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Geometric Analysis of Riemannian Submersions: Curvature Tensors and Total Umbilic Fibers

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Abstract

In this paper, we investigate the geometric properties of Riemannian submersions, providing a comprehensive analysis of various curvature tensors, all associated with a new type of semi-symmetric non-metric connection. We also study the behaviour of these curvatures in cases where Riemannian submersions have total umbilic fibers.

Keywords: *Curvature tensors; Riemannian submersion; Semi-symmetric non-metric connection; Total umbilic fibers.*
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1. Introduction

Riemannian submersions are fundamental concept in differential geometry, serving as a powerful tools for understanding the geometric properties of Riemannian manifolds. These submersions are special types of smooth mappings between Riemannian manifolds that preserve specific geometric structures, allowing for a systematic study of the interplay between different manifolds. With applications that span both pure mathematics and theoretical physics, Riemannian submersions provide a framework for examining various geometries, significantly impact our comprehension of the physical universe. Fundamental investigations into Riemannian submersions was pioneered by O'Neill [21] and Gray [13], and the topic has since been the subject of extensive research [4], [9], [10], [16], [19], [24], [25].
Curvature tensors are critical mathematical tools in mathematics and theoretical physics, especially in the study of geometry and gravitational theory. These tensors describe and quantify the curvature of spaces and the effects of gravity. Among them, the best- known curvature tensor is the Riemann curvature tensor, but there are also curvature tensors that play important roles in differential geometry, such as the Ricci tensor, scalar curvature, etc. The Riemann curvature tensor, denoted as R, is a mathematical object used to describe the curvature of a manifold in differential geometry. It captures the variation of the metric (a measure of distances) of a space from point to point. The Ricci tensor, denoted S, is a contraction of the Riemann tensor and is used to summarise the curvature information of a space. The scalar curvature, often denoted by r, is derived from the Ricci tensor and represents the total curvature of a space at a given point. It is a single number that characterizes the curvature at that point.
The Riemannian curvature tensor is an essential tool in differential geometry for quantifying the curvature of $n-$dimensional manifolds. This tensor is of great importance in both the domains of general relativity and gravitation. Mishra, in [18], introduced a novel set of curvature tensors on Riemannian manifolds, including the concircular curvature tensor. Extending Mishra's work, Pokhariyal and Mishra, in [23], introduced the Weyl projective curvature tensor for Riemannian manifolds. Subsequently, Ojhe extended this line of research further by defining the $M-$projective curvature tensor [20]. In 1988, M. Doric et al. [8] conducted an examination of the requirements for the conharmonic curvature tensor, particularly within the context of Kähler hypersurfaces in complex space forms. Additionally, in a separate study, Ahsan [1] explored the relativistic implications of the concircular curvature tensor. In 2018, G. Hall undertook a comprehensive investigation of projectively related connections on a space-time manifolds, closely examining the Weyl projective tensor based on Einstein's geodesic postulate [15]. These tensors play significant roles in the investigation and classification of geometric structures on manifolds, providing a deeper understanding of the curvature and conformal properties of spaces. In mathematics, these tensors are important for the classification of Riemannian and pseudo-Riemannian manifolds, enriching our understanding of the various possible geometric structures. More recently Akyol and Ayar studied new curvature tensors along Riemannian submersions [3].

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Moreover, consider M as an $n-$dimensional Riemannian manifold, where ∇ denotes the Levi-Civita connection associated with the Riemannian metric g_M on M. A linear connection $\widetilde{\nabla}$, defined on M is referred to as "symmetric" if the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$, defined by $\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]$, vanishes for all vector fields X and Y in M. If it doesn't equal zero, it is referred to as "non-symmetric". In 1924, Friedmann and Schouten [12] introduced the concept of a "semi-symmetric" linear connection on a differentiable manifold. When all vector fields X and Y on M satisfy the following equation, a linear connection $\widetilde{\nabla}$ on M is said to be semi-symmetric:

$$\widetilde{T}(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

where $1-$form η is linked to the vector field P and fulfills the condition:

$$\eta(X) = g_M(X,P). \tag{1.2}$$

A metric connection $\widetilde{\nabla}$ on a Riemannian manifold was first proposed by Hayden in 1932, and he subsequently called it a "Hayden connection". Later, Pak [22] examined the Hayden connection $\widetilde{\nabla}$ fitted with the torsion tensor $\widetilde{T}$ which is defined as in (1.1) and proved that it is a "semisymmetric metric connection". A linear connection $\widetilde{\nabla}$ is considered a metric on M if it satisfies the condition $\widetilde{\nabla}g = 0$; otherwise, it is classified as "non-metric". Yano [26] started the methodical investigation of semisymmetric metric connections $\widetilde{\nabla}$ on Riemannian manifolds in 1970. Furthermore, [2], [7], [6] and [17] contain recent research on Riemannian submersions with semi-symmetric metric or semi-symmetric non-metric connections.
Inspired by the above research, in this work, we compute and analyse these curvature tensors for Riemannian submersions according to a new type of semi-symmetric non-metric connection and investigate how they behave when total umbilic fibers are present. In the initial section, we establish the foundational definitions and theorems that are relevant to Riemannian submersions. In the following section, we conducted detailed calculations of various curvature tensors, which include the Weyl projective curvature tensor, Concircular curvature tensor, Conharmonic curvature tensor, Conformal curvature tensor, and $M-$projective curvature tensor, all within the framework of Riemannian submersions according to semi-symmetric metric connection. Additionally, we explore the behavior of these curvature tensors when Riemannian submersions exhibit total umbilic fibers. This exploration provides of a comprehensive analysis of the characteristics of these tensors under this specific geometric condition.

2. Preliminaries

Let (M,g_M) and (N,g_N) be Riemannian manifolds, where $dim(M) > dim(N)$ [21]. A surjective mapping $f:(M,g_M)\to(N,g_N)$ is called as a "Riemannian submersion" when the following conditions are satisfied:

1.) The rank of f is equal to the dimension of N. In this case, for each $q\in N$, $f^{-1}(q)$ represents a submanifold of dimension $dim(M)-dim(N)$ in M and is called a fiber. If a vector field on M is always orthogonal to the fibers, it is classified as horizontal; otherwise, it is classified as vertical. A vector field X on M is called basic if X is horizontal and is $f-$related to a vector field X' on N, meaning that $f_*(X_p) = X'_{f(p)}$ for all $p\in M$, where f_* is the differential map of f. The projections onto the vertical distribution $\ker f_*$ and the horizontal distribution $(\ker f_*)^{\perp}$ will be denoted $\mathscr{V}$ and $\mathscr{H}$, respectively. As usual, the manifolds (M,g_M) and (N,g_N) are denoted as the total manifold and the base manifold of the submersion, respectively.

2.) f_* preserves the lengths of the horizontal vectors.

Proposition 2.1. *Let $f:(M,g_M)\to(N,g_N)$ be a Riemannian submersion and denote ∇ and ∇' as the Levi-Civita connections on M and N, respectively. If X and Y are basic vector fields that are $f-$related to X' and Y', respectively, the following relation hold [11]:*

i.) $g_M(X,Y) = g_N(X',Y')\circ f$,
ii.) $h[X,Y]$ is the basic vector field $f-$related to $[X',Y']$,
iii.) $h(\nabla_X Y)$ is the basic vector field $f-$related to $\nabla'_{X'}Y'$,
iv.) for any vertical vector field V, $[X,V]$ is vertical.

