

Sigma Journal of Engineering and Natural Sciences Web page info: https://sigma.yildiz.edu.tr DOI: 10.14744/sigma.2024.00091



Research Article

Bayesian estimation of inverse weibull distribution scale parameter under the different loss functions

Esin KÖKSAL BABACAN^{1,*}

¹Department of Statistics, Faculty of Science, Ankara University, Ankara, 06100, Türkiye

ARTICLE INFO

Article history Received: 29 November 2022 Revised: 12 March 2023 Accepted: 26 June 2023

Keywords: Bayesian Estimation; Inverse Weibull Distribution, Loss Function

ABSTRACT

In this paper, the Bayesian estimators for the Inverse Weibull Distribution (IWD) scale parameter are derived when the shape parameter of distribution is known. The Bayesian estimators for the parameter are obtained by using the Gamma prior under the different types of loss functions such as square error loss function (Self), Entropy loss function (Elf), Precautionary loss function (Plf), Linear exponential loss function (Linexlf) and nonlinear exponential loss function (Nlinexlf). A classical maximum likelihood estimator (mle) for the parameter is also derived. To compare the efficiency of the parameter estimation methods, a simulation study is carried out. The comparison is based on mean square error.

Cite this article as: Köksal Babacan E. Bayesian estimation of inverse weibull distribution scale parameter under the different loss functions. Sigma J Eng Nat Sci 2024;42(4):1108–1115.

INTRODUCTION

The Weibull distribution is frequently used in reliability engineering, especially for analyzing lifetime data. The probability density function (pdf) of the Weibull is a uni-modal or decreasing function. Also, the Weibull hazard function depends on its shape parameter. Depending on the parameter's value, the hazard function decreases or increases. If the data have a non-monotone hazard function, the Weibull distribution is not considered the appropriate model, for example, lung and breast cancer patients' mortalities [1-3]. In these circumstances, the problem is to find an appropriate distribution for the analysis of such data sets. Kundu and Howlader (2010) remarked that the IWD is an appropriate model for these data sets [4]. The IWD might be considered a suitable model when the study concludes that the pdf of the data can be unimodal [3].

Keller and Kamath (1982) used the IWD to investigate the failures of mechanical components subject to deterioration [5]. Calabria and Pulcini (1994) studied parameter estimations of IWD based on classical and Bayesian methods [6]. In the Bayesian aspect, they used informative priors. Some crucial theoretical properties of the IWD are given in [5]. Kundu and Howlader (2010) considered Bayesian inference for IWD type II censored data in their study [4]. Helu and Samawi (2015) studied progressively the first failure censoring data in their work and used Lindey's methods to derive a Bayesian estimator for IWD parameters [7]. Bi and Gui (2017) studied the stress-strength reliability of IWD [8]. Nasar and Kaser (2017) described frequentist and Bayesian estimation for the parameters of the IWD based



Published by Yıldız Technical University Press, İstanbul, Turkey Copyright 2021, Yıldız Technical University. This is an open access article under the CC BY-NC license (http://creativecommons.org/licenses/by-nc/4.0/).

^{*}Corresponding author.

^{*}E-mail address: ekoksal@science.ankara.edu.tr

This paper was recommended for publication in revised form by Editor in-Chief Ahmet Selim Dalkilic

on an adaptive type-II progressive hybrid censoring scheme [9]. To obtain Bayesian estimation, they used the Lindley approximation. Singh and Tripathi considered the parameter estimation of an IWD when it is known that samples are progressive type-I interval censored [10]. They proposed an EM algorithm to obtain maximum likelihood estimates and mid-point estimates. They obtained Bayes estimates under the square error loss function. Under the entropy loss function, Jana and Bera (2022) developed Bayes estimators of the IWD parameters [11]. They, also investigated the reliability of multi-component stress-strength model using classical and Bayesian approaches.

Suppose that *Y* is a random variable from the Weibull distribution. The pdf of *Y* is given as follows:

$$f(y; \lambda, \alpha) = \alpha \lambda y^{\alpha - 1} e^{-\lambda y^{\alpha}}, y > 0; \lambda, \alpha > 0$$

where α is a shape and λ is a scale parameter of the distribution. If we take the transformation of *Y* with X = 1/Y, then *X* has the IWD, and the pdf of *X* is derived as follows:

$$f(x) = \alpha \lambda (x^{-(\alpha+1)}) (e^{-\lambda x^{-\alpha}}), x > 0; \alpha, \lambda > 0$$
(1)

The cumulative distribution function (cdf) of *X* is written as

$$F(x) = e^{-\lambda x^{-\alpha}}, x > 0$$
(2)

The expected value and the variance of the IWD are

$$E(X) = \lambda^{\frac{1}{\alpha}} \Gamma\left(1 - \frac{1}{\alpha}\right) \tag{3}$$

and

$$Var(X) = \lambda^{\frac{2}{\alpha}} \left(\Gamma\left(1 - \frac{2}{\alpha}\right) - \left(\Gamma\left(1 - \frac{1}{\alpha}\right)\right)^{2} \right).$$
(4)

respectively, where \varGamma is the Gamma function.

The reliability function of *X*,

$$R(x) = 1 - \left(e^{-\lambda x^{-\alpha}}\right), x \ge 0, \alpha, \lambda > 0$$
(5)

and the hazard function

$$H(x) = \frac{\lambda(\alpha^{-\alpha-1})(e^{-\lambda x^{-\alpha}})}{(1-e^{-\lambda x^{-\alpha}})}, x \ge 0, \alpha, \lambda > 0$$
(6)

is given. IWD has a scale parameter (α) and a shape parameter (λ). The λ is equal to the slope of the regression model, which is obtained via the graphical method. So it's known as a slope. When the slope takes a value between zero and one, failure rates increase. If $\alpha = 1$, this distribution is called the Inverse Exponential distribution, and if $\alpha = 2$, then the Inverse Rayleigh distribution. If the λ is a constant and the α increases, the kurtosis of the distribution increases, and parallel to this, the height decreases. If the value of the scale parameter decreases, the height of the curve increases, and this distribution is sharper. Density curves for different values of λ and α are shown in Figure 1 and Figure 2, respectively.



Figure 1. IWD density curves with various values of λ when $\alpha = 2$.



Figure 2. IWD density curves with various values α when $\lambda = 2$.

The IWD might be considered an alternative to the Log-Normal and the Gamma distributions, especially in life testing and reliability engineering. This study deals with the estimation of the scale parameter of the IWD. The estimation is considered in both classical and Bayesian aspects.

The rest of the study is as follows: In Section 2, the mle of λ is derived when the α is known. In Section 3, the Bayesian estimators with the Gamma prior under the Self, Elf, Plf, Linexlf, and Nlinexlf are discussed for the λ in IWD. In Section 4, the mle and the Bayes estimators for λ are compared based on the mean squared error criteria. In Section 5, the main observations of the simulation results are given. Concluding remarks are presented at the end of the paper.

MLE for the Scale Parameter of IWD

Mle is one of the most widely used techniques of estimation in statistics. In this section, for IWD, the mle for λ is derived when α is known. Suppose that $X_1, X_2, ..., X_n$ is a random sample from the IWD when α is known. The likelihood is given as

$$L(\lambda|\alpha, x) = \alpha^n \lambda^n \prod_{i=1}^n x_i^{-(\alpha+1)} e^{-\lambda(\sum_{i=1}^n x_i^{-\alpha})}$$
(7)

then, the log-likelihood function (*l*) corresponding to (7) can be obtained as

$$l(\lambda|\alpha, x) = n(\log \alpha) + n(\log \lambda) - (\alpha + 1)(\sum_{i=1}^{n} \log(x_i)) -\lambda(\sum_{i=1}^{n} (x_i^{-\alpha}))$$
(8)

By taking the first derivative of *l* relating to the parameter λ and setting the equation to zero, the following estimating equation can be derived:

$$\frac{dl(\lambda|\alpha,x)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{-\alpha} = 0.$$
(9)

Then mle for the $\boldsymbol{\lambda}$

$$\hat{\lambda}_{mle} = \frac{n}{\sum_{i=1}^{n} x_i^{-\alpha}} \tag{10}$$

is obtained.

