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HOM-ASSOCIATIVE MAGMAS WITH APPLICATIONS TO HOM-ASSOCIATIVE MAGMA ALGEBRAS

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Abstract. Let X be a magma, that is a set equipped with a binary operation, and consider a function $\alpha : X \to X$. We say that X is Hom-associative if, for all $x, y, z \in X$, the equality $\alpha(x)(yz) = (xy)\alpha(z)$ holds. For every isomorphism class of magmas of order two, we determine all functions α making X Hom-associative. Furthermore, we find all such α that are endomorphisms of X. We also consider versions of these results where the binary operation on X and the function α only are partially defined. We use our findings to construct numerous examples of two-dimensional Hom-associative as well as multiplicative magma algebras.

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1. Introduction

In the last decades there has risen an intense interest in various *Hom* versions of algebraical objects. The defining axioms of these objects are miscellaneous endomorphism deformations of its standard axioms. The first example of this seems to be [5] where Hartwig, Larsson and Silvestrov define *Hom–Lie algebras*. For such objects, the usual Jacobi identity is replaced by the so called Hom–Jacobi identity:

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$$

where α is an endomorphism of the Lie algebra. Another instance of this is [9] where Makhlouf and Silvestrov introduce *Hom-algebras*, where the usual associativity is replaced by so called Hom-associativity:

$$\alpha(x)(yz) = (xy)\alpha(z) \tag{1}$$

where α now is an algebra endomorphism. Similarly, Hom-coalgebras, Hom-bialgebras and Hom–Hopf algebras have been proposed, see [10,11,13]. In [8] Laurent-Gengoux, Makhlouf and Teles define a *Hom-group* as a nonempty set equipped with a binary operation satisfying (1), multiplicativity of α:

$$\alpha(xy) = \alpha(x)\alpha(y) \tag{2}$$

and having a distinguished member 1 satisfying the unital identity:

$$1x = x1 = \alpha(x) \tag{3}$$

as well as some Hom versions of invertibility axioms for X (see [8, Def. 0.1]).

An impetus for studying Hom versions of classical mathematical objects is that it potentially could give us a language to describe families of involved mathematical structures using well studied less complicated structures, looking at them through a Hom lens. There are many such instances. Indeed, in [4] Goze and Remm show that *all* three-dimensional algebras are Hom-associative Lie algebras. Another example is [6, Ex. 2.13-14] where Hassanzadeh describes several non-associative structures as Hom-groups. For other relevant results on various types of Hom-associative structures, see [1] and [7] and the references therein.

In this article, we apply this philosophy to the context of *magmas*, that is sets equipped with a binary operation (see [2, p. 1]). Classically, these objects have been categorized into many different types of families, such as groups, semigroups, Brandt groupoids, quasigroups, multigroups, hypergroups, loops etc. (see e.g. [3]). We wish to add a Hom perspective to this classification. More specifically, for a given magma X, we would like to answer the following questions:

- *For what functions $\alpha : X \to X$ is X Hom-associative in the sense of* (1)?
- *Which of these functions are magma endomorphisms in the sense of* (2)?

To fully answer both of these questions for all magmas is probably a difficult task. So a first step would be to consider some special classes of magmas. In this article, we completely answer these questions for magmas of order two (see Theorem 2.6). Note that finding such Hom structures on magmas is important not only from the magma perspective, but also from the point of view of algebras over a field K. Namely, given a function $\alpha : X \to X$, then it induces a natural Hom-algebra structure on the magma algebra $K[X]$ of X over K, and Hom properties of α reflects upon algebra properties of $K[X]$ (see Theorem 3.5).

Here is a detailed outline of the article.

In Section 2, we first state our conventions on sets, relations, (partial) functions and (partial) equality of (partial) functions. Then we define various concepts of (partial) magmas such as weak/partial homomorphisms, and (partially) Hom-associative magmas. Thereafter, we consider magmas of order two. We first find all non-isomorphic multiplication tables of partial magmas of order two. Then we give a complete characterisation of all (weak) partial endomorphisms of these structures as well as all (partial) Hom-associative structures defined on them (see Theorem 2.6).

In Section 3, we first recall some classical definitions of multiplicative and Hom-associative Hom-algebras (see Definition 3.1). After that, we introduce partial versions of these concepts (see Definition 3.3). Then we show how various Hom properties of a magma X reflect upon properties of the corresponding magma algebra $K[X]$ (see Theorem 3.5). At the end of this section, we exemplify our main results for some instances of two-dimensional magma algebras (see Example 3.6).