The distribution $\mathscr{V}$ corresponds to the foliation of M by setting $\mathscr{V}_p = \ker f_{*p}$ for any $p\in M$. Each $\mathscr{V}_p$ is defined as the vertical space at p, $\mathscr{V}$ is the vertical distribution, and the sections of $\mathscr{V}$ are determined as a Lie subalgebra $\chi^v(M)$ of $\chi(M)$. $\mathscr{H}$ represents the complementary distribution of $\mathscr{V}$ generated by the Riemannian metric g. Therefore, the orthogonal decomposition $T_pM = \mathscr{V}_p \oplus \mathscr{H}_p$ at each $p\in M$ is called the horizontal space at p. The sections of the horizontal distribution $\mathscr{H}$ are the horizontal vector fields, establishing a subspace $\chi^h(M)$ of $\chi(M)$. For any $E\in\chi(M)$, vE and hE represent the vertical and horizontal components of E, respectively [11].
A Riemannian submersion $f:M\to N$ determines two $(1,2)$ O'Neill tensor fields. The fundamental tensor fields are given by:

$$\mathscr{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF, \tag{2.1}$$
$$\mathscr{A}_E F = v\nabla_{hE} vF + h\nabla_{hE} vF, \tag{2.2}$$

for any E, $F\in\chi(M)$, where v and h are vertical and horizontal projections. Additionally

$$\nabla_U V = \mathscr{T}_U V + v\nabla_U V, \tag{2.3}$$
$$\nabla_U X = \mathscr{T}_U X + h\nabla_U X, \tag{2.4}$$
$$\nabla_X U = \mathscr{A}_X U + v\nabla_X U, \tag{2.5}$$
$$\nabla_X Y = \mathscr{A}_X Y + h\nabla_X Y, \tag{2.6}$$

for any $X,Y\in\chi^h(M)$; $U,V\in\chi^v(M)$. In addition, if X is a basic vector field then, $h\nabla_U X = h\nabla_X U = \mathscr{A}_X U$. We note that $\mathscr{T}_U V = \mathscr{T}_V U$, $\mathscr{A}_X Y = -\mathscr{A}_Y X$ [25].

Let $f: M \to N$ be a Riemannian submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) [25]. For any $E, F, H \in \chi(M)$ the covariant derivatives of $\mathscr{A}$ and $\mathscr{T}$ are given by

$$(\nabla_E \mathscr{A})_F H = \nabla_E(\mathscr{A}_F H) - \mathscr{A}_{\nabla_E F}(H) - \mathscr{A}_F(\nabla_E H), \tag{2.7}$$

$$(\nabla_E \mathscr{T})_F H = \nabla_E(\mathscr{T}_F H) - \mathscr{T}_{\nabla_E F}(H) - \mathscr{T}_F(\nabla_E H). \tag{2.8}$$

Definition 2.2. *Let (M, g_M) be a Riemannian manifold, at a point $p \in M$, the sectional curvature K_p is defined by:*

$$K_p = \frac{g_M(R(X,Y)Y,X)}{\|X\|^2 \|Y\|^2 - g_M(X,Y)^2}. \tag{2.9}$$

Also, the Ricci curvature denoted as Ric, is a mapping from $C_2^\infty(TM)$ to $C_2^\infty(TM)$, defined as:

$$Ric : C_2^\infty(TM) \to C_0^\infty(TM) \text{ by } Ric(X,Y) = \sum_{i=1}^{m} g_M(R(X,e_i)e_i,Y).$$

So, scalar curvature τ is given by

$$\tau = \sum_{j=1}^{m} R_i(e_j, e_j) = \sum_{j=1}^{m}\sum_{i=1}^{m} g_M(R(e_i,e_j)e_j,e_i), \tag{2.10}$$

where $\{e_1, e_2, ..., e_m\}$ represents any local orthonormal frame for the tangent bundle [14].

In this paper, we denote $Ric(X,Y)$ by $S(X,Y)$.

Definition 2.3. *Let (M, g_M) be a Riemannian manifold. An $f-$adapted local orthonormal frame $\{X_i, U_j\}_{1\le i\le n, 1\le j\le r}$ is defined such that each X_i is horizontal and each U_j is vertical [11].*

Lemma 2.4. *Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds (M, g_M) and (N, g_N). Then we get:*

$$\sum_{i=1}^{n} g_M(\mathscr{T}_U X_i, \mathscr{T}_V X_j) = \sum_{j=1}^{r} g_M(\mathscr{T}_U U_j, \mathscr{T}_V U_j) \tag{2.11}$$

$$\sum_{i=1}^{n} g_M(\mathscr{A}_X X_i, \mathscr{A}_Y X_i) = \sum_{j=1}^{r} g_M(\mathscr{A}_X U_j, \mathscr{A}_Y U_j) \tag{2.12}$$

$$\sum_{i=1}^{n} g_M(\mathscr{A}_X X_i, \mathscr{T}_U X_i) = \sum_{j=1}^{r} g_M(\mathscr{A}_X U_j, \mathscr{T}_U U_j) \tag{2.13}$$

where $X, Y \in \chi^h(M)$, $U, V \in \chi^v(M)$, and $\{X_i, U_j\}$ is an $f-$ adaptable frame on (M, g_M) [11].

Definition 2.5. *Let (M, g_M) be a Riemannian manifold and $\mathscr{V}$ be the local orthonormal frame of the vertical distribution. Then we define horizontal vector field $\mathscr{N}$ on (M, g_M) [11]:*

$$\mathscr{N} = \sum_{j=1}^{r} \mathscr{T}_{U_j} U_j. \tag{2.14}$$

Lemma 2.6. *Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds (M, g_M) and (N, g_N). Let $\{U_j\}_{1\le j\le r}$ be a local orthonormal frame of $\mathscr{V}$. For any $E \in \chi(M)$, we obtain [11]:*

$$g(\nabla_E \mathscr{N}, X) = \sum_{j=1}^{r} g_M((\nabla_E \mathscr{T})_{U_j} U_j, X). \tag{2.15}$$

Chaubey and Yildiz [5] introduced a new type of semi-symmetric non-metric connection on a Riemannian manifold as:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\eta(Y)X - \eta(X)Y\}, \tag{2.16}$$

for arbitrary vector fields X and Y on M and η is a $1-$form. Later, Karatas and others [17] obtained important results by Riemannian Submersions Endowed with a New Type of Semi-Symmetric Non-Metric Connection. Here, tensor fields $\widetilde{\mathscr{T}}$ and $\widetilde{\mathscr{A}}$ have been derived on manifold M for a new type of semi-symmetric non-metric connection. So let's recall some concepts:
Let M be a Riemannian manifold with metric g_M, f be a Riemannian submersion from M onto a Riemannian manifold N with metric g_N and $E, F \in \chi(M)$. In this case we have the following equations:

$$\widetilde{\mathscr{T}}_E F = \mathscr{T}_E F + \frac{1}{2}\eta(hF)vE, \tag{2.17}$$

$$\widetilde{\mathscr{A}}_E F = \mathscr{A}_E F + \frac{1}{2}\eta(vF)hE, \tag{2.18}$$

where $\widetilde{\mathscr{T}}$ and $\widetilde{\mathscr{A}}$ are tensor fields on M admitting to a new type of semi-symmetric non-metric connection $\widetilde{\nabla}$ [17].

Lemma 2.7. *Let $f:(M,g_M)\to(N,g_N)$ be a Riemannian submersion between Riemannian manifolds (M,g_M) and (N,g_N). Then we have:*

$$
\begin{aligned}
\widetilde{\nabla}_V X &= \mathscr{T}_V X + h\widetilde{\nabla}_V X + \frac{1}{2}\eta(X)V,\\
\widetilde{\nabla}_V W &= \mathscr{T}_V W + \widehat{\nabla}_V W,\\
\widetilde{\nabla}_X V &= \mathscr{A}_X V + v\widetilde{\nabla}_X V + \frac{1}{2}\eta(V)X,\\
\widetilde{\nabla}_X Y &= \mathscr{A}_X Y + h\widetilde{\nabla}_X Y,
\end{aligned}
$$

for $V,W\in\chi^v(M)$ and $X,Y\in\chi^h(M)$, where $\widehat{\nabla}_V W = v\,\widetilde{\nabla}_V W$ [17].

Definition 2.8. *Let M be a $n-$dimensional C^∞ manifold. In this case in the $n-$dimensional space V^n for every $X,Y,Z\in\chi(M)$, the Weyl projective curvature tensor field of M defined as follows:*

$$\tilde{P}(X,Y)Z=\tilde{R}(X,Y)Z-\frac{1}{n-1}\left\{\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y\right\}, \tag{2.19}$$

where $\widetilde{S}$ is the Ricci tensor of total space [18].

