_1)^2(x) \wedge (f_2)^2(x) \wedge (f_1)^2(z) \wedge (f_1)^2(z) \\
= & [(f_1)^2(x) \wedge (f_1)^2(z)] \wedge [(f_2)^2(x) \wedge (f_2)^2(z)] \\
= & \bigvee_{xyz=\sum_{i=1}^n a_ib_i}\{f_1^2(a_i) \wedge f_2^2(b_i)\} \\
= & (f_1 \lessdot f_2)^2(xyz)
\end{aligned}$$

for all $x, y, z \in H$.

Theorem 13. Let ϑ be a ring isomorphism from H to R and $\Im = (f, g)$ be a PF bi-ideal of H. $\vartheta(\Im)$ is a PF bi-ideal of R.

Proof. Since ϑ is a surjective homomorphism, there exist $x, y, z \in H$ such that $a = \vartheta(x)$, $b = \vartheta(y)$ and $c = \vartheta(z)$ for all $a, b, c \in R$.

$$\begin{aligned}
\vartheta(f^2)(a-b) &= \bigvee\{f^2(k)|k \in H, \vartheta(k) = a-b\} \\
&= \bigvee\{f^2(k)|k \in H, \vartheta(k) = \vartheta(x)-\vartheta(y)\} \\
&= \bigvee\{f^2(k)|k \in H, \vartheta(k) = \vartheta(x-y)\} \\
&= \bigvee\{f^2(k)|k \in H, k = x-y\} \\
&= \bigvee\{f^2(x-y)|x, y \in H, \vartheta(x) = a, \vartheta(y) = b\} \\
&\geq \bigvee\{f^2(x) \wedge f^2(y)|x, y \in H, \vartheta(x) = a, \vartheta(y) = b\} \\
&= \bigvee\{f^2(x)|x \in H, \vartheta(x) = a\} \wedge \bigvee\{f^2(y)|y \in H, \vartheta(y) = b\} \\
&= \vartheta(f^2)(a) \wedge \vartheta(f^2)(b)
\end{aligned}$$

and

$$\begin{aligned}
\vartheta(f^2)(ab) &= \bigvee\{f^2(k)|k \in H, \vartheta(k) = ab\} \\
&= \bigvee\{f^2(k)|k \in H, \vartheta(k) = \vartheta(x)\vartheta(y)\} \\
&= \bigvee\{f^2(k)|k \in H, \vartheta(k) = \vartheta(xy)\} \\
&= \bigvee\{f^2(k)|k \in H, k = xy\} \\
&= \bigvee\{f^2(xy)|x, y \in H, \vartheta(x) = a, \vartheta(y) = b\} \\
&\geq \bigvee\{f^2(x) \wedge f^2(y)|x, y \in H, \vartheta(x) = a, \vartheta(y) = b\} \\
&= \bigvee\{f^2(x)|x \in H, \vartheta(x) = a\} \wedge \bigvee\{f^2(y)|y \in H, \vartheta(y) = b\} \\
&= \vartheta(f^2)(a) \wedge \vartheta(f^2)(b)
\end{aligned}$$

Similarly, it is shown that $\vartheta(g^2)(a-b) \leq \vartheta(g^2)(a) \vee \vartheta(g^2)(b)$ and $\vartheta(g^2)(ab) \leq \vartheta(g^2)(a) \vee \vartheta(g^2)(b)$.

Thus, $\vartheta(\Im)$ is a PF ring of R. Moreover,

$$\begin{aligned}
\vartheta(f^2)(abc) &= \bigvee\{f^2(k)|k\in H,\vartheta(k)=abc\}\\
&= \bigvee\{f^2(k)|k\in H,\vartheta(k)=\vartheta(x)\vartheta(y)\vartheta(z)\}\\
&= \bigvee\{f^2(k)|k\in H,\vartheta(k)=\vartheta(xyz)\}\\
&= \bigvee\{f^2(k)|k\in H,k=xyz\}\\
&= \bigvee\{f^2(xyz)|x,y,z\in H,\vartheta(x)=a,\vartheta(y)=b,\vartheta(z)=c\}\\
&\geq \bigvee\{f^2(x)\wedge f^2(z)|x,z\in H,\vartheta(x)=a,\vartheta(z)=c\}\\
&= \bigvee\{f^2(x)|x\in H,\vartheta(x)=a\}\wedge\bigvee\{f^2(z)|z\in H,\vartheta(z)=c\}\\
&= \vartheta(f^2)(a)\wedge\vartheta(f^2)(c)
\end{aligned}$$

Similarly, it is shown that $\vartheta(g^2)(abc)\leq\vartheta(g^2)(a)\vee\vartheta(g^2)(c)$. Consequently, $\vartheta(\Im)$ is a PF bi-ideal of R.