2. Hom-associative magmas

2.1. Relations and functions. Let X and Y be sets. Suppose that f is a relation from X to Y. By this we mean that f is a subset of $X \times Y$ and we denote this by $f : X \to Y$. The *inverse relation of* f, denoted by $f^{-1} : Y \to X$, is the set $\{(y, x) \in Y \times X \mid (x, y) \in f\}$. Given $x \in X$ we put $f(x) := \{y \in Y \mid (x, y) \in f\}$ and we say that $f(x)$ is *defined* when $f(x) \neq \emptyset$. The *range* and *domain* of f are defined to be the sets $R_f := \cup_{x \in X} f(x)$ and $D_f := \cup_{y \in Y} f^{-1}(y)$ respectively. We say that f is a *partial function* if for all $x \in X$ the set $f(x)$ has at most one element. In that case, if $f(x)$ is defined and $f(x) = \{y\}$, then we will often, as customary, write $f(x) = y$. If f is a partial function with $D_f = X$, then f is called a *function*. If $g : Y \to Z$ is another relation, then the *composition of* g *and* f, denoted by $g \circ f : X \to Z$, is the set $\{(x, z) \in X \times Z \mid \exists y \in Y \ (x, y) \in f \text{ and } (y, z) \in g\}$. If $h : X' \to Y'$ is yet another relation, then $f \times h : X \times X' \to Y \times Y'$ is the relation $\{((x, x'), (y, y')) \mid (x, y) \in f \text{ and } (x', y') \in h\}$. The identity relation $\mathrm{id}_X : X \to X$ is the set $\{(x, x) \mid x \in X\}$; often we will skip the subscript and just write id. We let $\mathrm{Fun}(X, Y)$ ($\mathrm{Pfun}(X, Y)$) denote the set of (partial) functions from X to Y. Suppose that $f, g \in \mathrm{Pfun}(X, Y)$. We say that f and g are *partially equal*, denoted by $f \approx g$, if for all $x \in X$ such that $f(x)$ and $g(x)$ are defined, then $f(x) = g(x)$. Note that the relation $\approx$ is reflexive and symmetric but not necessarily transitive. Clearly, the restriction of $\approx$ to $\mathrm{Fun}(X, Y)$ coincides with the ordinary equality of functions.

2.2. Magmas. Let (X, ∇) be a *partial magma*. By this we mean that X is a set and $\nabla \in \mathrm{Pfun}(X \times X, X)$. Note that if $\nabla \in \mathrm{Fun}(X \times X, X)$, then (X, ∇) is a magma. Let (X', ∇') be another partial magma and suppose that $\alpha \in \mathrm{Pfun}(X, X')$.

Definition 2.1. With the above notations, we say that α is a:

- weak partial homomorphism of partial magmas if $\alpha \circ \nabla \approx \nabla' \circ (\alpha \times \alpha)$ as partial functions. In that case, if $\alpha \in \mathrm{Fun}(X, X')$, then α is called a weak homomorphism of partial magmas;
- partial homomorphism of partial magmas if $\alpha \circ \nabla = \nabla' \circ (\alpha \times \alpha)$ as partial functions. In that case, if $\alpha \in \mathrm{Fun}(X, X')$, then α is called a homomorphism of partial magmas;
- homomorphism of magmas if α is a homomorphism of partial magmas and (X, ∇) and (X', ∇') are indeed magmas.

We let M (PM) denote the category having (partial) magmas as objects and (partial) homomorphisms of (partial) magmas as morphisms. We let WPM denote the category having partial magmas as objects and weak partial homomorphisms of partial magmas as morphisms. Clearly, M is a subcategory of PM which, in turn, is a subcategory of WPM. The next result will not be used in the sequel in full generality. Nevertheless, we record it for its own interest.

Proposition 2.2. *Let (X, ∇) and (X', ∇') be partial magmas and suppose that $\alpha : (X, \nabla) \to (X', \nabla')$ is a morphism in* WPM.

(a) *The map α is an isomorphism in* M *if and only if α is bijective and for all $x, y \in X$ the equality $\alpha(\nabla(x, y)) = \nabla'(\alpha(x), \alpha(y))$ holds.*

(b) *The map α is an isomorphism in* PM *if and only if α is bijective and for all $x, y \in X$ $\alpha(\nabla(x, y))$ is defined $\Leftrightarrow$ $\nabla'(\alpha(x), \alpha(y))$ defined, and in that case the equality $\alpha(\nabla(x, y)) = \nabla'(\alpha(x), \alpha(y))$ holds.*

(c) *The map α is an isomorphism in* WPM *if and only if $\alpha|_{D_\alpha}$ is bijective and for all $x, y \in X$ $\alpha(\nabla(x, y))$ is defined $\Leftrightarrow$ $\nabla'(\alpha(x), \alpha(y))$ defined, and in that case the equality $\alpha(\nabla(x, y)) = \nabla'(\alpha(x), \alpha(y))$ holds.*