Definition 2.9. *Let M be an $n-$dimensional Riemannian manifold with metric tensor field g_M. The concircular curvature tensor $\widetilde{C}$ on a Riemannian manifold is defined as follows:*

$$\tilde{C}(X,Y)Z=\tilde{R}(X,Y)Z-\frac{\widetilde{\tau}}{n(n-1)}\left[g_M(Y,Z)X-g_M(X,Z)Y\right], \tag{2.20}$$

where $\widetilde{\tau}$ represents a scalar tensor [18].

Definition 2.10. *In the $n-$dimensional space V_n, the tensor*

$$\tilde{L}(X,Y)Z=\tilde{R}(X,Y)Z-\frac{1}{n-2}\{g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y\}, \tag{2.21}$$

is called conharmonic curvature tensor, where R and S denote the Riemannian curvature tensor and Ricci Tensor, respectively [18].

Definition 2.11. *In the $n-$dimensional space V_n, the tensor*

$$\tilde{V}(X,Y)Z=R^M(X,Y)Z-\frac{1}{n-2}[S(Y,Z)X-S(X,Z)Y+g_M(Y,Z)QX-g_M(X,Z)QY]+\frac{\tau^M}{(n-1)(n-2)}[g_M(Y,Z)X-g_M(X,Z)Y], \tag{2.22}$$

is called weyl conformal curvature tensor, where Q, R^M and S denote the Ricci operator, Riemannian curvature tensor and Ricci tensor, respectively [18]

Definition 2.12. *In the $n-$dimensional space V_n, the tensor*

$$\tilde{W}(X,Y,Z)=R^M(X,Y)Z-\frac{1}{2(n-1)}[S^M(Y,Z)X-S^M(X,Z)Y+g_M(Y,Z)QX-g_M(X,Z)QY], \tag{2.23}$$

is called $M-$projective curvature tensor, where Q, R^M and S denote the Ricci operator, Riemannian curvature tensor and Ricci tensor, respectively [23].

Definition 2.13. *A submanifold M of a Riemannian manifold (M,g) is totally umblical if, for any point $p\in M$ and any tangent vectors X,Y to M at p, the second fundamental form h satisfies:*

$$h(X,Y)=Hg(X,Y),$$

where H is the mean curvature tensor.

Theorem 2.14. *Let (M,g_M) and (N,g_N) Riemann Manifolds,*

$$f:(M,g_M)\to(N,g_N)$$

a Riemannian submersion and $\tilde{R},\acute{R}$ and $\hat{R}$ be Riemannian curvature tensors of M, N and $\left(f^{-1}(x),\hat{g}_x\right)$ fibre with respect to the new type of semi-symmetric non-metric connection, respectively. Then we have

$$
\begin{aligned}
g_M(\tilde{R}(X,Y)Z,H) &= g_M(\acute{R}(X,Y)Z,H)-g_M(\mathscr{A}_X H,\mathscr{A}_Y Z)+g_M(\mathscr{A}_X Z,\mathscr{A}_Y H)+2g_M(\mathscr{A}_X Y,\mathscr{A}_Z H)+\tfrac{1}{2}\eta(\mathscr{A}_Y Z)g_M(X,H)-\tfrac{1}{2}\eta(\mathscr{A}_X Z)g_M(Y,H),\\
g_M(\tilde{R}(X,Y)Z,V) &= -2g_M(\mathscr{A}_X Y,\mathscr{T}_V Z)+g_M(\tilde{\nabla}_X\mathscr{A}_Y Z,V)-g_M(\tilde{\nabla}_Y\mathscr{A}_X Z,V)+g_M(\mathscr{A}_X\acute{\nabla}_Y Z,V)-g_M(\mathscr{A}_Y\acute{\nabla}_X Z,V)-\tfrac{1}{2}\eta(Z)g_M([X,Y],V),\\
g_M(\tilde{R}(X,Y)V,W) &= \tfrac{1}{2}\left[g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),W)-g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),W)-\eta(V)g_M(\mathscr{A}_X Y,W)\right]\\
&\quad +g_M(\tilde{\nabla}_X v\tilde{\nabla}_Y V,W)-g_M(\tilde{\nabla}_Y v\tilde{\nabla}_X V,W)-g_M(\tilde{\nabla}_{[X,Y]}V,W),\\
g_M(\tilde{R}(X,V)Y,W) &= g_M((\tilde{\nabla}_X\mathscr{T})_V Y,W)-g_M((\tilde{\nabla}_V\mathscr{A})_X Y,W)+g_M(\mathscr{A}_Y W,h\tilde{\nabla}_V X)+g_M(\mathscr{T}_W Y,v\tilde{\nabla}_V X)+\tfrac{1}{2}\eta(Y)g_M(\tilde{\nabla}_V X,W)-\tfrac{1}{2}\eta(h\tilde{\nabla}_X Y)g_M(V,W)\\
g_M(\tilde{R}(U,V)W,X) &= g_M((\tilde{\nabla}_U\mathscr{T})_V W,X)-g_M((\tilde{\nabla}_V\mathscr{T})_U W,X),\\
g_M(\tilde{R}(U,V)W,F) &= g_M(\hat{R}(U,V)W,F)-g_M(\mathscr{T}_U F,\mathscr{T}_V W)+g_M(\mathscr{T}_V F,\mathscr{T}_U W)+\tfrac{1}{2}\eta(\mathscr{T}_V W)g_M(U,F)-\tfrac{1}{2}\eta(\mathscr{T}_U W)g_M(V,F),\\
g_M(\tilde{R}(X,Y)V,H) &= g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),H)-g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),H)+\tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_X Y,H)-\tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X,H)+g_M(\mathscr{T}_V H,[X,Y]),
\end{aligned}
$$

for all $X,Y,Z,H\in\chi^h(M)$ and $U,V,W,F\in\chi^v(M)$ [17].

Proposition 2.15. *Let (M,g_M) and (N,g_N) be Riemannian manifolds,*

$$f:(M,g_M)\to(N,g_N)$$

be a Riemannian submersion, $\{X_i,U_j\}$ and $\{X_i,V_j\}$ be $f-$adaptable frame.Then the Ricci tensor $\tilde{S}$ of (M,g_M) with respect to the new type of semi-symmetric non-metric connection is given by [17];

$$\begin{aligned}\tilde{S}(U,V)=&\hat{S}(U,V)+g_M(\mathscr{N},\mathscr{T}_UV)-\tfrac{1}{2}\eta(\mathscr{T}_UV)+\sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_UX_i,V)+g_M(\mathscr{A}_{X_i}V,h\tilde{\nabla}_UX_i)+g_M(\mathscr{T}_VX_i,v\tilde{\nabla}_UX_i)+\tfrac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_UX_i,V)\\&-\tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,V)\}-\sum_j\{g_M(\mathscr{T}_UU_j,\mathscr{T}_VU_j)-\tfrac{1}{2}\eta(\mathscr{T}_{U_j}V)g_M(U,U_j)\},\\ \tilde{S}(X,Y)=&\acute{S}(\acute{X},\acute{Y})\circ f-\tfrac{1}{2}\eta(\mathscr{A}_XY)-\tfrac{1}{2}\eta(h\tilde{\nabla}_XY)+\sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}Y)+\tfrac{1}{2}\eta(\mathscr{A}_{X_i}Y)g_M(X,X_i)\}+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Y,U_j)-g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XY,U_j)\\&+g_M(\mathscr{A}_YU_j,h\tilde{\nabla}_{U_j}X)+g_M(\mathscr{T}_{U_j}Y,v\tilde{\nabla}_{U_j}X)+\tfrac{1}{2}\eta(Y)g_M(\tilde{\nabla}_{U_j}X,U_j),\\ \tilde{S}(X,V)=&g_M(\tilde{\nabla}_V\mathscr{N},X)+\sum_i\{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,V),X_i)-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i)+\tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_XX_i,X_i)-\tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X^i}X,X_i)+g_M(\mathscr{T}_VX_i,[X,X_i]\}\\&-\sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_VU_j,X)\},\end{aligned}$$

where $X,Y,\in\chi^h(M)$ and $U,V\in\chi^v(M)$ [17].