Bayesian Estimator for the Scale Parameter of IWD Under The Different Loss Functions

The selection of appropriate priors and appropriate loss functions are the most important issues in Bayesian inference. The symmetric square loss function is frequently used in Bayesian theory. In this loss function, positive and negative prediction errors are given equal weight. It may not be appropriate to use a symmetric loss function for a different valuation of estimation errors [6], [12], [13], [14]. This paper deals with comparing estimation results for different loss functions in IWD. Parsian and Kinmari (2002) and Misra and van der Meulen (2003) studied asymmetric Linexlf, and they estimated the Normal distribution location parameter θ [15], [16]. Nematollahi and Shariati (2009) estimated the scale parameter of the Gamma distribution under the Elf [17]. Azimi et al. (2012) studied the comparison of Bayesian estimation methods under different loss functions for a progressive censored Rayleigh distribution [18].

Suppose that X_i , i = 1, ..., n is an iid random sample from the IWD, and the shape parameter α is known. The likelihood is given as

$$L(\lambda | x, \alpha) = \prod_{i=1}^{n} f(x; \alpha, \lambda)$$
(11)

$$L = \prod_{i=1}^{n} (\alpha \lambda) (x_i^{-(\alpha+1)}) e^{-\lambda x_i^{-\alpha}} = \alpha^n \lambda^n e^{-\lambda T} \prod_{i=1}^{n} (x_i^{-(\alpha+1)})$$
(12)

where T= $\sum_{i=1}^{n} (x_i^{-\alpha})$.

In this study, to compute the Bayesian estimate of the IWD scale parameter λ , a Gamma prior and five different loss functions are used. Choosing a prior distribution for the unknown model parameter is essential in Bayesian theory. Being a natural conjugate prior, in this study, the Gamma distribution is considered a prior for the scale parameter λ in IWD with hyperparameters *a* and *b*. Then it has the following density function:

$$\pi(\lambda) = \frac{b^a \lambda^{a-1}}{\Gamma(a)} e^{-b\lambda}, \lambda > 0, a, b > 0.$$
(13)

For the $\boldsymbol{\lambda},$ the posterior distribution is computed as follows:

$$\pi \left(\lambda \left| x \right) = \frac{L(\lambda | x, \alpha) \pi(\lambda)}{\int_0^\infty L(\lambda | x, \alpha) \pi(\lambda) d\lambda}$$
(14)

$$\pi(\lambda|x) = \frac{\lambda^{n+a-1}e^{-\lambda(T+b)}}{\int_0^\infty \alpha^{n+a-1}e^{-\alpha(T+b)}d\lambda} = \frac{(T+b)^{n+a}}{\Gamma(n+a)}\lambda^{n+a-1}e^{-\lambda(T+b)}.$$
 (15)

This distribution is appropriate for the Gamma distribution with parameters (n + a) and (T + b).

Bayesian Estimator under the Self

In Bayesian estimation, a frequently used loss function is the symmetric Self. The Self is given as

$$L(\lambda, \hat{\lambda}) = \left(\lambda - \hat{\lambda}\right)^2.$$
⁽¹⁶⁾

The Self gives equal weight to both overestimation and underestimation. If the Self is used as the loss function in Bayesian inference, the estimator is the expected value of the posterior distribution. Then, for the λ , it is given as

$$\hat{\lambda}_{Self} = E(\lambda | x) \tag{17}$$

then the Bayes estimator

$$\hat{\lambda}_{Self} = \frac{(n+a)}{(T+b)} \tag{18}$$

is obtained.

Bayesian Estimator under the Elf

The Elf is a useful asymmetric loss function that was pointed out by [6]. Let $f(x, \lambda)$ represent the pdf of the random variable X, and λ is the parameter. If $\hat{\lambda}$ is an estimator for λ , then general Elf is defined by

$$L(\lambda,\hat{\lambda}) = \left(\left(\frac{\hat{\lambda}}{\lambda}\right)^{c_1} - c_1 ln \frac{\hat{\lambda}}{\lambda} - 1 \right)$$
(19)

[3]. In Elf, if $c_1 > 0$, then an overestimation error is more important than an underestimation error. If $c_1 < 0$, then an underestimation error is more important than an overestimation error. Under the general Elf, the Bayesian estimator of λ is given as

$$\hat{\lambda}_{Gelf}(X) = \left(E(\lambda^{-c_1} | X)\right)^{-c_1}$$
(20)

when $E_{\lambda}(.)$ exists and is finite. If $c_1 = -1$ then the Bayesian estimator under the Elf is the same as under the Self. If in the Elf, c_1 is taken as 1 then for the λ , the Bayesian estimator becomes

$$\hat{\lambda}_{Elf}(X) = \left(E\left(\frac{1}{\lambda} \middle| X\right)\right)^{-1}.$$
(21)

The expectation is computed as follows:

$$E\left(\frac{1}{\lambda}\left|X=x\right) = \int_0^\infty \frac{1}{\lambda} \frac{(T+b)^{n+a}}{\Gamma(n+a)} \lambda^{n+a-1} e^{-\lambda(T+b)} d\lambda = \frac{T+b}{n+a-1}$$
(22)

Then the estimator for the λ under the Elf

$$\hat{\lambda}_{Elf} (X = x) = \left(\frac{T+b}{n+a-1}\right)^{-1} = \frac{n+a-1}{T+b}$$
 (23)

is obtained.

Bayesian Estimator under the Linexlf

The Linexlf for the parameter λ is expressed as the following:

$$L(\lambda, \hat{\lambda}) = (e^{c(\hat{\lambda} - \lambda)} - 1 - c(\hat{\lambda} - \lambda))$$
(24)

where λ is an estimate of λ and $c \neq 0$ [19]. This loss function is asymmetric. Many authors have discussed the Linex loss function [15],[20], [21], [22]. Under the Linexlf, the Bayesian estimator for λ is obtained following

$$\hat{\lambda}_{Linex} = \frac{-1}{c} \log(E[e^{-c\lambda} | X])$$
(25)

[23]. For IWD, the $\hat{\lambda}_{Linexlf}$ is

$$E(e^{-c\lambda}|x) = \int_0^\infty \frac{(T+b)^{n+a}}{\Gamma(n+a)} e^{-c\lambda} \lambda^{n+a-1} e^{-\lambda(T+b)} d\lambda$$

= $\int_0^\infty e^{-\lambda(c+T+b)} \frac{(T+b)^{n+a}}{\Gamma(n+a)} \lambda^{n+a-1} d\lambda$ (26)

$$\int_{0}^{\infty} (e^{-t})(t^{n+a-1})d\lambda = \Gamma(n+a)$$

$$E(e^{-c\lambda}|x) = \frac{(T+b)^{n+a}}{(c+T+b)^{n+a-1}}$$
(28)

$$\hat{\lambda}_{Linexlf} = -\frac{1}{c} \log\left(E\left(e^{-c\lambda}|x\right)\right)$$
$$\hat{\lambda}_{Linexlf} = -\frac{1}{c} \log\left(\frac{(T+b)^{n+a}}{(c+T+b)^{n+a-1}}\right)$$
(29)

computed.