Proof. (a) Follows from (b). Now we prove (b). First we show the "only if" statement. Suppose that $\alpha : (X, \nabla) \to (X', \nabla')$ is an isomorphism in PM. Then there is a morphism $\beta : (X', \nabla') \to (X, \nabla)$ in PM such that $\beta \circ \alpha = \mathrm{id}_X$ and $\alpha \circ \beta = \mathrm{id}_{X'}$. Take $x, y \in X$. If $\alpha(\nabla(x, y))$ is defined, then, since α is morphism in PM it follows that $\nabla'(\alpha(x), \alpha(y))$ is also defined. If $\nabla'(\alpha(x), \alpha(y))$ is defined, then $\beta(\nabla'(\alpha(x), \alpha(y)))$ is defined, which, since β is a morphism in PM, implies that $\nabla(\beta(\alpha(x)), \beta(\alpha(y))) = \nabla(x, y)$ is defined. Thus $\alpha(\nabla(x, y))$ is defined. Now we show the "if" statement. Suppose that α is bijective and for all $x, y \in X$ $\alpha(\nabla(x, y))$ is defined $\Leftrightarrow$ $\nabla'(\alpha(x), \alpha(y))$ defined, and in that case the equality $\alpha(\nabla(x, y)) = \nabla'(\alpha(x), \alpha(y))$ holds. Put $\beta = \alpha^{-1}$ and take $x', y' \in X'$ such that $\beta(\nabla'(x', y'))$ is defined. Take $x, y \in X$ with $\alpha(x) = x'$ and $\alpha(y) = y'$. Then $\beta(\nabla'(\alpha(x), \alpha(y)))$ is defined. From the assumptions it follows that $\nabla(x, y) = \beta(\alpha(\nabla(x, y)))$ also is defined and that $\beta(\nabla'(x', y')) = \nabla(\beta(x'), \beta(y'))$. Thus, β is a morphism in PM. Clearly, $\beta \circ \alpha = \mathrm{id}_X$ and $\alpha \circ \beta = \mathrm{id}_{X'}$ so that α is an isomorphism in PM. The statement in (c) follows from (b) by restriction. $\square$

Definition 2.3. Suppose that (X, ∇) is a partial magma and $\alpha \in \mathrm{Pfun}(X, X)$. We say that the triple (X, ∇, α) is:

- partially Hom-associative if $\nabla \circ (\alpha \times \nabla) \approx \nabla \circ (\nabla \times \alpha)$ as partial functions;
- Hom-associative if $\nabla \circ (\alpha \times \nabla) = \nabla \circ (\nabla \times \alpha)$ as partial functions;
- partially associative if (X, ∇, id) is partially Hom-associative;
- associative if (X, ∇, id) is Hom-associative.

2.3. Magmas of order two. For the rest of this section, (X, ∇) denotes a partial magma with $X = \{1, 2\}$. We will write $\nabla(x, y) = 3$ when $\nabla(x, y)$ is not defined. A multiplication table defined by $\nabla(1, 1) = a$, $\nabla(1, 2) = b$, $\nabla(2, 1) = c$ and $\nabla(2, 2) = d$ will be written in short hand as $abcd$. So, for instance, the multiplication table 2131 is to be interpreted as $\nabla(1, 1) = 2$, $\nabla(1, 2) = 1$, $\nabla(2, 1)$ is undefined and $\nabla(2, 2) = 1$. Clearly, there are $3^4 = 81$ different such multiplication tables. Put $\overline{X} = X \cup \{3\}$ and define the function $t : \overline{X} \to \overline{X}$ by $t(1) = 2$, $t(2) = 1$ and $t(3) = 3$. The next result is probably folklore. Nonetheless, for the convenience of the reader, we include it as well as a proof of it here.

Proposition 2.4. *Suppose that $a, b, c, d, e, f, g, h \in \overline{X}$. The multiplication tables $abcd$ and $efgh$ yield isomorphic partial magmas if and only if $a = e$, $b = f$, $c = g$ and $d = h$, or $t(a) = h$, $t(b) = g$, $t(c) = f$ and $t(d) = e$.*

Proof. Let ∇ and ∇' denote the partial maps $X \times X \to X$ defined by the multiplication tables $abcd$ and $efgh$, respectively. First we show the "if" statement. We consider two cases. Case 1: $a = e$, $b = f$, $c = g$ and $d = h$. By Proposition 2.2, id is an isomorphism of partial magmas $(X, \nabla) \to (X, \nabla')$. Case 2: $t(a) = h$, $t(b) = g$, $t(c) = f$ and $t(d) = e$. Then:

$$\begin{array}{rcccccccl} t(\nabla(1,1)) &=& t(a) &=& h &=& \nabla'(2,2) &=& \nabla'(t(1),t(1)) \\ t(\nabla(1,2)) &=& t(b) &=& g &=& \nabla'(2,1) &=& \nabla'(t(1),t(2)) \\ t(\nabla(2,1)) &=& t(c) &=& f &=& \nabla'(1,2) &=& \nabla'(t(2),t(1)) \\ t(\nabla(2,2)) &=& t(d) &=& e &=& \nabla'(1,1) &=& \nabla'(t(2),t(2)). \end{array}$$

Therefore, by Proposition 2.2 again, t is an isomorphism of partial magmas $(X, \nabla) \to (X, \nabla')$. Now we show the "only if" statement. Suppose that $F : (X, \nabla) \to (X, \nabla')$

is an isomorphism of partial magmas. By Proposition 2.2, F is a bijection $X \to X$ so that $F = \mathrm{id}$ or $F = t$. Case 1: $F = \mathrm{id}$. Then:

$$\begin{array}{rclcrcl} F(\nabla(1,1)) & = & \nabla'(F(1),F(1)) & \Rightarrow & a & = & e \\ F(\nabla(1,2)) & = & \nabla'(F(1),F(2)) & \Rightarrow & b & = & f \\ F(\nabla(2,1)) & = & \nabla'(F(2),F(1)) & \Rightarrow & c & = & g \\ F(\nabla(2,2)) & = & \nabla'(F(2),F(2)) & \Rightarrow & d & = & h. \end{array}$$