Proposition 2.16. *Let (M,g_M) and (N,g_N) be Riemannian manifolds,*

$$f:(M,g_M)\to(N,g_N)$$

be a Riemannian submersion. Let $\widetilde{\tau}$, τ' , $\widehat{\tau}$, the scalar curvature of the manifold (M,g_M), (N,g_N), and $\left(f^{-1}(x),\hat{g}_x\right)$ fibre, respectively. Then the scalar curvature $\widetilde{\tau}$ of (M,g_M) manifold with respect to the new type of semi-symmetric non-metric connection is given by

$$\begin{aligned}\widetilde{\tau}=&\widehat{\tau}+\tau'\circ f-\sum_i\eta(h\widetilde{\nabla}_{X_i}X_i)+\sum_{i,j}\{2g_M((\widetilde{\nabla}_{X_i}\mathscr{T})_{U_j}X_i,U_j)+2g_M(\mathscr{A}_{X_i}U_j,h\nabla_{U_j}X_i)\\&+2g_M(\mathscr{T}_{U_j}X_i,v\widetilde{\nabla}_{U_j}X_i)+\eta(X_i)g_M(\widetilde{\nabla}_{U_j}X_i,U_j)+3g_M(\mathscr{A}_{X_i}U_j,\mathscr{A}_{X_i}U_j)\}-\eta(\mathscr{N}),\end{aligned}$$

where $\{X_i,U_j\}$ $f-$ adaptable frame [17].

3. Calculations of Curvature Tensors in Riemannian Submersions with respect to a new type of semi-symmetric non- metric connection

3.1. Weyl projective curvature tensor

In this section, we investigate the relationships involving the Weyl projective curvature tensor among the total space, the base space, and the fibers within the context of a Riemannian submersion with a new type of semi-symmetric non-metric connection. Additionally, we provide a corollary for cases in which the Riemannian submersion with a new type semi-symmetric non-metric connection features completely umbilical fibers.

Theorem 3.1. *Let (M,g_M) and (N,g_N) Riemannian manifolds,*

$$f:(M,g_M)\to(N,g_N)$$

a Riemannian submersion and $\tilde{R}$, $\acute{R}$, and $\hat{R}$ be Riemannian curvature tensors, $\tilde{S}$, $\acute{S}$, and $\hat{S}$ be Ricci tensors of M,N, and the fibre respectively. Then for every $U,V,W,F\in\chi^v(M)$ and $X,Y,Z,H\in\chi^h(M)$, we get the following Weyl projective curvature tensor relations:

$$\begin{aligned}g_M(\tilde{P}(X,Y)Z,H)=&g_M(\acute{R}(X,Y)Z,H)-g_M(\mathscr{A}_XH,\mathscr{A}_YZ)+g_M(\mathscr{A}_XZ,\mathscr{A}_YH)+2g_M(\mathscr{A}_XY,\mathscr{A}_ZH)+\tfrac{1}{2}\eta(\mathscr{A}_YZ)g_M(X,H)-\tfrac{1}{2}\eta(\mathscr{A}_XZ)g_M(Y,H)\\&-\tfrac{1}{n-1}\{g_M(X,H)[\acute{S}(Y,\acute{Z})\circ f-\tfrac{1}{2}\eta(\mathscr{A}_YZ)-\tfrac{1}{2}\eta(h\tilde{\nabla}_YZ)+\sum_i\{3g_M(\mathscr{A}_{X_i}Y,\mathscr{A}_{X_i}Z)+\tfrac{1}{2}\eta(\mathscr{A}_{X_i}Z)g_M(Y,X_i)\}\\&+\sum_j g_M((\tilde{\nabla}_Y\mathscr{T})_{U_j}Z,U_j)-g_M((\tilde{\nabla}_{U_j}\mathscr{A})_YZ,U_j)+g_M(\mathscr{A}_ZU_j,h\tilde{\nabla}_{U_j}Y)+g_M(\mathscr{T}_{U_j}Z,v\tilde{\nabla}_{U_j}Y)+\tfrac{1}{2}\eta(Z)g_M(\tilde{\nabla}_{U_j}Y,U_j)]\\&-g(Y,H)[\acute{S}(\acute{X},\acute{Z})\circ f-\tfrac{1}{2}\eta(\mathscr{A}_XZ)-\tfrac{1}{2}\eta(h\tilde{\nabla}_XZ)+\sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}Z)+\tfrac{1}{2}\eta(\mathscr{A}_{X_i}Z)g_M(X,X_i)\}\\&+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Z,U_j)-g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XZ,U_j)+g_M(\mathscr{A}_ZU_j,h\tilde{\nabla}_{U_j}X)+g_M(\mathscr{T}_{U_j}Z,v\tilde{\nabla}_{U_j}X)+\tfrac{1}{2}\eta(Z)g_M(\tilde{\nabla}_{U_j}X,U_j)]\},\end{aligned}$$

$$g_M(\tilde{P}(X,Y)Z,V)=-2g_M(\mathscr{A}_XY,\mathscr{T}_VZ)+g_M(\tilde{\nabla}_X\mathscr{A}_YZ,V)-g_M(\tilde{\nabla}_Y\mathscr{A}_XZ,V)+g_M(\mathscr{A}_X\acute{\nabla}_YZ,V)-g_M(\mathscr{A}_Y\acute{\nabla}_XZ,V)-\tfrac{1}{2}\eta(Z)g_M([X,Y],V),$$

$$\begin{aligned}g_M(\tilde{P}(X,Y)V,W)=&\tfrac{1}{2}\left[g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),W)-g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),W)-\eta(V)g_M(\mathscr{A}_XY,W)\right]\\&+g_M(\tilde{\nabla}_Xv\tilde{\nabla}_YV,W)-g_M(\tilde{\nabla}_Yv\tilde{\nabla}_XV,W)-g_M(\tilde{\nabla}_{[X,Y]}V,W),\end{aligned}$$

$$\begin{aligned}g_M(\tilde{P}(X,V)Y,W)=&g_M((\tilde{\nabla}_X\mathscr{T})_VY,W)-g_M((\tilde{\nabla}_V\mathscr{A})_XY,W)+g_M(\mathscr{A}_YW,h\tilde{\nabla}_VX)+g_M(\mathscr{T}_WY,v\tilde{\nabla}_VX)+\tfrac{1}{2}\eta(Y)g_M(\tilde{\nabla}_VX,W)\\&-\tfrac{1}{2}\eta(h\tilde{\nabla}_XY)g_M(V,W)-\tfrac{1}{n-1}g_M(V,W)[\acute{S}(\acute{X},\acute{Y})\circ f-\tfrac{1}{2}\eta(\mathscr{A}_XY)-\tfrac{1}{2}\eta(h\tilde{\nabla}_XY)\\&+\sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}Y)+\tfrac{1}{2}\eta(\mathscr{A}_{X_i}Y)g_M(X,X_i)\}+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Y,U_j)-g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XY,U_j)\\&+g_M(\mathscr{A}_YU_j,h\tilde{\nabla}_{U_j}X)+g_M(\mathscr{T}_{U_j}Y,v\tilde{\nabla}_{U_j}X)+\tfrac{1}{2}\eta(Y)g_M(\tilde{\nabla}_{U_j}X,U_j)]\end{aligned}$$

$$g_M(\tilde{P}(U,V)W,X)=g_M((\tilde{\nabla}_U\mathscr{T})_VW,X)-g_M((\tilde{\nabla}_V\mathscr{T})_UW,X),$$