Bayesian Estimator under the Plf

The Plf is introduced in [14] as follows

$$L(\lambda, \hat{\lambda}) = \frac{(\hat{\lambda} - \lambda)^2}{\hat{\lambda}}.$$
 (30)

This function is asymmetric. By solving the equation that follows, the Bayesian estimator under the Plf is obtained as

$$\hat{\lambda}_{plf}^2 = E(\lambda^2 | X) \,. \tag{31}$$

Then

$$E(\lambda^{2}|x) = \int_{0}^{\infty} \lambda^{2} \frac{(T+b)^{n+a}}{\Gamma(n+a)} \lambda^{n+a-1} e^{-\lambda(T+b)} d\lambda$$

=
$$\int_{0}^{\infty} \lambda^{n+a+1} \frac{(T+b)^{n+a}}{\Gamma(n+a)} e^{-\lambda(T+b)} d\lambda$$
 (32)

$$E(\lambda^2 | \mathbf{x}) = \frac{\Gamma(n+a+2)}{\Gamma(n+a)(T+b)^2}$$
(33)

is computed. Finally, under the Plf, the Bayesian estimator of λ can be obtained as

$$\hat{\lambda}_{plf} = \frac{\sqrt{(n+a+1)(n+a)}}{(T+b)}.$$
(34)

Bayesian Estimator under the Nlinexlf

The Nlinex loss function is given as

$$L_{Nlinexlf}(\hat{\lambda}, \lambda) = k \left[e^{cD} - cD^2 - cD - 1 \right], k > 0, c > 0.$$
(35)

In Equation (35) $D = \hat{\lambda} - \lambda$ represents estimation error. For lack of generality, one can assume k = 1. The estimator under Nlinexlf $\hat{\lambda}_{Nlinexlf}$ is obtained as follows:

$$\hat{\lambda}_{Nlinexlf} = -\frac{\left[\ln E\left(e^{-c\lambda}\right) - 2E\left(\lambda\right)\right]}{(c+2)}$$
(36)

[24]. Then at first,

$$E\left(e^{-c\lambda}\right) = \int_0^\infty e^{-c\lambda} \pi(\lambda \mid x) d\lambda \tag{37}$$

is computed. From (15), it is known that the posterior distribution is the Gamma, then

$$E((\lambda|x)) = \frac{(n+a)}{(T+b)}$$
(38)

and

$$E(e^{-c\lambda}|x) = \frac{(T+b)^{n+a}}{(c+T+b)^{n+a-1}}.$$
(39)

Then, the Bayes estimator under Nlinexlf is as follows:

$$\hat{\lambda}_{Nlinexlf} = -\frac{\left[\ln\left(\frac{(T+b)^{n+a}}{(c+T+b)^{n+a-1}}\right) - 2(n+a)(T+b)\right]}{(c+2)}$$
(40)

SIMULATION STUDY

In a simulation study, different random samples from the IWD with the sample sizes n=10,30,50,70,90,110 are generated in Matlab. A simulation study is carried out 1000 times for the parameter values $\alpha = 2$ and $\lambda = 2,3,4$ where the prior hyperparameters are chosen as a = 2, b = 2. This is iterated 1000 times, and the λ is estimated using each of the methods given in the previous sections. The mean squared error (MSE) is used as a criterion to compare the efficiency of the methods. The MSE is calculated as follows:

$$MSE = \frac{\sum_{i=1}^{1000} (\lambda_i - \hat{\lambda}_i)^2}{1000}$$

The results are given in Table 1.

RESULTS AND DISCUSSION

The main observations of the results from Table 1 are summarized below:

- From Table 1, it can be said that the Bayes estimate using the Gamma prior under the Plf provides the smallest Mse values in most cases as compared to the other loss functions and the classical mle. Especially for small sample sizes, Bayes estimates give better results than classical mle, except for the linexlf and the Nonlinexlf.
- The Linexlf and the Nlinexlf give the worst conclusions in all cases.
- 3) The Bayesian estimates under linexlf and nonlinexlf are sensitive to the values of the corresponding shape parameter *c*.