Case 2: $F = t$. Then:

$$\begin{array}{rclcrcl} F(\nabla(1,1)) & = & \nabla'(F(1),F(1)) & \Rightarrow & t(a) & = & h \\ F(\nabla(1,2)) & = & \nabla'(F(1),F(2)) & \Rightarrow & t(b) & = & g \\ F(\nabla(2,1)) & = & \nabla'(F(2),F(1)) & \Rightarrow & t(c) & = & f \\ F(\nabla(2,2)) & = & \nabla'(F(2),F(2)) & \Rightarrow & t(d) & = & e. \end{array}$$

□

Proposition 2.5. *The 81 multiplication tables of partial magma structures defined on X is partitioned into the following 45 isomorphism classes:*

(1) 3333 (2) $1333 \cong 3332$ (3) $2333 \cong 3331$ (4) $3133 \cong 3323$
(5) $3233 \cong 3313$ (6) $1133 \cong 3322$ (7) $1233 \cong 3312$ (8) $2133 \cong 3321$
(9) $2233 \cong 3311$ (10) $1313 \cong 3232$ (11) $1323 \cong 3132$ (12) $2313 \cong 3231$
(13) $2323 \cong 3131$ (14) $1331 \cong 2332$ (15) 1332 (16) 2331
(17) $3113 \cong 3223$ (18) 3123 (19) 3213 (20) $1113 \cong 3222$
(21) $1123 \cong 3122$ (22) $1213 \cong 3212$ (23) $2113 \cong 3221$ (24) $1223 \cong 3112$
(25) $2123 \cong 3121$ (26) $2213 \cong 3211$ (27) $2223 \cong 3111$ (28) $1131 \cong 2322$
(29) $1132 \cong 1322$ (30) $1231 \cong 2312$ (31) $2131 \cong 2321$ (32) $1232 \cong 1312$
(33) $2132 \cong 1321$ (34) $2231 \cong 2311$ (35) $2232 \cong 1311$ (36) $1111 \cong 2222$
(37) $1112 \cong 1222$ (38) $1121 \cong 2122$ (39) $1211 \cong 2212$ (40) $2111 \cong 2221$
(41) 1122 (42) 1212 (43) $1221 \cong 2112$ (44) 2121 (45) 2211

Proof. This is a straightforward but tedious application of Proposition 2.4. □

A partial map $\alpha : X \to X$ will below be encoded by a binary word ab where $a, b \in \overline{X}$ meaning that $\alpha(1) = a$ and $\alpha(2) = b$ where we put $a = 3$ or $b = 3$ when $\alpha(1)$ respectively $\alpha(2)$ is not defined. So, for instance, the word 23 means the partial map α with $\alpha(1) = 2$ and $\alpha(2)$ is undefined. Note that with this notation $\mathrm{Pfun}(X, X) = \{33, 13, 23, 31, 32, 11, 12, 21, 22\}$. In the next result, we determine the (weak) partial endomorphisms and the (weak) Hom-associative structures for the first representative in each of the 45 isomorphism classes in Proposition 2.5. This is achieved by an elementary case by case analysis using simple MATLAB-programs, the codes of which can be obtained from the author upon request.

Theorem 2.6. (a) *The set of weak partial endomorphisms for each of the first representatives in the 45 isomorphism classes in Proposition 2.5 is:*

(1) $\mathrm{Pfun}(X,X)$ (2) $\mathrm{Pfun}(X,X)$ (3) $\{33,13,23,31,32,12,21,22\}$
(4) $\mathrm{Pfun}(X,X)$ (5) $\mathrm{Pfun}(X,X)$ (6) $\mathrm{Pfun}(X,X)$ (7) $\mathrm{Pfun}(X,X)$
(8) $\{33,13,23,31,32,12,21,22\}$ (9) $\{33,13,23,31,32,12,21,22\}$
(10) $\mathrm{Pfun}(X,X)$ (11) $\mathrm{Pfun}(X,X)$ (12) $\{33,13,23,31,32,12,21,22\}$
(13) $\{33,13,23,31,32,12,21,22\}$ (14) $\{33,13,31,32,11,12\}$
(15) $\mathrm{Pfun}(X,X)$ (16) $\{33,13,23,31,32,12,21\}$
(17) $\{33,13,23,31,32,11,12,22\}$ (18) $\mathrm{Pfun}(X,X)$ (19) $\mathrm{Pfun}(X,X)$
(20) $\{33,13,23,31,32,11,12,22\}$ (21) $\mathrm{Pfun}(X,X)$ (22) $\mathrm{Pfun}(X,X)$
(23) $\{33,13,23,31,32,12,22\}$ (24) $\{33,13,23,31,32,11,12,22\}$
(25) $\{33,13,23,31,32,12,21,22\}$ (26) $\{33,13,23,31,32,12,21,22\}$
(27) $\{33,13,23,31,32,12,21,22\}$ (28) $\{33,13,31,32,11,12\}$
(29) $\mathrm{Pfun}(X,X)$ (30) $\{33,13,31,32,11,12\}$ (31) $\{33,13,23,31,32,12,21\}$
(32) $\mathrm{Pfun}(X,X)$ (33) $\{33,13,23,32,12,22\}$ (34) $\{33,13,23,31,32,12,21\}$
(35) $\{33,13,23,32,12,22\}$ (36) $\{33,13,31,32,11,12\}$
(37) $\{33,13,23,31,32,11,12,22\}$ (38) $\{33,13,31,32,11,12\}$
(39) $\{33,13,31,32,11,12\}$ (40) $\{33,13,23,31,32,12\}$ (41) $\mathrm{Pfun}(X,X)$
(42) $\mathrm{Pfun}(X,X)$ (43) $\{33,13,31,32,11,12\}$
(44) $\{33,13,23,31,32,12,21\}$ (45) $\{33,13,23,31,32,12,21\}$