$$g_M(\tilde{P}(U,V)W,F) = g_M(\hat{R}(U,V)W,F) - g_M(\mathscr{T}_U F, \mathscr{T}_V W) + g_M(\mathscr{T}_V F, \mathscr{T}_U W) + \frac{1}{2}\eta(\mathscr{T}_V W) g_M(U,F) - \frac{1}{2}\eta(\mathscr{T}_U W) g_M(V,F)$$
$$-\frac{1}{n-1}\{g(U,F)[\hat{S}(V,W) + g_M(\mathscr{N}, \mathscr{T}_V W) - \frac{1}{2}\eta(\mathscr{T}_V W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i, W) + g_M(\mathscr{A}_{X_i} W, h\tilde{\nabla}_V X_i)$$
$$+ g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_V X_i) + \frac{1}{2}\eta(X_i) g_M(\tilde{\nabla}_V X_i, W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i} X_i) g_M(V,W)\} - \sum_j \{g_M(\mathscr{T}_V U_j, \mathscr{T}_W U_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j} W) g_M(V,U_j)\}]$$
$$-g(V,F)[\hat{S}(U,W) + g_M(\mathscr{N}, \mathscr{T}_U W) - \frac{1}{2}\eta(\mathscr{T}_U W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_U X_i, W) + g_M(\mathscr{A}_{X_i} W, h\tilde{\nabla}_U X_i)$$
$$+ g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_U X_i) + \frac{1}{2}\eta(X_i) g_M(\tilde{\nabla}_U X_i, W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i} X_i) g_M(U,W)\} - \sum_j \{g_M(\mathscr{T}_U U_j, \mathscr{T}_W U_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j} W) g_M(U,U_j)\}]\},$$

$$g_M(\tilde{P}(X,Y)V,H) = g_M((\tilde{\nabla}_X \mathscr{A})(Y,V),H) - g_M((\tilde{\nabla}_Y \mathscr{A})(X,V),H) + \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_X Y,H) - \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_Y X,H) + g_M(\mathscr{T}_V H, [X,Y])$$
$$-\frac{1}{n-1}\{g_M(X,H)[g_M(\tilde{\nabla}_V N, Y) + \sum_i \{g_M((\tilde{\nabla}_Y \mathscr{A})(X_i,V), X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V), X_i)$$
$$+ \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_Y X_i, X_i) - \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_{X_i} Y, X_i) + g_M(\mathscr{T}_V X_i, [Y,X_i]\}$$
$$-\sum_j \{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j, Y)\}] - g(Y,H)[g_M(\tilde{\nabla}_V \mathscr{N}, X) + \sum_i \{g_M((\tilde{\nabla}_X \mathscr{A})(X_i,V), X_i)$$
$$-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V), X_i) + \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_X X_i, X_i) - \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_{X i} X, X_i) + g_M(\mathscr{T}_V X_i, [X,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{V_i}\mathscr{T})_V U_j, X)\}]\}.$$

Proof. To avoid repetition, we will only give the proof of the first equation. We can easily show by making an inner production between $\tilde{P}$ and H and using Theorem 2.14 (1) and Proposition 2.15 (2) in (2.19). □

Corollary 3.2. *Let $f:(M,g_M)\to(N,g_N)$ be a Riemannian submersion, where (M,g_M) and (N,g_N) are Riemannian manifolds. If the Riemannian submersion has total umblical fibres ($\mathscr{N}=0$), then the Weyl projective curvature tensor is given by*

$$g_M(\tilde{P}(U,V)W,F) = g_M(\hat{R}(U,V)W,F) - g_M(\mathscr{T}_U F, \mathscr{T}_V W) + g_M(\mathscr{T}_V F, \mathscr{T}_U W) + \frac{1}{2}\eta(\mathscr{T}_V W) g_M(U,F) - \frac{1}{2}\eta(\mathscr{T}_U W) g_M(V,F) - \frac{1}{n-1}\{g_M(U,F)[\hat{S}(V,W)$$
$$-\frac{1}{2}\eta(\mathscr{T}_V W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i, W) + g_M(\mathscr{A}_{X_i} W, h\tilde{\nabla}_V X_i) + g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_V X_i) + \frac{1}{2}\eta(X_i) g_M(\tilde{\nabla}_V X_i, W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i} X_i) g_M(V,W)\}$$
$$-\sum_j \{g_M(\mathscr{T}_V U_j, \mathscr{T}_W U_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j} W) g_M(V,U_j)\}] - g(V,F)[\hat{S}(U,W) - \frac{1}{2}\eta(\mathscr{T}_U W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_U X_i, W) + g_M(\mathscr{A}_{X_i} W, h\tilde{\nabla}_U X_i)$$
$$+ g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_U X_i) + \frac{1}{2}\eta(X_i) g_M(\tilde{\nabla}_U X_i, W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i} X_i) g_M(U,W)\} - \sum_j \{g_M(\mathscr{T}_U U_j, \mathscr{T}_W U_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j} W) g_M(U,U_j)\}]\},$$

$$g_M(\tilde{P}(X,Y)V,H) = g_M((\tilde{\nabla}_X \mathscr{A})(Y,V),H) - g_M((\tilde{\nabla}_Y \mathscr{A})(X,V),H) + \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_X Y,H) - \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_Y X,H) + g_M(\mathscr{T}_V H, [X,Y])$$
$$-\frac{1}{n-1}\{g(X,H)[\sum_i \{g_M((\tilde{\nabla}_Y \mathscr{A})(X_i,V), X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V), X_i) + \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_Y X_i, X_i) - \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_{X_i} Y, X_i) + g_M(\mathscr{T}_V X_i, [Y,X_i]\}$$
$$-\sum_j \{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j, Y)\} - g(Y,H)[\sum_i \{g_M((\tilde{\nabla}_X \mathscr{A})(X_i,V), X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V), X_i) + \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_X X_i, X_i) - \frac{1}{2}\eta(V) g_M(\tilde{\nabla}_{X i} X, X_i)$$
$$+ g_M(\mathscr{T}_V X_i, [X,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j, X)\}]\}.$$

for all $X,Y,Z,H \in \chi^h(M)$ and $U,V,W,F \in \chi^v(M)$.

3.2. Concircular curvature tensor

In this section we give curvature relations of concircular curvature tensor in Riemannian submersions endowed with a new type of semi-symmetric non-metric connection.

Theorem 3.3. *Let (M,g_M) and (N,g_N) Riemannian manifolds,*

$$f:(M,g_M)\to(N,g_N)$$

a Riemannian submersion and $\tilde{R}, \acute{R}$ and $\hat{R}$ be Riemannian curvature tensors, $\tilde{\tau}, \tau'$ and $\hat{\tau}$ be scalar curvature tensors of M,N and the fibre respectively. Then we get the following relations for every $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$.

$$g_M(\tilde{C}(X,Y)Z,H) = g_M(\acute{R}(X,Y)Z,H) - g_M(\mathscr{A}_X H, \mathscr{A}_Y Z) + g_M(\mathscr{A}_X Z, \mathscr{A}_Y H) + 2g_M(\mathscr{A}_X Y, \mathscr{A}_Z H) + \frac{1}{2}\eta(\mathscr{A}_Y Z) g_M(X,H) - \frac{1}{2}\eta(\mathscr{A}_X Z) g_M(Y,H)$$
$$-\frac{\tilde{\tau}}{n(n-1)}[g_M(X,H) g_M(Y,Z) - g_M(Y,H) g_M(X,Z)],$$

$$g_M(\tilde{C}(X,Y)Z,V) = -2g_M(\mathscr{A}_X Y, \mathscr{T}_V Z) + g_M(\tilde{\nabla}_X \mathscr{A}_Y Z, V) - g_M(\tilde{\nabla}_Y \mathscr{A}_X Z, V) + g_M(\mathscr{A}_X \acute{\nabla}_Y Z, V) - g_M(\mathscr{A}_Y \acute{\nabla}_X Z, V) - \frac{1}{2}\eta(Z) g_M([X,Y],V),$$

$$g_M(\tilde{C}(X,Y)V,W) = \frac{1}{2}\left[g_M((\tilde{\nabla}_X \mathscr{A})(Y,V),W) - g_M((\tilde{\nabla}_Y \mathscr{A})(X,V),W) - \eta(V) g_M(\mathscr{A}_X Y,W)\right]$$
$$+ g_M(\tilde{\nabla}_X v\tilde{\nabla}_Y V, W) - g_M(\tilde{\nabla}_Y v\tilde{\nabla}_X V, W) - g_M(\tilde{\nabla}_{[X,Y]} V, W),$$