$\begin{array}{c c c c c c c c c c c c c c c c c c c $				λ=2		λ=3		λ=4	
Mic n = 10 2.20644 0.62068 3.32654 1.53461 4.45114 2.90048 Eif 1.20709 0.30685 2.04200 0.07607 2.17539 2.17539 3.47012 Eif 1.442129 0.43547 1.84773 1.45778 2.17539 3.47012 Linedif c = 10 1.13154 1.58873 0.03683 7.43790 0.04522 1.641679 c = 5 0.66579 1.82877 0.95972 4.63050 0.97563 9.10847 Kinadi c = 10 1.02170 2.43911 0.12471 0.18770 0.832261 1.21378 7.07972 Mic n = 30 2.07868 0.15796 3.09597 0.33262 3.16607 0.94392 Self 1.84922 0.1318 2.59618 3.0562 0.66412 Linedif c = 10 3.16380 2.46902 6.31791 1.89742 7.49366 2.36038 e = 5 0.99492 1.07413 1.59993 2.07548 5.4872 3.840			$\alpha = 2$	λ	MSE	λ	MSE	λ	MSE
Self1.370900.306982.042001.076072.440372.173393.70102Pif1.441010.201292.137030.918102.516263.39000C1.313541.588730.308630.7437900.045221.64479cc0.0663741.824770.950724.247381.144158.19917cc0.0663731.7943700.085274.247381.144158.19917c0.021702.499110.1247110.185730.657622.191668c0.923931.220251.262523.091591.501226.30034c-0.818801.4276411.0919183.332411.2137887.794792Self1.849250.11382.501680.332563.273390.64412Linedifc = 103.163802.400792.631400.2383263.326620.23038c = 51.379320.107992.631400.2383463.326620.36036C = 101.13182.400482.040342.400462.30031c = 101.23120.631381.7947320.118873.44228Ninexifc = 102.349240.323842.041472.973733.24028C = 101.23120.631381.7947331.664742.148773.51711Minexifc = 102.349140.369392.073448.548723.84028Self1.232400.673381.67474 <t< th=""><th>Mle</th><th></th><th><i>n</i> = 10</th><th>2.20644</th><th>0.62068</th><th>3.32654</th><th>1.53461</th><th>4.45114</th><th>2.80048</th></t<>	Mle		<i>n</i> = 10	2.20644	0.62068	3.32654	1.53461	4.45114	2.80048
Elf	Self			1.57090	0.30698	2.04200	1.07607	2.40437	2.71816
Pif	Elf			1.42129	0.43547	1.84753	1.45778	2.17539	3.47012
Linesifc = 101.131541.588730.308637.437900.0452216.44679c = 100.668371.794570.852704.247381.144158.19977Ninexifc = 100.668371.294570.852704.247381.594226.30244c = 50.818801.427670.852704.303500.057622.191668Selfm = 302.078680.157963.095970.350624.130270.61825Self1.849250.113182.596180.322563.367620.90439Pif1.788620.129302.511060.388263.366622.30638C = 103.163802.469026.3179118.954627.498462.32608C = 50.9991210.71311.590932.075454.34625Linexifc = 101.111090.827981.551962.116801.918774.38625C = 101.234120.651131.378141.366522.420062.63801C = 101.234120.631581.734331.661742.148773.81625Self1.906120.072182.738200.215683.3498300.40755Self1.924900.072182.738200.216683.3498300.40755Linexifc = 101.234120.656131.378141.366522.305400.36835C = 101.294200.057381.679783.256412.71715Minexifc = 102.684040.378450.40674	Plf			1.64401	0.26129	2.13703	0.91810	2.51626	2.39000
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	Linexlf	<i>c</i> = -10		1.13154	1.58873	0.30863	7.43790	0.04522	16.44679
c = 10 0.66837 1.79457 0.88270 4.63050 0.07562 21.91668 Ninedf c = 5 0.92593 1.22029 1.26252 3.09159 1.50422 6.3024 Mie n = 30 0.87868 0.15796 3.09597 0.35062 4.13027 0.61825 Self 1.844252 0.11318 2.39061 0.32256 3.32739 0.75132 Elf 1.84923 0.10293 2.31166 0.38256 3.16662 0.29448 C = 5 0.99492 1.07413 1.59003 2.07454 8.54872 3.844228 c = 5 0.99492 1.07413 1.59003 2.07454 8.54872 3.48625 Ninexif c = 10 3.10302 0.65013 1.87814 3.06562 2.10437 3.5772 Mie n = 50 2.04451 0.08798 3.06739 0.20560 4.08864 0.03833 Self 1.39042 0.07218 2.75382 0.16860 3.37827 0.348423 Linexif </th <th></th> <th><i>c</i> = 5</th> <th></th> <th>0.66794</th> <th>1.82347</th> <th>0.95072</th> <th>4.24738</th> <th>1.14415</th> <th>8.19917</th>		<i>c</i> = 5		0.66794	1.82347	0.95072	4.24738	1.14415	8.19917
Ninexif c = 10 1.02170 2.43911 0.12471 10.18573 0.657c2 2.191668 c = 5 0.92593 1.22024 1.26222 3.0219 1.55422 6.30024 c = 10 0.81880 1.427641 1.050918 3.33341 1.21578 7.794792 Self 1.84925 0.11318 2.59618 0.32256 3.15607 0.51081 Elf 1.84925 0.10799 2.63840 0.22848 3.16607 0.9439 Linexif c = -10 3.16301 2.46026 6.31508 2.07454 8.4872 3.43625 Vinexif c = -10 3.49244 3.79986 7.24834 3.001407 2.07873 3.51741 Mie n = 50 2.04431 0.06513 1.87814 1.66647 2.14487 3.34864 0.36833 Elf 1.92400 0.07702 2.70978 1.52877 2.43634 2.5164 1.41111 C = 5 1.05226 0.95238 1.99778 1.52877 2.43634 <		c = 10		0.66837	1.79457	0.85270	4.63050	0.97563	9.16387
c = 5 0.92593 1.2029 1.20252 3.09159 1.59422 6.30244 e = 0 0.8180 1.42741 1.059918 3.33241 1.213758 7.794792 Mie n = 30 2.07868 0.15796 3.09597 0.35062 4.13027 0.61825 Elf 1.78862 0.12930 2.51106 0.23826 3.1607 0.90439 Plf 1.87952 0.10799 2.63840 0.23854 8.54872 3.49366 2.23208 Linexlf c = 10 1.11109 0.82798 1.56196 2.11680 1.91857 3.44228 Nlinexlf c = 5 1.23020 0.05013 1.87814 1.36662 2.43064 2.36001 c = 5 1.23020 0.05013 1.87814 1.36626 2.4406 2.36001 C = 6 1.23412 0.037978 3.001407 2.07833 3.79721 Self 1.90612 0.07218 2.73382 0.18680 3.43948 1.31714 Self 1.99240 <th>Nlinexlf</th> <th>c = -10</th> <th></th> <th>1.02170</th> <th>2.43911</th> <th>0.12471</th> <th>10.18573</th> <th>0.65762</th> <th>21.91668</th>	Nlinexlf	c = -10		1.02170	2.43911	0.12471	10.18573	0.65762	21.91668
c = 10 0.81880 1.427641 1.090918 3.832341 1.213758 7.794922 Self 1.84925 0.11318 2.59618 0.32256 3.23739 0.51312 Elf 1.78862 0.12390 2.51106 0.38826 3.16607 0.99439 Plf 1.87932 0.10799 2.63840 0.29548 3.3662 0.68412 Linedf c = 10 3.16300 2.46020 6.31711 18.95462 7.49366 2.326038 Ninedf c = 10 3.49244 3.79986 7.24834 3.001407 2.07873 3.79572 c = 5 1.23902 0.65013 1.74333 1.66474 2.14437 3.8462 Self 1.90612 0.07718 2.75334 1.69647 2.4484 3.09986 0.49884 0.498568 Elf 1.90612 0.0726 2.78095 0.176663 3.57482 0.38086 Linextf e = 10 2.69097 0.63340 4.260984 0.4049568 Plf 1.92340 <		<i>c</i> = 5		0.92593	1.22029	1.26252	3.09159	1.50422	6.30024
Mie n = 30 2.07868 0.157%0 3.09597 0.33062 4.13027 0.051825 Eif 1.78862 0.12930 2.51166 0.32826 3.16077 0.90439 Pif 1.78862 0.12930 2.51166 0.38826 3.16077 0.90439 Linexif c = 10 3.16380 2.46902 6.31791 118.95462 7.49366 2.322038 c = 5 0.99492 1.07413 1.59093 2.07344 8.54872 3.379572 c = 10 1.11109 0.82798 1.56196 2.11680 1.91857 4.38625 Ninexif c = 10 1.23412 0.65113 1.87814 1.36562 2.42006 2.63001 c = 5 1.23902 0.07218 2.73582 0.18680 3.51741 Mie n = 50 2.04412 0.097344 2.639994 0.211681 3.499854 0.469568 Pif 1.96412 0.07262 2.78050 1.36633 0.55848 1.14111 c = 5 1.0		<i>c</i> = 10		0.81880	1.427641	1.050918	3.832341	1.213758	7.794792