Theorem 14. Let ϑ be a ring homomorphism from H to R and $\Gamma=(n,t)$ be a PF bi-ideal of R. $\vartheta^{-1}(\Gamma)$ is a PF bi-ideal of H.

Proof. It is easily shown that $\vartheta^{-1}(\Gamma)$ is a PF ring of H.

$$\begin{aligned}
\vartheta^{-1}(n^2)(xyz) &= n^2(\vartheta(xyz))\\
&= n^2(\vartheta(x)\vartheta(y)\vartheta(z))\\
&\geq n^2(\vartheta(x))\wedge n^2(\vartheta(z))\\
&= \vartheta^{-1}(n^2)(x)\wedge\vartheta^{-1}(n^2)(z)
\end{aligned}$$

and

$$\begin{aligned}
\vartheta^{-1}(t^2)(xyz) &= n^2(\vartheta(xyz))\\
&= t^2(\vartheta(x)\vartheta(y)\vartheta(z))\\
&\leq t^2(\vartheta(x))\vee t^2(\vartheta(z))\\
&= \vartheta^{-1}(t^2)(x)\vee\vartheta^{-1}(t^2)(z)
\end{aligned}$$

for all $x,y,z\in H$. Finally, $\vartheta^{-1}(\Gamma)$ is a PF bi-ideal of H.

Theorem 15. Let H be a field and $\Im=(f,g)$ be a PF bi-ideal of H. $f^2(x)=f^2(1_H)$ and $g^2(x)=g^2(1_H)$ for all $x\in H$.

Proof. For all $\in H$,

$$f^2(x)=f^2(1_Hx1_H)\geq f^2(1_H)\wedge f^2(1_H)=f^2(1_H)$$

$$f^2(1_H)=f^2(1_H1_H)=f^2(xx^{-1}x^{-1}x)=f^2(x(x^{-1}x^{-1})x)\geq f^2(x)\wedge f^2(x)=f^2(x)$$

and

$$g^2(x)=g^2(1_Hx1_H)\leq g^2(1_H)\vee g^2(1_H)=g^2(1_H)$$

$$g^2(1_H)=g^2(1_H1_H)=g^2(xx^{-1}x^{-1}x)=g^2(x(x^{-1}x^{-1})x)\leq g^2(x)\vee g^2(x)=g^2(x)$$

Therefore, $f^2(x)=f^2(1_H)$ and $g^2(x)=g^2(1_H)$.

Theorem 16. Let $\Im=(f,g)$ be a PF bi-ideal of H. $\Im_{(\alpha,\beta)}$ is a bi-ideal of H where $\alpha\leq f(0_H)$ and $\beta\geq g(0_H)$.

Proof. For all $x, y \in \Im_{(\alpha,\beta)}$, we have $f^2(x) \geq \alpha$, $f^2(y) \geq \alpha$ and $g^2(x) \leq \beta$, $g^2(y) \leq \beta$.

$$f^2(x-y) \geq f^2(x) \wedge f^2(y) \geq \alpha \wedge \alpha \geq \alpha$$

and

$$g^2(x-y) \leq g^2(x) \vee g^2(y) \leq \beta \vee \beta \leq \beta$$

Thus, it is obtained that $x-y \in \Im_{(\alpha,\beta)}$. For all $x, y \in \Im_{(\alpha,\beta)}$ and $h \in H$, we have $f^2(x) \geq \alpha$, $f^2(y) \geq \alpha$ and $g^2(x) \leq \beta$, $g^2(y) \leq \beta$.

$$f^2(xhy) \geq f^2(x) \wedge f^2(y) \geq \alpha \wedge \alpha \geq \alpha$$

and

$$g^2(xhy) \leq g^2(x) \vee g^2(y) \leq \beta \vee \beta \leq \beta$$

Thus, it is obtained that $xhy \in \Im_{(\alpha,\beta)}$. Hence, $\Im_{(\alpha,\beta)}$ is a bi-ideal of H.

Example 7. Consider the PF bi-ideal in Example 2. For $\alpha = 0.6, \beta = 0.7$, it is obtained that $\Im_{(0.6,0.7)} = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$.

Hence, $\Im_{(0.6,0.7)}$ is a bi-ideal of H.