$$g_M(\tilde{C}(X,V)Y,W) = g_M((\tilde{\nabla}_X \mathscr{T})_V Y, W) - g_M((\tilde{\nabla}_V \mathscr{A})_X Y, W) + g_M(\mathscr{A}_Y W, h\tilde{\nabla}_V X) + g_M(\mathscr{T}_W Y, v\tilde{\nabla}_V X) + \frac{1}{2}\eta(Y) g_M(\tilde{\nabla}_V X, W)$$
$$-\frac{1}{2}\eta(h\tilde{\nabla}_X Y) g_M(V,W) + \frac{\tilde{\tau}}{n(n-1)}\{g(X,Y) g(V,W)\},$$

$$g_M(\tilde{C}(U,V)W,X) = g_M((\tilde{\nabla}_U \mathscr{T})_V W, X) - g_M((\tilde{\nabla}_V \mathscr{T})_U W, X),$$

$$g_M(\tilde{C}(U,V)W,F) = g_M(\hat{R}(U,V)W,F) - g_M(\mathscr{T}_U F, \mathscr{T}_V W) + g_M(\mathscr{T}_V F, \mathscr{T}_U W) + \frac{1}{2}\eta(\mathscr{T}_V W) g_M(U,F) - \frac{1}{2}\eta(\mathscr{T}_U W) g_M(V,F)$$
$$-\frac{\tilde{\tau}}{n(n-1)}[g(V,W) g(U,F) - g(U,W) g(V,F)],$$

$$g_M(\tilde{C}(X,Y)V,H) = g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),H) - g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),H) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_XY,H) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_YX,H) + g_M(\mathscr{T}_VH,[X,Y]),$$

where

$$\widetilde{\tau} = \widehat{\tau} + \tau' \circ f - \sum_i \eta(h\widetilde{\nabla}_{X_i}X_i) + \sum_{i,j}\{2g_M((\widetilde{\nabla}_{X_i}\mathscr{T})_{U_j}X_i,U_j) + 2g_M(\mathscr{A}_{X_i}U_j,h\nabla_{U_j}X_i)$$
$$+2g_M(\mathscr{T}_{U_j}X_i,v\widetilde{\nabla}_{U_j}X_i) + \eta(X_i)g_M(\widetilde{\nabla}_{U_j}X_i,U_j) + 3g_M(\mathscr{A}_{X_i}U_j,\mathscr{A}_{X_i}U_j)\} - \eta(\mathscr{N}).$$

Proof. We can easily show the proof of the first equation in Theorem (3.3) by making an inner production between $\tilde{C}$ and H and using Theorem 2.14 (1) in (2.20). We can similarly perform other proofs. □

Corollary 3.4. *Let $f:(M,g_M)\to(N,g_N)$ be a Riemannian submersion, where (M,g_M) and (N,g_N) are Riemannian manifolds. Then the concircular curvature tensor of the Riemannian submersion has no totally umbilical fibres.*

3.3. Conharmonic curvature tensor

In this section we give curvature relations of concircular curvature tensor in Riemannian submersions endowed with a new type of semi-symmetric non-metric connection.

Theorem 3.5. *Let (M,g_M) and (N,g_N) Riemannian manifolds,*

$$f:(M,g_M)\to(N,g_N)$$

a Riemannian submersion and $\tilde{R},\acute{R}$ and $\hat{R}$ be Riemannian curvature tensors, $\tilde{S},\acute{S}$ and $\hat{S}$ be Ricci tensors of M,N and the fibre respectively. Then for any $U,V,W,F\in\chi^v(M)$ and $X,Y,Z,H\in\chi^h(M)$, we have the following relations

$$\begin{aligned}
g_M(\tilde{L}(X,Y)Z,H) &= g_M(\acute{R}(X,Y)Z,H) - g_M(\mathscr{A}_XH,\mathscr{A}_YZ) + g_M(\mathscr{A}_XZ,\mathscr{A}_YH) + 2g_M(\mathscr{A}_XY,\mathscr{A}_ZH) + \tfrac{1}{2}\eta(\mathscr{A}_YZ)g_M(X,H) - \tfrac{1}{2}\eta(\mathscr{A}_XZ)g_M(Y,H)\\
&-\tfrac{1}{n-2}\{g(Y,Z)[\acute{S}(\acute{X},\acute{H})\circ f - \tfrac{1}{2}\eta(\mathscr{A}_XH) - \tfrac{1}{2}\eta(h\tilde{\nabla}_XH) + \sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}H) + \tfrac{1}{2}\eta(\mathscr{A}_{X_i}H)g_M(X,X_i)\}\\
&+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}H,U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XH,U_j) + g_M(\mathscr{A}_HU_j,h\tilde{\nabla}_{U_j}X) + g_M(\mathscr{T}_{U_j}H,v\tilde{\nabla}_{U_j}X) + \tfrac{1}{2}\eta(H)g_M(\tilde{\nabla}_{U_j}X,U_j)\}]\\
&-g(X,Z)[\acute{S}(\acute{Y},\acute{H})\circ f - \tfrac{1}{2}\eta(\mathscr{A}_YH) - \tfrac{1}{2}\eta(h\tilde{\nabla}_YH) + \sum_i\{3g_M(\mathscr{A}_{X_i}Y,\mathscr{A}_{X_i}H) + \tfrac{1}{2}\eta(\mathscr{A}_{X_i}H)g_M(Y,X_i)\}\\
&+\sum_j g_M((\tilde{\nabla}_Y\mathscr{T})_{U_j}H,U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_YH,U_j) + g_M(\mathscr{A}_HU_j,h\tilde{\nabla}_{U_j}Y) + g_M(\mathscr{T}_{U_j}H,v\tilde{\nabla}_{U_j}Y) + \tfrac{1}{2}\eta(H)g_M(\tilde{\nabla}_{U_j}Y,U_j)\}]\\
&+g_M(X,H)[\acute{S}(\acute{Y},\acute{Z})\circ f - \tfrac{1}{2}\eta(\mathscr{A}_YZ) - \tfrac{1}{2}\eta(h\tilde{\nabla}_YZ) + \sum_i\{3g_M(\mathscr{A}_{X_i}Y,\mathscr{A}_{X_i}Z) + \tfrac{1}{2}\eta(\mathscr{A}_{X_i}Z)g_M(Y,X_i)\}\\
&+\sum_j g_M((\tilde{\nabla}_Y\mathscr{T})_{U_j}Z,U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_YZ,U_j) + g_M(\mathscr{A}_ZU_j,h\tilde{\nabla}_{U_j}Y) + g_M(\mathscr{T}_{U_j}Z,v\tilde{\nabla}_{U_j}Y) + \tfrac{1}{2}\eta(Z)g_M(\tilde{\nabla}_{U_j}Y,U_j)\}]\\
&-g_M(Y,H)[\acute{S}(\acute{X},\acute{Z})\circ f - \tfrac{1}{2}\eta(\mathscr{A}_XZ) - \tfrac{1}{2}\eta(h\tilde{\nabla}_XZ) + \sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}Z) + \tfrac{1}{2}\eta(\mathscr{A}_{X_i}Z)g_M(X,X_i)\}\\
&+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Z,U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XZ,U_j) + g_M(\mathscr{A}_ZU_j,h\tilde{\nabla}_{U_j}X) + g_M(\mathscr{T}_{U_j}Z,v\tilde{\nabla}_{U_j}X) + \tfrac{1}{2}\eta(Z)g_M(\tilde{\nabla}_{U_j}X,U_j)\}]\}
\end{aligned}$$