Set $1.18492.5$ 0.11318 2.39018 0.34256 3.74399 0.75132 Pif 1.87932 0.10799 2.63840 0.29548 3.36667 0.90439 Pif $c = 10$ 3.16380 2.4692 6.31791 18.95462 7.49356 23.26038 c = 5 0.99492 1.07413 1.59093 2.07454 8.54872 38.44228 c = 5 0.99492 1.07413 1.59093 2.07454 8.54872 38.44228 c = 10 1.1109 0.82798 1.56196 2.11680 1.91857 4.38625 c = 5 1.23902 0.65013 1.87814 1.36562 2.42006 2.63001 c = 10 1.23412 0.63158 1.73433 1.66474 2.14437 3.51741 Mie $n = 50$ 2.04451 0.08798 3.06739 0.20508 4.088864 0.66833 Self 1.90612 0.07218 2.7382 0.18680 3.35995 0.40785 Elf 1.90612 0.07218 2.7382 0.18680 3.35995 0.40785 Elf 1.90612 0.07218 2.7382 0.18680 3.35995 0.40785 Elf $c = -10$ 2.69097 0.63340 4.26068 2.30543 4.265848 11.14111 c = 5 1.05226 0.95238 1.79978 1.52877 2.4364 2.05848 1.0469568 Pif 1.25260 0.52388 1.85760 1.36063 2.35641 2.77175 Ninexif $c = -10$ 2.869178 0.93784 4.63739 3.62416 7.31312 17.64494 c = 5 1.05226 0.95238 1.79787 1.52877 2.44564 2.56540 c = 10 1.25340 0.059238 1.79787 1.52877 2.44564 1.269494 c = 5 1.05226 0.95238 1.17978 1.52877 2.44564 1.269494 c = 5 1.05226 0.95765 3.04745 0.1407 4.04641 0.24060 Self 1.99695 c = 10 1.36218 0.44724 2.00697 1.05143 2.25366 1.169695 c = 10 1.36218 0.44724 2.00697 1.05143 2.55366 1.169695 c = 10 1.36218 0.44724 2.00697 1.05143 2.55366 2.30511 Linexif $c = -10$ 2.58868 0.40574 3.83915 1.0675 3.54611 0.24744 Self 1.92921 0.05087 2.82030 0.13190 3.56433 0.31314 Pif 1.94284 0.04978 0.12657 0.12657 3.54010 2.673484 c = 5 1.06035 0.92384 1.88711 1.31728 2.60159 2.06333 c = 10 1.31867 0.49517 2.01267 1.02877 2.59702 2.03842 c = 5 1.30860 0.52177 2.15374 0.80662 2.30059 1.32891 c = 10 1.31867 0.49517 2.10267 1.02877 2.59702 2.03842 c = 5 1.30360 0.92377 1.216374 0.8663 0.09373 3.7318 0.23114 c = 5 1.30360 0.92377 1.216374 0.27649 1.53452 c = 10 1.31867 0.49517 2.10215 0.8470 2.76497 1.58458 c = 5 1.30360 0.92377 2.31932 0.10984 3.260159 2.05633 c = 10 1.45691 0.32548 2.22693 0.40573 3.77417 1.7598	Mle		n=30	2.07868	0.15796	3.09597	0.35062	4.13027	0.61825
Lin Linexif c = -10 Lin Lin Linexif c = -10 Lin Lin Lin Lin Lin Lin Lin Lin	Self			1.84925	0.11318	2.59618	0.32256	3.27339	0.75132
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	Elf			1.78862	0.12930	2.51106	0.38826	3.16607	0.90439
Linexif c = 10 3.16380 2.46302 6.31791 18.59462 7.4936b 2.32003 c = 10 1.11109 0.82798 1.56196 2.11680 1.91857 4.38625 Ninexif c = -10 3.49244 3.79986 7.24834 3.01407 2.07873 3.79572 Mile n = 50 2.04451 0.08798 3.06739 0.20508 4.08864 0.36833 Self 1.80638 0.07214 2.75382 0.18680 3.35995 0.40785 Elf 1.80838 0.078214 2.609244 0.211881 3.469856 0.469568 Plf 1.82490 0.07026 2.78095 0.17668 3.57482 0.38086 Linexif c = 10 2.5340 0.93288 1.85876 1.36063 2.3564 2.77175 Ninexif c = 10 2.88718 0.97274 4.63739 3.62416 7.31312 17.6494 c = 5 1.20622 0.55002 2.07236 0.95889 2.75166 1.69695 <		10		1.8/932	0.10/99	2.63840	0.29548	3.32662	0.68412
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	Linexif	c = -10		3.16380	2.46902	6.31/91	18.95462	7.49366	23.26038
linexif c = 10 1.1119 0.32/39 1.36139 2.11680 1.91837 4.30823 sinexif c = 5 1.23902 0.65013 1.87814 1.36562 2.42006 2.63001 c = 5 1.23902 0.65013 1.87814 1.36562 2.42006 2.63001 Mic n = 50 2.04431 0.063158 1.73333 1.66474 2.14473 3.51741 Mic n = 50 2.04431 0.079218 2.75382 0.18680 3.53995 0.40785 Elf 1.96438 0.077226 2.78095 0.17668 3.57482 0.38086 Linexif c = 5 1.05226 0.95338 1.79978 1.52877 2.43634 2.56440 c = 5 1.05226 0.95388 1.85760 1.36632 2.75166 1.69695 c = 10 1.236218 0.47274 0.46733 3.62416 7.31312 17.64494 c = 5 1.23622 0.55202 2.07236 0.95883 2.55364 1.79809		c = 5		0.99492	1.0/413	1.59093	2.0/454	8.54872	38.44228
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	Nilia and	c = 10		1.11109	0.82/98	1.56196	2.11680	1.91857	4.38625
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	Nimexii	c = -10		3.49244	5./9980 0.65013	1.24634	30.01407	2.07873	5./95/2 2.62001
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		c = 5		1.23902	0.63015	1.8/814	1.30302	2.42006	2.03001
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	Mle	$\iota = 10$	n - 50	2 04451	0.03138	3.06739	0.20508	4.08864	0.36833
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	Self		n = 50	1 90612	0.03798	2 75382	0.20500	3 53995	0.30833
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	Flf			1.96838	0.07210	2.79902	0.211681	3 469854	0.469568
In1.12100.030202.100582.304300.510200.50300Linexlf $c = 10$ 2.690970.633404.200882.305430.558481.114111 $c = 5$ 1.052260.952381.799781.528772.436342.55340Nlinexlf $c = -10$ 2.887180.972744.637393.624167.3131217.64494 $c = 5$ 1.296220.552022.072360.958892.751661.69695 $c = 10$ 1.362180.447242.006971.051432.553662.17880Mle $n = 70$ 2.027970.057763.047450.140474.046410.24060Self1.901840.054202.780300.145073.596330.31314Plf1.942840.049780.126540.126543.673850.26311Linexlf $c = -10$ 2.358680.426773.893151.067525.410102.67484 $c = 5$ 1.060350.925841.887111.317282.601592.06323 $c = 10$ 1.240430.521772.153740.800622.900591.32891 $c = 5$ 1.306600.521772.153740.800622.900591.32891 $c = 10$ 1.420430.364773.247720.101935.494500.18797Self 1.99620 0.043572.147270.778732.772201.58935Mle $n = 90$ 2.025670.946733.741790.710435.093931.57450 $c = 10$ 1.358	Plf			1.00050	0.070214	2.099294	0.17668	3 57482	0.38086
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	Linexlf	c = -10		2 69097	0.63340	4 26068	2 30543	6 55848	11 14111
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	Linean	c = 5		1.05226	0.95238	1.79978	1.52877	2.43634	2.56340
$\begin{split} & \text{Nlinexlf} \begin{array}{c} \mathbf{c} = -10 & 2.88718 & 0.97274 & 4.63739 & 3.62416 & 7.31312 & 17.64494 \\ \mathbf{c} = 5 & 1.29622 & 0.55202 & 2.07236 & 0.95889 & 2.75166 & 1.69695 \\ \hline \mathbf{c} = 10 & 1.36218 & 0.44724 & 2.00697 & 1.05143 & 2.55366 & 2.17880 \\ \hline \text{Mle} & \mathbf{n} = 70 & 2.02797 & 0.05776 & 3.04745 & 0.14047 & 4.04641 & 0.24060 \\ \hline \text{Self} & 1.90184 & 0.05420 & 2.78030 & 0.14507 & 3.59633 & 0.31314 \\ \hline \text{Plf} & 1.90184 & 0.05420 & 2.78030 & 0.14057 & 3.59633 & 0.31314 \\ \hline \text{Plf} & 1.94284 & 0.04978 & 0.12654 & 0.12654 & 3.67385 & 0.26311 \\ \hline \text{Linexlf} \mathbf{c} = -10 & 2.58868 & 0.42674 & 3.89315 & 1.06752 & 5.41010 & 2.67484 \\ \hline \mathbf{c} = 10 & 1.31867 & 0.49517 & 2.01267 & 1.02877 & 2.59702 & 2.03842 \\ \hline \text{Nlinexlf} \mathbf{c} = -10 & 2.75354 & 0.65812 & 4.16136 & 1.67429 & 5.85061 & 4.30444 \\ \hline \mathbf{c} = 10 & 1.42043 & 0.36915 & 2.14727 & 0.78773 & 2.77220 & 1.58935 \\ \hline \text{Mle} & \mathbf{n} = 90 & 2.02507 & 0.04677 & 3.02781 & 0.00255 & 4.04501 & 0.18797 \\ \hline \text{Self} & 1.942620 & 0.04357 & 2.81932 & 0.10984 & 3.68762 & 0.22695 \\ \hline \text{Plf} & 1.95846 & 0.04114 & 2.86653 & 0.09762 & 3.74936 & 0.19656 \\ \hline \text{Linexlf} \mathbf{c} = -10 & 2.55689 & 0.36793 & 3.7479 & 0.71043 & 5.09393 & 1.57450 \\ \hline \mathbf{c} = 5 & 1.05983 & 0.92237 & 1.92404 & 1.22483 & 2.71174 & 1.75989 \\ \hline \mathbf{c} = 10 & 1.45691 & 0.32548 & 2.2693 & 0.0762 & 3.74936 & 0.19656 \\ \hline \text{Linexlf} \mathbf{c} = -10 & 2.70919 & 0.56603 & 3.96453 & 1.1153 & 5.4521 & 2.51891 \\ \hline \mathbf{c} = 10 & 1.45691 & 0.32548 & 2.22693 & 0.05075 & 2.92561 & 21.03268 \\ \hline \text{Mie} & \mathbf{n} = 110 & 2.0365 & 0.037582 & 3.02873 & 0.08849 & 4.04084 & 0.15622 \\ \hline \text{Self} & 1.95074 & 0.03501 & 2.88234 & 0.08498 & 3.77911 & 0.16572 \\ \hline \text{Elf} & 1.95074 & 0.03501 & 2.88234 & 0.08498 & 3.77911 & 0.16572 \\ \hline \text{Self} & 1.95075 & 0.03457 & 2.8953 & 0.08847 & 2.76497 & 1.59458 \\ \hline \text{Miexlf} & \mathbf{c} = -10 & 1.45691 & 0.037582 & 3.02873 & 0.08849 & 4.04084 & 0.15622 \\ \hline \text{Self} & 1.95075 & 0.03457 & 0.58632 & 4.93338 & 1.13273 \\ \hline \mathbf{c} = 5 & 1.04366 & 0.97781 & 1.93373 & 0.15882 & 2.77927 & 1.58$		c = 10		1.25340	0.59388	1.85760	1.36063	2.35641	2.77175
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	Nlinexlf	c = -10		2.88718	0.97274	4.63739	3.62416	7.31312	17.64494