(b) *The set of partial endomorphisms for each of the first representatives in the 45 isomorphism classes in Proposition 2.5 is:*

(1) $\mathrm{Pfun}(X,X)$ (2) $\{33,13,32,12\}$ (3) $\{33,23,12\}$ (4) $\{33,31,32,12\}$
(5) $\{33,13,23,12\}$ (6) $\{33,32,12\}$ (7) $\{33,13,12\}$ (8) $\{33,12\}$
(9) $\{33,23,12\}$ (10) $\{33,32,12\}$ (11) $\{33,13,12\}$ (12) $\{33,12\}$
(13) $\{33,23,12\}$ (14) $\{33,12\}$ (15) $\{33,13,23,31,32,12,21\}$
(16) $\{33,12,21\}$ (17) $\{33,31,32,12\}$ (18) $\{33,12,21\}$ (19) $\{33,12,21\}$
(20) $\{33,32,12\}$ (21) $\{33,12\}$ (22) $\{33,12\}$ (23) $\{33,12\}$
(24) $\{33,13,12\}$ (25) $\{33,12\}$ (26) $\{33,12\}$ (27) $\{33,23,12\}$
(28) $\{33,12\}$ (29) $\{33,31,32,12\}$ (30) $\{33,12\}$ (31) $\{33,12\}$
(32) $\{33,13,23,12\}$ (33) $\{33,12\}$ (34) $\{33,12\}$ (35) $\{33,12\}$
(36) $\{33,11,12\}$ (37) $\{33,31,32,11,12,22\}$ (38) $\{33,11,12\}$
(39) $\{33,11,12\}$ (40) $\{33,12\}$ (41) $\{33,11,12,21,22\}$
(42) $\{33,11,12,21,22\}$ (43) $\{33,11,12\}$ (44) $\{33,12,21\}$ (45) $\{33,12,21\}$

(c) *The set of partial Hom-associative structures for each of the first representatives in the 45 isomorphism classes in Proposition 2.5 is:*

(1) $\mathrm{Pfun}(X,X)$ (2) $\mathrm{Pfun}(X,X)$ (3) $\mathrm{Pfun}(X,X)$ (4) $\mathrm{Pfun}(X,X)$
(5) $\mathrm{Pfun}(X,X)$ (6) $\mathrm{Pfun}(X,X)$ (7) $\{33,13,23,31,32,12,21,22\}$
(8) $\mathrm{Pfun}(X,X)$ (9) $\mathrm{Pfun}(X,X)$ (10) $\mathrm{Pfun}(X,X)$
(11) $\{33,13,23,31,32,12,21,22\}$ (12) $\mathrm{Pfun}(X,X)$ (13) $\mathrm{Pfun}(X,X)$
(14) $\mathrm{Pfun}(X,X)$ (15) $\mathrm{Pfun}(X,X)$ (16) $\mathrm{Pfun}(X,X)$ (17) $\mathrm{Pfun}(X,X)$
(18) $\mathrm{Pfun}(X,X)$ (19) $\mathrm{Pfun}(X,X)$ (20) $\mathrm{Pfun}(X,X)$ (21) $\{33,13,32,12\}$
(22) $\{33,13,32,12\}$ (23) $\{33,13,23,31,32,12,21,22\}$
(24) $\{33,13,23,31,32,12,21,22\}$ (25) $\{33,23,31,32,21,22\}$
(26) $\{33,23,31,32,21,22\}$ (27) $\mathrm{Pfun}(X,X)$ (28) $\mathrm{Pfun}(X,X)$
(29) $\{33,13,23,31,32,11,12\}$ (30) $\{33,13,23,31,32,12,21\}$
(31) $\{33,13,23,31,32,11,21,22\}$ (32) $\{33,13,23,31,32,12,22\}$
(33) $\{33,13,23,31,32,12,21\}$ (34) $\{33,13,23,31,32,11,21,22\}$
(35) $\mathrm{Pfun}(X,X)$ (36) $\mathrm{Pfun}(X,X)$ (37) $\{33,13,23,31,32,11,12\}$
(38) $\{33,13\}$ (39) $\{33,13\}$ (40) $\{33,13,23,31,32,22\}$
(41) $\{33,13,32,12\}$ (42) $\{33,13,32,12\}$ (43) $\{33,13,23,31,32,12,21\}$
(44) $\{33,23,31,21\}$ (45) $\{33,23,31,21\}$