$$\begin{aligned}
g_M(\acute{L}(X,Y)Z,V) &= -2g_M(\mathscr{A}_XY,\mathscr{T}_VZ) + g_M(\tilde{\nabla}_X\mathscr{A}_YZ,V) - g_M(\tilde{\nabla}_Y\mathscr{A}_XZ,V) + g_M(\mathscr{A}_X\acute{\nabla}_YZ,V) - g_M(\mathscr{A}_Y\acute{\nabla}_XZ,V) - \tfrac{1}{2}\eta(Z)g_M([X,Y],V)\\
&-\tfrac{1}{n-2}\{g_M(Y,Z)[g_M(\tilde{\nabla}_V\mathscr{N},X) + \sum_i\{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i) + \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_XX_i,X_i) - \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}X,X_i)\\
&+g_M(\mathscr{T}_VX_i,[X,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_VU_j,X)\}] - g_M(X,Z)[g_M(\tilde{\nabla}_V\mathscr{N},Y) + \sum_i\{g_M((\tilde{\nabla}_Y\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V),X_i)\\
&+\tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_YX_i,X_i) - \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}Y,X_i) + g_M(\mathscr{T}_VX_i,[Y,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_VU_j,Y)\}]\},
\end{aligned}$$

$$\begin{aligned}
g_M(\tilde{L}(X,Y)V,W) &= \tfrac{1}{2}\left[g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),W) - g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),W) - \eta(V)g_M(\mathscr{A}_XY,W)\right]\\
&+g_M(\tilde{\nabla}_Xv\tilde{\nabla}_YV,W) - g_M(\tilde{\nabla}_Yv\tilde{\nabla}_XV,W) - g_M(\tilde{\nabla}_{[X,Y]}V,W),
\end{aligned}$$

$$\begin{aligned}
g_M(\tilde{L}(X,V)Y,W) &= g_M((\tilde{\nabla}_X\mathscr{T})_VY,W) - g_M((\tilde{\nabla}_V\mathscr{A})_XY,W) + g_M(\mathscr{A}_YW,h\tilde{\nabla}_VX) + g_M(\mathscr{T}_WY,v\tilde{\nabla}_VX) + \tfrac{1}{2}\eta(Y)g_M(\tilde{\nabla}_VX,W) - \tfrac{1}{2}\eta(h\tilde{\nabla}_XY)g_M(V,W)\\
&-\tfrac{1}{n-2}\{-g_M(X,Y)[\hat{S}(V,W) + g_M(\mathscr{N},\mathscr{T}_VW) - \tfrac{1}{2}\eta(\mathscr{T}_VW) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_VX_i,W) + g_M(\mathscr{A}_{X_i}W,h\tilde{\nabla}_VX_i)\\
&+g_M(\mathscr{T}_WX_i,v\tilde{\nabla}_VX_i) + \tfrac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_VX_i,W) - \tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j\{g_M(\mathscr{T}_VU_j,\mathscr{T}_WU_j) - \tfrac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(V,U_j)\}]\\
&-g_M(V,W)[\acute{S}(\acute{X},\acute{Y})\circ f - \tfrac{1}{2}\eta(\mathscr{A}_XY) - \tfrac{1}{2}\eta(h\tilde{\nabla}_XY) + \sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}Y) + \tfrac{1}{2}\eta(\mathscr{A}_{X_i}Y)g_M(X,X_i)\}\\
&+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Y,U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XY,U_j) + g_M(\mathscr{A}_YU_j,h\tilde{\nabla}_{U_j}X) + g_M(\mathscr{T}_{U_j}Y,v\tilde{\nabla}_{U_j}X) + \tfrac{1}{2}\eta(Y)g_M(\tilde{\nabla}_{U_j}X,U_j)]\},
\end{aligned}$$

$$\begin{aligned}
g_M(\tilde{L}(U,V)W,X) &= g_M((\tilde{\nabla}_U\mathscr{T})_VW,X) - g_M((\tilde{\nabla}_V\mathscr{T})_UW,X) - \tfrac{1}{n-2}\{g(V,W)[g_M(\tilde{\nabla}_X\mathscr{N},U) + \sum_i\{g_M((\tilde{\nabla}_U\mathscr{A})(X_i,X),X_i)\\
&-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(U,X),X_i) + \tfrac{1}{2}\eta(X)g_M(\tilde{\nabla}_UX_i,X_i) - \tfrac{1}{2}\eta(X)g_M(\tilde{\nabla}_{X_i}U,X_i) + g_M(\mathscr{T}_XX_i,[U,X_i]\}\\
&-\sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_XU_j,U)\}] - g_M(U,W)[g_M(\tilde{\nabla}_X\mathscr{N},V) + \sum_i\{g_M((\tilde{\nabla}_V\mathscr{A})(X_i,X),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(V,X),X_i)\\
&+\tfrac{1}{2}\eta(X)g_M(\tilde{\nabla}_VX_i,X_i) - \tfrac{1}{2}\eta(X)g_M(\tilde{\nabla}_{Xi}V,X_i) + g_M(\mathscr{T}_XX_i,[V,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_i}\mathscr{T})_XU_j,V)\}]\},
\end{aligned}$$

$$g_M(\tilde{L}(U,V)W,F) = g_M(\hat{R}(U,V)W,F) - g_M(\mathscr{T}_UF,\mathscr{T}_VW) + g_M(\mathscr{T}_VF,\mathscr{T}_UW) + \frac{1}{2}\eta(\mathscr{T}_VW)g_M(U,F) - \frac{1}{2}\eta(\mathscr{T}_UW)g_M(V,F)$$
$$-\frac{1}{n-2}\{g_M(V,W)[\hat{S}(U,F) + g_M(\mathscr{N},\mathscr{T}_UF) - \frac{1}{2}\eta(\mathscr{T}_UF) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_UX_i,F) + g_M(\mathscr{A}_{X_i}F,h\tilde{\nabla}_UX_i)$$
$$+g_M(\mathscr{T}_FX_i,v\tilde{\nabla}_UX_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_UX_i,F) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,F)\} - \sum_j\{g_M(\mathscr{T}_UU_j,\mathscr{T}_FU_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}F)g_M(U,U_j)\}]$$
$$-g_M(U,W)[\hat{S}(V,F) + g_M(\mathscr{N},\mathscr{T}_VF) - \frac{1}{2}\eta(\mathscr{T}_VF) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_VX_i,F) + g_M(\mathscr{A}_{X_i}F,h\tilde{\nabla}_VX_i)$$
$$+g_M(\mathscr{T}_FX_i,v\tilde{\nabla}_VX_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_VX_i,F) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,F)\} - \sum_j\{g_M(\mathscr{T}_VU_j,\mathscr{T}_FU_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}F)g_M(V,U_j)\}]$$
$$+g_M(U,F)[\hat{S}(V,W) + g_M(\mathscr{N},\mathscr{T}_VW) - \frac{1}{2}\eta(\mathscr{T}_VW) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_VX_i,W) + g_M(\mathscr{A}_{X_i}W,h\tilde{\nabla}_VX_i)$$
$$+g_M(\mathscr{T}_WX_i,v\tilde{\nabla}_VX_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_VX_i,W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j\{g_M(\mathscr{T}_VU_j,\mathscr{T}_WU_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(V,U_j)\}]$$
$$-g_M(V,F)[\hat{S}(U,W) + g_M(\mathscr{N},\mathscr{T}_UW) - \frac{1}{2}\eta(\mathscr{T}_UW) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_UX_i,W) + g_M(\mathscr{A}_{X_i}W,h\tilde{\nabla}_UX_i)$$
$$+g_M(\mathscr{T}_WX_i,v\tilde{\nabla}_UX_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_UX_i,W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,W)\} - \sum_j\{g_M(\mathscr{T}_UU_j,\mathscr{T}_WU_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(U,U_j)\}]\},$$

$$g_M(\tilde{L}(X,Y)V,H) = g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),H) - g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),H) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_XY,H) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_YX,H)$$
$$+g_M(\mathscr{T}_VH,[X,Y]) - \frac{1}{n-2}\{g_M(X,H)[g_M(\tilde{\nabla}_V\mathscr{N},Y) + \sum_i\{g_M((\tilde{\nabla}_Y\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V),X_i)$$
$$+\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_YX_i,X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}Y,X_i) + g_M(\mathscr{T}_VX_i,[Y,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_i}\mathscr{T})_VU_j,Y)\}] - g_M(Y,H)[g_M(\tilde{\nabla}_V\mathscr{N},X)$$
$$+\sum_i\{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_XX_i,X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}X,X_i)$$
$$+g_M(\mathscr{T}_VX_i,[X,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_VU_j,X)\}]\}.$$