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		c = 5		1.29622	0.55202	2.07236	0.95889	2.75166	1.69695
Mle n = 70 2.02797 0.05776 3.04745 0.14047 4.04641 0.24060 Self 1.92921 0.05087 2.82030 0.13190 3.64807 0.27840 Elf 1.94284 0.04978 0.12654 0.14507 3.59633 0.31314 Plf 1.94284 0.04978 0.12654 0.12654 3.67385 0.26311 Linexlf c = 10 2.58868 0.42674 3.89315 1.06752 5.41010 2.67844 c = 5 1.06035 0.92584 1.88711 1.31728 2.60159 2.06323 c = 10 1.31867 0.49517 2.01267 1.02877 2.59702 2.03842 Nlinexlf c = 5 1.30860 0.52177 2.15374 0.80062 2.90059 1.32891 c = 10 1.42043 0.36915 2.14727 0.78773 2.77220 1.58935 Mle n = 90 2.02507 0.04677 3.02781 0.10255 4.04501 0.18797 Self		c = 10		1.36218	0.44724	2.00697	1.05143	2.55366	2.17880
Self 1.92921 0.05087 2.82030 0.13190 3.64807 0.27840 Elf 1.90184 0.05420 2.78030 0.14507 3.59633 0.31314 Plf 1.94284 0.04978 0.12654 0.12654 3.67385 0.26311 Linexlf <i>c</i> = -10 2.58868 0.42674 3.89315 1.06752 5.41010 2.67484 <i>c</i> = 5 1.06035 0.92584 1.88711 1.31728 2.60159 2.06323 <i>c</i> = 10 1.31867 0.49517 2.01267 1.02877 2.59702 2.03842 Nlinexlf <i>c</i> = -10 2.75354 0.65812 4.16136 1.67429 5.85061 4.30444 <i>c</i> = 5 1.30860 0.52177 2.15374 0.80062 2.90059 1.32891 <i>c</i> = 10 1.42043 0.36915 2.14727 0.78773 2.77220 1.58935 Mle <i>n</i> = 90 2.05507 0.04677 3.02781 0.10255 4.04501 0.18797 Self 1.92620 0.04357 2.81932 0.10199 3.78479 0.71043	Mle		n = 70	2.02797	0.05776	3.04745	0.14047	4.04641	0.24060
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	Self			1.92921	0.05087	2.82030	0.13190	3.64807	0.27840
Plf 1.94284 0.04978 0.12654 0.12654 3.67385 0.26311 Linexlf c = 10 2.58868 0.42674 3.89315 1.06752 5.41010 2.67484 c = 5 1.06035 0.92584 1.88711 1.31728 2.60159 2.06323 c = 10 1.31867 0.49517 2.01267 1.02877 2.59702 2.03842 Nlinexlf c = -10 2.75354 0.65812 4.16136 1.67429 5.85061 4.30444 c = 5 1.30860 0.52177 2.15374 0.80062 2.90059 1.32891 Mle n = 90 2.02507 0.04677 3.02781 0.10255 4.04501 0.18797 Self 1.94772 0.04171 2.85082 0.1019 3.72882 0.22695 Plf 1.95264 0.36793 3.74179 0.71043 5.09393 1.57450 Linexlf c = -10 2.55689 0.36793 3.74179 0.71043 5.09393 1.57450 Linexlf c = 5 1.05983 0.92237 1.92404 1.22483 2.71174<	Elf			1.90184	0.05420	2.78030	0.14507	3.59633	0.31314
Linexlf c = -10 2.58868 0.42674 3.89315 1.06752 5.41010 2.67484 c = 5 1.06035 0.92584 1.88711 1.31728 2.60159 2.06323 c = 10 1.31867 0.49517 2.01267 1.02877 2.59702 2.03842 Nlinexlf c = -10 2.75354 0.65812 4.16136 1.67429 5.85061 4.30444 c = 5 1.30860 0.52177 2.15374 0.80062 2.90059 1.32891 c = 10 1.42043 0.36915 2.14727 0.78773 2.77220 1.58935 Mle n = 90 2.02507 0.04677 3.02781 0.10255 4.04501 0.18797 Self 1.92620 0.04357 2.81932 0.10984 3.68762 0.22695 Plf 1.95846 0.04114 2.86653 0.09762 3.74936 0.19656 Linexlf c = 10 1.35875 0.44017 2.10215 0.85470 2.76497 1.59458	Plf			1.94284	0.04978	0.12654	0.12654	3.67385	0.26311
c = 5 1.06035 0.92584 1.88711 1.31728 2.60159 2.06323 Nlinexlf c = 10 1.31867 0.49517 2.01267 1.02877 2.59702 2.03842 Nlinexlf c = -10 2.75354 0.65812 4.16136 1.67429 5.85061 4.30444 c = 5 1.30860 0.52177 2.15374 0.80062 2.90059 1.32891 c = 10 1.42043 0.36915 2.14727 0.78773 2.77220 1.58935 Mle n = 90 2.02507 0.04677 3.02781 0.10255 4.04501 0.18797 Self 1.94772 0.04171 2.85082 0.10119 3.72882 0.22695 Plf 1.92620 0.04357 3.74179 0.71043 5.09393 1.57450 Linexlf c = -10 2.55689 0.36793 3.74179 0.71043 5.09393 1.57450 Linexlf c = -10 1.35875 0.44017 2.10215 0.85470 2.76497 1.59458	Linexlf	<i>c</i> = -10		2.58868	0.42674	3.89315	1.06752	5.41010	2.67484
c = 10 1.31867 0.49517 2.01267 1.02877 2.59702 2.03842 Nlinexlf c = -10 2.75354 0.65812 4.16136 1.67429 5.85061 4.30444 c = 5 1.30860 0.52177 2.15374 0.80062 2.90059 1.32891 c = 10 1.42043 0.36915 2.14727 0.78773 2.77220 1.58935 Mle n = 90 2.02507 0.04677 3.02781 0.10255 4.04501 0.18797 Self 1.94772 0.04171 2.85082 0.10119 3.72882 0.20581 Elf 1.92620 0.04357 2.81932 0.10984 3.68762 0.22695 Plf 1.95846 0.04114 2.86653 0.09762 3.74936 0.19656 Linexlf c = -10 2.55689 0.36793 3.74179 0.71043 5.09393 1.57450 c = 5 1.05983 0.92237 1.92404 1.22483 2.71174 1.75989 c = 10 2.70		<i>c</i> = 5		1.06035	0.92584	1.88711	1.31728	2.60159	2.06323
NlinexIf c = -10 2.75354 0.65812 4.16136 1.67429 5.85061 4.30444 c = 5 1.30860 0.52177 2.15374 0.80062 2.90059 1.32891 c = 10 1.42043 0.36915 2.14727 0.78773 2.77220 1.58935 Mle n = 90 2.02507 0.04677 3.02781 0.10255 4.04501 0.18797 Self 1.94772 0.04171 2.85082 0.1019 3.72882 0.20581 Elf 1.92620 0.04357 2.81932 0.10984 3.68762 0.22695 Plf 1.95846 0.04114 2.86653 0.09762 3.74936 0.19656 LinexIf c = -10 2.55689 0.36793 3.74179 0.71043 5.09393 1.57450 c = 5 1.05983 0.92237 1.92404 1.22483 2.71174 1.75989 c = 6 1.35875 0.44017 2.10215 0.85470 2.76497 1.59458 NlinexIf c =		c = 10		1.31867	0.49517	2.01267	1.02877	2.59702	2.03842
c = 5 1.30860 0.52177 2.15374 0.80062 2.90059 1.32891 c = 10 1.42043 0.36915 2.14727 0.78773 2.77220 1.58935 Mle n = 90 2.02507 0.04677 3.02781 0.10255 4.04501 0.18797 Self 1.92620 0.04357 2.81932 0.10984 3.68762 0.22695 Plf 1.95846 0.04114 2.86653 0.09762 3.74936 0.19656 Linexlf c = -10 2.55689 0.36793 3.74179 0.71043 5.09393 1.57450 c = 5 1.05983 0.92237 1.92404 1.22483 2.71174 1.75989 c = 10 1.35875 0.44017 2.10215 0.85470 2.76497 1.59458 Nlinexlf c = 5 1.31351 0.50986 2.18883 0.72840 3.0023 1.10430 c = 5 1.31351 0.50986 2.18883 0.72840 3.0023 1.10430 c = 5 1.31350	Nlinexlf	<i>c</i> = -10		2.75354	0.65812	4.16136	1.67429	5.85061	4.30444
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		<i>c</i> = 5		1.30860	0.52177	2.15374	0.80062	2.90059	1.32891
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		<i>c</i> = 10		1.42043	0.36915	2.14727	0.78773	2.77220	1.58935
Self $1,94772$ 0.04171 2.85082 0.10119 3.72882 0.20581 Elf 1.92620 0.04357 2.81932 0.10984 3.68762 0.22695 Plf 1.95846 0.04114 2.86653 0.09762 3.74936 0.19656 Linexlf $c = -10$ 2.55689 0.36793 3.74179 0.71043 5.09393 1.57450 $c = 5$ 1.05983 0.92237 1.92404 1.22483 2.71174 1.75989 $c = 10$ 1.35875 0.44017 2.10215 0.85470 2.76497 1.59458 Nlinexlf $c = -10$ 2.70919 0.56603 3.96453 1.11530 5.43521 2.51891 $c = 5$ 1.31351 0.50986 2.18883 0.72840 3.0023 1.10430 $c = 10$ 1.45691 0.32548 2.22693 0.65075 2.92561 21.23268 Mle $n = 110$ 2.01365 0.037582 3.02873 0.08849 4.04084 0.15622 Self 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Elf 1.99309 0.03647 0.09052 0.09052 3.74491 0.17989 Plf 1.95955 0.03452 2.89535 0.08273 3.79617 0.15953 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 </th <th>Mle</th> <th></th> <th>n = 90</th> <th>2.02507</th> <th>0.04677</th> <th>3.02781</th> <th>0.10255</th> <th>4.04501</th> <th>0.18797</th>	Mle		n = 90	2.02507	0.04677	3.02781	0.10255	4.04501	0.18797
Elf 1.92620 0.04357 2.81932 0.10984 3.68762 0.22695 Plf 1.95846 0.04114 2.86653 0.09762 3.74936 0.19656 Linexlf c = -10 2.55689 0.36793 3.74179 0.71043 5.09393 1.57450 c = 5 1.05983 0.92237 1.92404 1.22483 2.71174 1.75989 c = 10 1.35875 0.44017 2.10215 0.85470 2.76497 1.59458 Nlinexlf c = -10 2.70919 0.56603 3.96453 1.11530 5.43521 2.51891 c = 5 1.31351 0.50986 2.18883 0.72840 3.0023 1.10430 c = 10 1.45691 0.32548 2.22693 0.65075 2.92561 21.23268 Mle n = 110 2.01365 0.037582 3.02873 0.08849 4.04084 0.15622 Self 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Elf 1.93309 0.03647 0.09052 0.09052 3.74491 0.17989	Self			1.94772	0.04171	2.85082	0.10119	3.72882	0.20581