Remark 3. The converse of the Theorem 16 is not always valid. To prove this, we give a counterexample.

Example 8. Let $H = \left\{ \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{0} & \bar{0} \end{pmatrix} : \bar{x} \in \mathbb{Z}_2 \right\}$ be a ring. Consider that the PF set $\Im = (f, g)$ of ring H as follows:
$f \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.8$, $f \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.6$, $f \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.2$, $f \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.7$,
and
$g \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.3$, $g \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.7$, $g \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.8$, $g \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.4$
Then, $\Im_{(0.6,0.5)} = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$.
Thus, it is followed that $\Im_{(0.6,0.5)}$ is a bi-ideal of H.
Since $f^2 \left(\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} - \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right) \ngeq f^2 \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \cap f^2 \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$ and
$g^2 \left(\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} - \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right) \nleq g^2 \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \cup g^2 \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$, $\Im = (f, g)$ is not a PF ring of ring H. Therefore, $\Im = (f, g)$ is not a PF bi-ideal of ring H.

Theorem 17. Let $\Im = (f, g)$ be a PF bi-ideal of H. Then, $\Im_*$ is a bi-ideal of H.

Proof. We have $0_H \in \Im_{*+}$ and $0_H \in \Im_{*-}$ by definition of $\Im_*$. Therefore, it is followed that $\Im_{*+} \neq \emptyset$ and $\Im_{*-} \neq \emptyset$. For all $x, y \in \Im_{*+}$ and $h \in H$, we have $f^2(x) = f^2(0_H)$ and $f^2(y) = f^2(0_H)$. Then,

$$f^2(x-y) \geq f^2(x) \wedge f^2(y) = f^2(0_H) \wedge f^2(0_H) = f^2(0_H)$$

and

$$f^2(xhy) \geq f^2(x) \wedge f^2(y) = f^2(0_H) \wedge f^2(0_H) = f^2(0_H)$$

Also, it is known that $f^2(0_H) \geq f^2(x-y)$ and $f^2(0_H) \geq f^2(xhy)$. Thus, it is obtained that $x-y \in \Im_{*+}$ and $xhy \in \Im_{*+}$. We have $g^2(x) = g^2(0_H)$ and $g^2(y) = g^2(0_H)$ for all $x, y \in \Im_{*-}$ and $h \in H$. Eventually,

$$g^2(x-y) \leq g^2(x) \vee g^2(y) = g^2(0_H) \vee g^2(0_H) = g^2(0_H)$$

and

$$g^2(xhy) \leq g^2(x) \vee g^2(y) = g^2(0_H) \vee g^2(0_H) = g^2(0_H)$$

Also, it is known that $g^2(0_H) \leq g^2(x-y)$ and $g^2(0_H) \leq g^2(xhy)$. Thus, it is obtained that $x-y \in \Im_{*-}$ and $xhy \in \Im_{*-}$. Consequently, $\Im_*$ is a bi-ideal of H.

Example 9. Let $H = \left\{ \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{0} & \bar{0} \end{pmatrix} : \bar{x} \in \mathbb{Z}_2 \right\}$ be a ring. Consider that the PF set $\Im = (f, g)$ of ring H as follows:
$f \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.69$, $f \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.04$, $f \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.04$, $f \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.69$,
and
$g \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.35$, $g \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.77$, $g \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.77$, $g \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = 0.63$
Then, $\Im = (f, g)$ is a PF bi-ideal of H. It is followed that $\Im_* = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$. Hence, $\Im_*$ is a bi-ideal of H.

Conclusion

This paper defined the Pythagorean fuzzy product and investigated some basic properties. Then, we examined the connection between the Pythagorean fuzzy product and the Pythagorean fuzzy ideal. Moreover, we introduced the Pythagorean fuzzy bi-ideal and examined its algebraic properties. We proved that the intersection, Cartesian product, and Pythagorean fuzzy product of Pythagorean fuzzy bi-ideals are also Pythagorean fuzzy bi-ideals. Additionally, we showed that the image and preimage of Pythagorean fuzzy bi-ideal are Pythagorean fuzzy bi-ideal. In actuality, some of the findings in this paper have already established a foundation for additional discourse over the Pythagorean fuzzy ideal's future evolution. In our future study of the structure of the Pythagorean fuzzy ideal, we can describe many other ideal types, such as the Pythagorean fuzzy interior ideal, Pythagorean fuzzy quasi ideal, etc. The algebraic properties of these ideals can be studied. Furthermore, the relationship between the Pythagorean fuzzy bi-ideal and these new concepts can be examined.

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