In the 37 cases (1)-(24), (27)-(30), (32), (33), (35)-(37) *and* (41)-(43) (X,∇) *is partially associative.*

(d) *The set of Hom-associative structures for each of the first representatives in each the 45 isomorphism classes in Proposition 2.5 is:*

(1) $\mathrm{Pfun}(X,X)$ (2) $\{33,13,23,32,12,22\}$ (3) $\mathrm{Pfun}(X,X)$
(4) $\{33,13,31,11\}$ (5) $\{33,23,32,22\}$ (6) $\{33\}$ (7) $\{33,12\}$ (8) $\{33\}$
(9) $\{33,23,32,22\}$ (10) $\{33\}$ (11) $\{33,12\}$ (12) $\{33\}$
(13) $\{33,23,32,22\}$ (14) $\{33,23,32,22\}$ (15) $\{33,13,32,12\}$
(16) $\{33,23,31,21\}$ (17) $\{33,13,31,11\}$ (18) $\{33\}$ (19) $\{33\}$ (20) $\{33\}$
(21) $\{33\}$ (22) $\{33\}$ (23) $\{33\}$ (24) $\{33,23,12\}$ (25) $\{33\}$ (26) $\{33\}$
(27) $\{33,23,32,22\}$ (28) $\{33\}$ (29) $\{33\}$ (30) $\{33\}$ (31) $\{33\}$ (32) $\{33\}$
(33) $\{33\}$ (34) $\{33\}$ (35) $\{33\}$ (36) $\{33,11,12,21,22\}$ (37) $\{33,11,12\}$
(38) $\{33\}$ (39) $\{33\}$ (40) $\{33,22\}$ (41) $\{33,12\}$ (42) $\{33,12\}$
(43) $\{33,12,21\}$ (44) $\{33,21\}$ (45) $\{33,21\}$

In the 13 cases (1)-(3), (7), (11), (15), (24), (36), (37) *and* (41)-(43) (X,∇) *is associative.*

3. Hom-associative magma algebras

3.1. Hom-algebras. For the rest of this paper, K denotes a field and A denotes a K-vector space. Suppose that μ is a K-bilinear map $A \times A \to A$ and that τ is a K-linear map $A \to A$. Recall from [11] that the triple (A, μ, τ) is then called a *Hom-algebra*. From [11,12], we extract the following:

Definition 3.1. A Hom-algebra (A, μ, τ) is said to be:

- multiplicative if $\tau \circ \mu = \mu \circ (\tau \times \tau)$;
- Hom-associative if $\mu \circ (\tau \times \mu) = \mu \circ (\mu \times \tau)$;
- associative if (A, μ, id) is Hom-associative.

From now on, we fix a K-vector space basis $B = \{e_i\}_{i \in I}$ for A. The maps μ and τ are uniquely determined by their *structure constants* $C_{ij}^k, t_{ij} \in K$ which are defined by $\mu(e_i, e_j) = \sum_{k \in I} C_{ij}^k e_k$ and $\tau(e_i) = \sum_{j \in I} t_{ij} e_j$ for $i, j \in I$. Note that given $i, j \in I$ ($i \in I$), then $C_{ij}^k = 0$ ($t_{ij} = 0$) for all but finitely many $k \in I$ ($j \in I$) making the sums above well defined. The properties in Definition 3.1 can now be formulated using structure constants. Indeed, from the discussion in [12, Section 4.4] we extract the following:

Proposition 3.2. *A Hom-algebra (A, μ, τ) is:*

(a) *multiplicative if and only if for all $i, j, s \in I$*

$$\sum_{p \in I} t_{sp} C_{ij}^p = \sum_{p,q \in I} t_{pi} t_{qj} C_{pq}^s;$$

(b) *Hom-associative if and only if for all $i, j, k, s \in I$*

$$\sum_{l,m \in I} t_{il} C_{jk}^m C_{lm}^s = \sum_{l,m \in I} t_{mk} C_{ij}^l C_{lm}^s;$$

(c) *associative if and only if for all $i, j, k, s \in I$*

$$\sum_{m \in I} C_{jk}^m C_{im}^s = \sum_{l \in I} C_{ij}^l C_{lk}^s.$$

We now introduce the following weakening of Definition 3.1:

Definition 3.3. We say that a Hom-algebra (A, μ, τ) is:

- partially B-multiplicative if $\tau(\mu(a, b)) = \mu(\tau(a), \tau(b))$ holds for all $a, b \in B$ with $\tau(\mu(a, b)) \neq 0 \neq \mu(\tau(a), \tau(b))$;
- partially B-Hom-associative if $\mu(\tau(a), \mu(b, c)) = \mu(\mu(a, b), \tau(c))$ holds for all $a, b, c \in B$ with $\mu(\tau(a), \mu(b, c)) \neq 0 \neq \mu(\mu(a, b), \tau(c))$;
- partially B-associative if (A, μ, id) is partially B-Hom-associative.