Pir1.95846 0.04114 2.86653 $0.09/62$ 3.74936 0.19656 Linexlf $c = -10$ 2.55689 0.36793 3.74179 0.71043 5.09393 1.57450 $c = 5$ 1.05983 0.92237 1.92404 1.22483 2.71174 1.75989 $c = 10$ 1.35875 0.44017 2.10215 0.85470 2.76497 1.59458 Nlinexlf $c = -10$ 2.70919 0.56603 3.96453 1.11530 5.43521 2.51891 $c = 5$ 1.31351 0.50986 2.18883 0.72840 3.0023 1.10430 $c = 10$ 1.45691 0.32548 2.22693 0.65075 2.92561 21.23268 Mle $n = 110$ 2.01365 0.037582 3.02873 0.08849 4.04084 0.15622 Self 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Elf 1.95955 0.03452 2.89535 0.08273 3.79617 0.15953 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.1582 2.77927 1.58430 $c = 10$ 1.37505 0.41641 2.17095 0.73494 2.88021 1.32190 Ninexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.0	Elf			1.92620	0.04357	2.81932	0.10984	3.68762	0.22695
Linexif $c = -10$ 2.556890.367935.741790.710435.095931.57450 $c = 5$ 1.059830.922371.924041.224832.711741.75989 $c = 10$ 1.358750.440172.102150.854702.764971.59458Nlinexlf $c = -10$ 2.709190.566033.964531.115305.435212.51891 $c = 5$ 1.313510.509862.188830.728403.00231.10430 $c = 10$ 1.456910.325482.226930.650752.9256121.23268Mle $n = 110$ 2.013650.0375823.028730.088494.040840.15622Self1.950740.035012.882340.084983.779110.16572Elf1.953090.036470.090520.090523.744910.17989Plf1.959550.034522.895350.082733.796170.15953Linexlf $c = -10$ 2.533180.327993.680570.586324.933381.13273 $c = 5$ 1.043660.947871.953371.158822.779271.58430Nlinexlf $c = -10$ 2.678780.507513.880130.913015.221941.79988 $c = 5$ 1.302830.519132.218790.675843.064940.97472 $c = 10$ 1.471000.306762.289510.556003.030031.01606		- 10		1.95846	0.04114	2.86653	0.09/62	3./4936	0.19656
c = 51.05983 0.92237 1.92404 1.22483 2.71174 1.73989 $c = 10$ 1.35875 0.44017 2.10215 0.85470 2.76497 1.59458 Nlinexlf $c = -10$ 2.70919 0.56603 3.96453 1.11530 5.43521 2.51891 $c = 5$ 1.31351 0.50986 2.18883 0.72840 3.0023 1.10430 $c = 10$ 1.45691 0.32548 2.22693 0.65075 2.92561 21.23268 Mle $n = 110$ 2.01365 0.037582 3.02873 0.08849 4.04084 0.15622 Self 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Elf 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Plf 1.95955 0.03452 2.89535 0.08273 3.79617 0.15953 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 Nlinexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606	Linexii	c = -10		2.55689	0.36/93	3./41/9	0./1045	5.09393	1.5/450
c = 101.33873 0.44017 2.10213 0.03470 2.70497 1.39438 Nlinexlf $c = -10$ 2.70919 0.56603 3.96453 1.11530 5.43521 2.51891 $c = 5$ 1.31351 0.50986 2.18883 0.72840 3.0023 1.10430 $c = 10$ 1.45691 0.32548 2.22693 0.65075 2.92561 21.23268 Mle $n = 110$ 2.01365 0.037582 3.02873 0.08849 4.04084 0.15622 Self 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Elf 1.95955 0.03452 2.89535 0.08273 3.79617 0.15953 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 Nlinexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606		c = 5		1.05985	0.92237	1.92404	1.22485	2.71174	1./3989
Nimexit $c = 10$ 2.70919 0.36003 3.90433 1.11330 3.43321 2.31891 $c = 5$ 1.31351 0.50986 2.18883 0.72840 3.0023 1.10430 $c = 10$ 1.45691 0.32548 2.22693 0.65075 2.92561 21.23268 Mle $n = 110$ 2.01365 0.037582 3.02873 0.08849 4.04084 0.15622 Self 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Elf 1.93309 0.03647 0.09052 0.09052 3.74491 0.17989 Plf 1.95955 0.03452 2.89535 0.08273 3.79617 0.15953 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 Nlinexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606	Minovlf	c = 10		1.338/3	0.44017	2.10215	0.854/0	2./049/	1.39438
c = 3 1.31331 0.30360 2.18863 0.72640 5.0025 1.10430 $c = 10$ 1.45691 0.32548 2.22693 0.65075 2.92561 21.23268 Mle $n = 110$ 2.01365 0.037582 3.02873 0.08849 4.04084 0.15622 Self 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Elf 1.93309 0.03647 0.09052 0.09052 3.74491 0.17989 Plf 1.95955 0.03452 2.89535 0.08273 3.79617 0.15953 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 Nlinexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606	NIIIEXII	c = -10		2.70919	0.50005	2 18883	0.72840	3.0023	1 10/30
Mle $n = 110$ 2.01365 0.32346 2.22035 0.05073 2.52501 21.23208 Mle $n = 110$ 2.01365 0.037582 3.02873 0.08849 4.04084 0.15622 Self 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Elf 1.93309 0.03647 0.09052 0.09052 3.74491 0.17989 Plf 1.95955 0.03452 2.89535 0.08273 3.79617 0.15953 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 Nlinexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606		c = 3		1.51551	0.30548	2.10003	0.72840	2 92561	21 23268
Kit $i = 110$ 2.01503 0.057302 5.02075 0.06017 1.01011 0.15022 Self 1.95074 0.03501 2.88234 0.08498 3.77911 0.16572 Elf 1.93309 0.03647 0.09052 0.09052 3.74491 0.17989 Plf 1.95955 0.03452 2.89535 0.08273 3.79617 0.15953 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 Nlinexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606	Mle	t = 10	n = 110	2 01365	0.037582	3.02873	0.03073	4 04084	0.15622
Elf 1.93309 0.03647 0.09052 0.09052 3.74491 0.17989 Plf 1.95955 0.03452 2.89535 0.08273 3.79617 0.15953 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 Nlinexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606	Self		<i>n</i> = 110	1 95074	0.03501	2 88234	0.08498	3 77911	0.16572
Plf 1.95955 0.03452 2.89535 0.08273 3.79617 0.179573 Linexlf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 Nlinexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606	Elf			1,93309	0.03647	0.09052	0.09052	3.74491	0.17989
LinexIf $c = -10$ 2.53318 0.32799 3.68057 0.58632 4.93338 1.13273 $c = 5$ 1.04366 0.94787 1.95337 1.15882 2.77927 1.58430 $c = 10$ 1.37505 0.41641 2.17095 0.73494 2.88021 1.32190 NlinexIf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606	Plf			1 95955	0.03452	2.89535	0.08273	3 79617	0 15953
c = 51.043660.947871.953371.158822.779271.58430 $c = 10$ 1.375050.416412.170950.734942.880211.32190Nlinexlf $c = -10$ 2.678780.507513.880130.913015.221941.79988 $c = 5$ 1.302830.519132.218790.675843.064940.97472 $c = 10$ 1.471000.306762.289510.556003.030031.01606	Linexlf	c = -10		2.53318	0.32799	3.68057	0.58632	4,93338	1.13273
c = 101.375050.416412.170950.734942.880211.32190Nlinexlf $c = -10$ 2.678780.507513.880130.913015.221941.79988 $c = 5$ 1.302830.519132.218790.675843.064940.97472 $c = 10$ 1.471000.306762.289510.556003.030031.01606	21110/11	c = 5		1.04366	0.94787	1.95337	1.15882	2.77927	1.58430
Nlinexlf $c = -10$ 2.67878 0.50751 3.88013 0.91301 5.22194 1.79988 $c = 5$ 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606		c = 0		1.37505	0.41641	2.17095	0.73494	2.88021	1.32190
c = 5 1.30283 0.51913 2.21879 0.67584 3.06494 0.97472 $c = 10$ 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606	Nlinexlf	c = -10		2.67878	0.50751	3.88013	0.91301	5.22194	1.79988
c = 10 1.47100 0.30676 2.28951 0.55600 3.03003 1.01606		c = 5		1.30283	0.51913	2,21879	0.67584	3.06494	0.97472
		<i>c</i> = 10		1.47100	0.30676	2.28951	0.55600	3.03003	1.01606