In analogy with Proposition 3.2, it is a straightforward exercise to see that the properties in Definition 3.3 can be formulated using structure constants:

Proposition 3.4. *A Hom-algebra (A, μ, τ) is:*

(a) *partially B-multiplicative if and only if for all $i, j \in I$ with both*

$$\left(\sum_{p \in I} t_{sp} C_{ij}^p\right)_{s \in I} \quad \text{and} \quad \left(\sum_{p,q \in I} t_{pi} t_{qj} C_{pq}^s\right)_{s \in I} \quad \text{nonzero,}$$

then $\left(\sum_{p \in I} t_{sp} C_{ij}^p\right)_{s \in I} = \left(\sum_{p,q \in I} t_{pi} t_{qj} C_{pq}^s\right)_{s \in I}$;

(b) *partially B-Hom-associative if and only if for all $i, j, k \in I$ with both*

$$\left(\sum_{l,m \in I} t_{il} C_{jk}^m C_{lm}^s\right)_{s \in I} \quad \text{and} \quad \left(\sum_{l,m \in I} t_{mk} C_{ij}^l C_{lm}^s\right)_{s \in I} \quad \text{nonzero,}$$

then $\left(\sum_{l,m \in I} t_{il} C_{jk}^m C_{lm}^s\right)_{s \in I} = \left(\sum_{l,m \in I} t_{mk} C_{ij}^l C_{lm}^s\right)_{s \in I}$;

(c) *partially B-associative if and only if for all $i, j, k \in I$ with both*

$$\left(\sum_{m \in I} C_{jk}^m C_{im}^s\right)_{s \in I} \quad \text{and} \quad \left(\sum_{l \in I} C_{ij}^l C_{lk}^s\right)_{s \in I} \quad \text{nonzero,}$$

then $\left(\sum_{m \in I} C_{jk}^m C_{im}^s\right)_{s \in I} = \left(\sum_{l \in I} C_{ij}^l C_{lk}^s\right)_{s \in I}$.

3.2. Magma algebras. Suppose (X, ∇) is a partial magma and $\alpha \in \mathrm{Pfun}(X, X)$. We now show that ∇ and α induce, in a natural way, a Hom-algebra structure on the so called magma algebra $K[X]$ of X over K (see Theorem 3.5 below). Recall that the elements of $K[X]$ are formal sums $\sum_{x \in X} k_x x$, for some $k_x \in K$, satisfying $k_x = 0$ for all but finitely many $x \in X$. Take $k \in K$. Suppose that $a := \sum_{x \in X} l_x x \in K[X]$ and $b := \sum_{x \in X} m_x x \in K[X]$. If we put

$$ka = \sum_{x \in X} (k l_x) x \qquad \text{and} \qquad a + b = \sum_{x \in X} (l_x + m_x) x,$$

then, with these operations, $K[X]$ is a K-vector space having the elements of $B := X$ as a basis. Let $\tau_\alpha : K[X] \to K[X]$ be defined in the following way. Take $x \in X$. Put $\tau_\alpha(x) = \alpha(x)$, if $\alpha(x)$ is defined, and $\tau_\alpha(x) = 0$, otherwise. Using structure constants this means that for all $x, y \in X$:

$$t_{xy} = \begin{cases} 1 & \text{if } \alpha(x) \text{ is defined and } y = \alpha(x); \\ 0 & \text{if } \alpha(x) \text{ is not defined, or } \alpha(x) \text{ is defined but } y \neq \alpha(x). \end{cases}$$

Then we K-linearly extend τ_α to $K[X]$. Let $\mu_\nabla : K[X] \times K[X] \to K[X]$ be defined in the following way. Take $x, y \in X$. Put $\mu_\nabla(x, y) = \nabla(x, y)$, if $\nabla(x, y)$ is defined, and $\mu_\nabla(x, y) = 0$, otherwise. In the language of structure constants, this amounts to saying that for all $x, y, z \in X$:

$$C_{xy}^z = \begin{cases} 1 & \text{if } \nabla(x,y) \text{ is defined and } z = \nabla(x,y); \\ 0 & \text{if } \nabla(x,y) \text{ is not defined, or } \nabla(x,y) \text{ is defined but } z \neq \nabla(x,y). \end{cases}$$

Then we K-bilinearly extend μ_∇ to $K[X] \times K[X]$. With the above notations:

Theorem 3.5. *Let (X, ∇) be a partial magma and let $\alpha \in \mathrm{Pfun}(X, X)$.*

(a) *$(K[X], \mu_\nabla, \tau_\alpha)$ is a Hom-algebra;*

(b) *$(K[X], \mu_\nabla, \tau_\alpha)$ is (partially X-)multiplicative if and only if α is a (weak) partial endomorphism of partial magmas;*

(c) *$(K[X], \mu_\nabla, \tau_\alpha)$ is (partially X-)Hom-associative if and only if (X, ∇, α) is (partially) Hom-associative;*

(d) *$(K[X], \mu_\nabla)$ is partially X-associative if and only if (X, ∇) is partially associative;*

(e) *$(K[X], \mu_\nabla)$ is associative if and only if (X, ∇) is associative.*

Proof. (a) This is clear. The statement in (b) follows from K-linearity and the fact that the equalities

$$\begin{aligned} (\tau_\alpha \circ \mu_\nabla)(x,y) &= (\alpha \circ \nabla)(x,y) \\ (\mu_\nabla \circ (\tau_\alpha \times \tau_\alpha))(x,y) &= (\nabla \circ (\alpha \times \alpha))(x,y) \end{aligned}$$

hold for all $x, y \in X$ for which the left hand sides above are nonzero. The statement in (c) follows from K-bilinearity and the fact that the equalities

$$\begin{aligned} (\mu_\nabla \circ (\tau_\alpha \times \mu_\nabla))(x,y,z) &= (\nabla \circ (\alpha \times \nabla))(x,y,z) \\ (\mu_\nabla \circ (\mu_\nabla \times \tau_\alpha))(x,y,z) &= (\nabla \circ (\nabla \times \alpha))(x,y,z) \end{aligned}$$

hold for all $x, y, z \in X$ for which the left hand sides above are nonzero. The statements in (d) and (e) follow from (c). □

Example 3.6. We now exemplify Theorems 2.6 and 3.5 for some instances of Hom-algebras $H := (K[X], \mu_\nabla, \tau_\alpha)$ when (X, ∇) is a magma of order two.

(a) Let $\nabla = 2232$. This is magma (35) in Proposition 2.5. Then:
   - H is partially X-multiplicative $\Leftrightarrow \alpha \in \{33, 13, 23, 32, 12, 22\}$;
   - H is multiplicative $\Leftrightarrow \alpha \in \{33, 12\}$;
   - H is partially X-Hom-associative $\Leftrightarrow \alpha \in \mathrm{Pfun}(X, X)$;
   - H is Hom-associative $\Leftrightarrow \alpha = 33$;
   - H is multiplicative and Hom-associative $\Leftrightarrow \alpha = 33$;
   - H is partially X-associative but not associative.

(b) Let $\nabla = 2111$. This is magma (40) in Proposition 2.5. Then:
   - H is partially X-multiplicative $\Leftrightarrow \alpha \in \{33, 13, 23, 31, 32, 12\}$;
   - H is multiplicative $\Leftrightarrow \alpha \in \{33, 12\}$;
   - H is partially X-Hom-associative $\Leftrightarrow \alpha \in \{33, 13, 23, 31, 32, 22\}$;
   - H is Hom-associative $\Leftrightarrow \alpha \in \{33, 22\}$;
   - H is multiplicative and Hom-associative $\Leftrightarrow \alpha = 33$;
   - H is not partially X-associative and hence not associative.

(c) Let $\nabla = 1221$. This is magma (43) in Proposition 2.5. Note that (X, ∇) is a group so that H is the group ring $K[X]$. Then:
   - H is partially X-multiplicative $\Leftrightarrow \alpha \in \{33, 13, 31, 32, 11, 12\}$;
   - H is multiplicative $\Leftrightarrow \alpha \in \{33, 11, 12\}$;
   - H is partially X-Hom-associative $\Leftrightarrow \alpha \in \{33, 13, 23, 31, 32, 12, 21\}$;
   - H is Hom-associative $\Leftrightarrow \alpha \in \{33, 12, 21\}$;
   - H is multiplicative and Hom-associative $\Leftrightarrow \alpha \in \{33, 12\}$;
   - H is associative and hence partially X-associative.

(d) Let $\nabla = 2121$. This is magma (44) in Proposition 2.5. Then:
   - H is partially X-multiplicative $\Leftrightarrow \alpha \in \{33, 13, 23, 31, 32, 12, 21\}$;
   - H is multiplicative $\Leftrightarrow \alpha \in \{33, 12, 21\}$;
   - H is partially X-Hom-associative $\Leftrightarrow \alpha \in \{33, 23, 31, 21\}$;
   - H is Hom-associative $\Leftrightarrow \alpha \in \{33, 21\}$;
   - H is multiplicative and Hom-associative $\Leftrightarrow \alpha \in \{33, 21\}$;
   - H is not partially X-associative and hence not associative.

(e) Let $\nabla = 2211$. This is magma (45) in Proposition 2.5. Then:
   - H is partially X-multiplicative $\Leftrightarrow \alpha \in \{33, 13, 23, 31, 32, 12\}$;
   - H is multiplicative $\Leftrightarrow \alpha \in \{33, 12\}$;
   - H is partially X-Hom-associative $\Leftrightarrow \alpha \in \{33, 13, 23, 31, 32, 22\}$;
   - H is Hom-associative $\Leftrightarrow \alpha \in \{33, 22\}$;
   - H is multiplicative and Hom-associative $\Leftrightarrow \alpha = 33$;
   - H is not partially X-associative and hence not associative.

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