Table 1. Estimation results for the scale parameter $\boldsymbol{\lambda}$

- 4) The results show that estimators of the different methods are closer to each other as the sample size increases, except for Linexlf and Nlinexlf.
- 5) Moreover, it is seen that when the sample size increases, the MSE decreases significantly.

CONCLUSION

This study deals with the investigation of the mle and the Bayesian estimator for the scale parameter of IWD when the shape parameter is known. For Bayesian inference, five different loss functions are used respectively: Self, Plf, Elf, Linexlf, and Nlinexlf. In the simulation study, the Bayesian estimates and the mle are computed. To compare the results of the estimations, the MSE values are calculated. The results show that the Bayesian method of estimation for the Gamma prior under the Self, Plf, and Elf gives better results than the mle method. Also, the Bayesian estimators that are obtained under the Plf have the smallest MSE as compared with the Bayesian estimators that are obtained under the other loss functions. Also, Bayesian estimators, which are obtained under Linexlf and Nlinexlf, are worse than the other loss functions and the mle.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

- Langlands AO, Pocock SJ, Kerr GR, Gore SM. Long-term survival of patients with breast cancer: A study of the curability of the disease. Br Med J 1979;2:1247–1251. [CrossRef]
- [2] Bennett S. Log-logistic regression models for survival data. J R Stat Soc 1983;32:165–171. [CrossRef]
- [3] Kumar SS, Umesh S, Kumar D. Bayesian estimation of parameters of inverse Weibull distribution. J Appl Stat 2013;40:1597–1607. [CrossRef]

- [4] Kundu D, Howlader H. Bayesian inference and prediction of the inverse weibull distribution for type-II censored data. Comput Stat Data Anal 2010;54:1547–1558. [CrossRef]
- [5] Keller AZ, Kamath ARR. Alternative reliability models for mechanical systems. In proceeding of the 3rd International Conference on Reliability and Maintainability; 1982. pp. 411–415.
- [6] Calabria R, Pulcini G. Bayes 2-sample prediction for the inverse Weibull distribution. Commun Stat Theory Methods 1994;23:1811–1824. [CrossRef]
- [7] Helu A, Samawi H. The inverse weibull distribution as a failure model under various loss functions and based on progressive first-failure censored data. Quality Technol Quant Manage 2015;12:517–535. [CrossRef]
- [8] Bi Q, Gui W. Bayesian and classical estimation of stress-strength reliability for inverse weibull lifetime models. Algorithms 2017;10:71. [CrossRef]
- [9] Nassar M, Abo-Kasem EE. Estimation of the inverse Weibull parameters under adaptive type-II progressive hybrid censoring scheme. J Comput Appl Math 2017;315:228–239. [CrossRef]
- [10] Singh S, Tripathi YM. Estimating the parameters of an inverse Weibull distribution under progressive type-I interval censoring. Stat Papers 2018;59:21–56. [CrossRef]
- [11] Jana N, Bera S. Estimation of parameters of inverse Weibull distribution and application to multi-component stress-strength model. J Appl Stat 2022;49:169-194. [CrossRef]
- [12] Basu AP, Ebrahimi N. Bayesian approach to life testing and reliability estimation using asymmetric loss function. J Stat Plan Inference 1992;29:21–31. [CrossRef]
- [13] Berger JO. Statistical decision theory, foundation, concepts and method. New York, NY, USA: Springer; 1985. [CrossRef]
- [14] Norstrom JG. The use of precautionary loss functions in risk analysis. IEEE Trans Reliab vol. 1996;45:400-403. [CrossRef]
- [15] Parsian A, Kirmani SNU. Estimation Under LINEX Loss Function. In: Ullah A, Wan ATK, Chaturvedi A, Dekker M, editors. Handbook of applied econometrics and statistical inference. Boca Raton, FL, USA: CRC Press; 2020. pp. 53–76. [CrossRef]
- [16] Misra N, Meulen EC. On estimating the mean of the selected normal population under the LINEX loss function. Metrika 2003;58:173–183. [CrossRef]
- [17] Nematollahi N, Motamed-Shariati F. Estimation of the scale parameter of the selected gamma population under the entropy loss function, communications in statistics. Theory Methods 2009;38;208–221. [CrossRef]
- [18] Azimi R, Yaghmaei F, Azimi D. Comparison of bayesian estimation methods for rayleigh progressive censored data under the different asymmetric loss function. Int J Appl Math Res 2012;1:452–461. [CrossRef]

- [19] Ahmad K, Ahmad SP, Ahmed A. Classical and bayesian approach in estimation of scale parameter of inverse weibull distribution. Math Theory Model 2015;5.
- [20] Calabria R, Pulcini G. Point estimation under asymmetric loss functions for left-truncated exponential samples. Commun Stat Aeory Methods 1996;25:585–600. [CrossRef]
- [21] Gencer G, Saraçoğlu B. Comparison of approximate Bayes estimators under different loss functions for parameters of odd Weibull distribution. J Selçuk Univ Nat Appl Sci 2016;5:18–32.
- [22] Khatun N, Matin MA. A study on LINEX loss function with different estimating methods. Open J Stat 2020;10:52–63. [CrossRef]
- [23] Sultan KS. Bayesian estimates based on record values from the inverse Weibull lifetime model. Qual Technol Quant Manage 2008;5:363–374. [CrossRef]
- [24] Islam SAFM, Roy MK, Ali MM. A non-linear exponential (NLINEX) loss function in bayesian analysis. J Korean Data Inform Sci Soc 2004;15:899–910.