Proof. We can easily show by making an inner production between $\tilde{L}$ and H and using teorem 2.14 in 2.21. We can do other proofs similarly. □

Corollary 3.6. *Let $f:(M,g_M)\to(N,g_N)$ be a Riemannian submersion, where (M,g_M) and (N,g_N) are Riemann manifolds. If the Riemannian submersion has total umblical fibres ($\mathscr{N}=0$), then the conharmonic curvature tensor is given by*

$$g_M(\acute{L}(X,Y)Z,V) = -2g_M(\mathscr{A}_XY,\mathscr{T}_VZ) + g_M(\tilde{\nabla}_X\mathscr{A}_YZ,V) - g_M(\tilde{\nabla}_Y\mathscr{A}_XZ,V) + g_M(\mathscr{A}_X\acute{\nabla}_YZ,V) - g_M(\mathscr{A}_Y\acute{\nabla}_XZ,V) - \frac{1}{2}\eta(Z)g_M([X,Y],V)$$
$$-\frac{1}{n-2}\{g_M(Y,Z)[\sum_i\{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_XX_i,X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}X,X_i) + g_M(\mathscr{T}_VX_i,[X,X_i]\}$$
$$-\sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_VU_j,X)\}] - g_M(X,Z)[\sum_i\{g_M((\tilde{\nabla}_Y\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V),X_i) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_YX_i,X_i)$$
$$-\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}Y,X_i) + g_M(\mathscr{T}_VX_i,[Y,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_VU_j,Y)\}]\},$$

$$g_M(\tilde{L}(X,V)Y,W) = g_M((\tilde{\nabla}_X\mathscr{T})_VY,W) - g_M((\tilde{\nabla}_V\mathscr{A})_XY,W) + g_M(\mathscr{A}_YW,h\tilde{\nabla}_VX) + g_M(\mathscr{T}_WY,v\tilde{\nabla}_VX) + \frac{1}{2}\eta(Y)g_M(\tilde{\nabla}_VX,W)$$
$$-\frac{1}{2}\eta(h\tilde{\nabla}_XY)g_M(V,W) - \frac{1}{n-2}\{-g_M(X,Y)[\hat{S}(V,W) - \frac{1}{2}\eta(\mathscr{T}_VW) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_VX_i,W) + g_M(\mathscr{A}_{X_i}W,h\tilde{\nabla}_VX_i)$$
$$+g_M(T_WX_i,v\tilde{\nabla}_VX_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_VX_i,W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j\{g_M(\mathscr{T}_VU_j,\mathscr{T}_WU_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(V,U_j)\}]$$
$$-g_M(V,W)[\acute{S}(\acute{X},\acute{Y})\circ f - \frac{1}{2}\eta(\mathscr{A}_XY) - \frac{1}{2}\eta(h\tilde{\nabla}_XY) + \sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}Y) + \frac{1}{2}\eta(\mathscr{A}_{X_i}Y)g_M(X,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Y,U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XY,U_j) + g_M(\mathscr{A}_YU_j,h\tilde{\nabla}_{U_j}X) + g_M(\mathscr{T}_{U_j}Y,v\tilde{\nabla}_{U_j}X) + \frac{1}{2}\eta(Y)g_M(\tilde{\nabla}_{U_j}X,U_j)]\},$$

$$g_M(\tilde{L}(U,V)W,X) = g_M((\tilde{\nabla}_U\mathscr{T})_VW,X) - g_M((\tilde{\nabla}_V\mathscr{T})_UW,X) - \frac{1}{n-2}\{g_M(V,W)[\sum_i\{g_M((\tilde{\nabla}_U\mathscr{A})(X_i,X),X_i)$$
$$-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(U,X),X_i) + \frac{1}{2}\eta(X)g_M(\tilde{\nabla}_UX_i,X_i) - \frac{1}{2}\eta(X)g_M(\tilde{\nabla}_{X_i}U,X_i) + g_M(\mathscr{T}_XX_i,[U,X_i]\}$$
$$-\sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_XU_j,U)\}] - g_M(U,W)[\sum_i\{g_M((\tilde{\nabla}_V\mathscr{A})(X_i,X),V_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(V,X),X_i) + \frac{1}{2}\eta(X)g_M(\tilde{\nabla}_VX_i,X_i)$$
$$-\frac{1}{2}\eta(X)g_M(\tilde{\nabla}_{Xi}V,X_i) + g_M(\mathscr{T}_XX_i,[V,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_i}\mathscr{T})_XU_j,V)\}]\},$$

$$g_M(\tilde{L}(U,V)W,F) = g_M(\hat{R}(U,V)W,F) - g_M(\mathscr{T}_UF,\mathscr{T}_VW) + g_M(\mathscr{T}_VF,\mathscr{T}_UW) + \frac{1}{2}\eta(\mathscr{T}_VW)g_M(U,F) - \frac{1}{2}\eta(\mathscr{T}_UW)g_M(V,F)$$
$$-\frac{1}{n-2}\{g_M(V,W)[\hat{S}(U,F) - \frac{1}{2}\eta(\mathscr{T}_UF) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_UX_i,F) + g_M(\mathscr{A}_{X_i}F,h\tilde{\nabla}_UX_i)$$
$$+g_M(\mathscr{T}_FX_i,v\tilde{\nabla}_UX_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_UX_i,F) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,F)\} - \sum_j\{g_M(\mathscr{T}_UU_j,\mathscr{T}_FU_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}F)g_M(U,U_j)\}]$$
$$-g_M(U,W)[\hat{S}(V,F) - \frac{1}{2}\eta(\mathscr{T}_VF) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_VX_i,F) + g_M(\mathscr{A}_{X_i}F,h\tilde{\nabla}_VX_i)$$
$$+g_M(\mathscr{T}_FX_i,v\tilde{\nabla}_VX_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_VX_i,F) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,F)\} - \sum_j\{g_M(\mathscr{T}_VU_j,\mathscr{T}_FU_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}F)g_M(V,U_j)\}]$$
$$+g_M(U,F)[\hat{S}(V,W) - \frac{1}{2}\eta(\mathscr{T}_VW) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_VX_i,W) + g_M(\mathscr{A}_{X_i}W,h\tilde{\nabla}_VX_i)$$
$$+g_M(\mathscr{T}_WX_i,v\tilde{\nabla}_VX_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_VX_i,W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j\{g_M(\mathscr{T}_VU_j,\mathscr{T}_WU_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(V,U_j)\}]$$
$$-g_M(V,F)[\hat{S}(U,W) - \frac{1}{2}\eta(\mathscr{T}_UW) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_UX_i,W) + g_M(\mathscr{A}_{X_i}W,h\tilde{\nabla}_UX_i) + g_M(\mathscr{T}_WX_i,v\tilde{\nabla}_UX_i)$$
$$+\frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_UX_i,W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,W)\} - \sum_j\{g_M(\mathscr{T}_UU_j,\mathscr{T}_WU_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(U,U_j)\}]\},$$

$$g_M(\tilde{L}(X,Y)V,H)=g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),H)-g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),H)+\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_XY,H)-\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_YX,H)$$
$$+g_M(\mathscr{T}_VH,[X,Y])-\frac{1}{n-2}\{g(X,H)+\sum_i\{g_M((\tilde{\nabla}_Y\mathscr{A})(X_i,V),X_i)-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V),X_i)+\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_YX_i,X_i)$$
$$-\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}Y,X_i)+g_M(\mathscr{T}_VX_i,[Y,X_i]\}-\sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_VU_j,Y)\}]-g_M(Y,H)[\sum_i\{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,V),X_i)$$
$$-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i)+\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_XX_i,X_i)-\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}X,X_i)+g_M(\mathscr{T}_VX_i,[X,X_i]\}-\sum_j\{g_M((\tilde{\nabla}_{Uj}\mathscr{T})_VU_J,X)\}]\}.$$

3.4. Weyl conformal curvature tensor

In this section we give curvature relations of Weyl conformal curvature tensor in Riemannian submersions endowed with a new type of semi-symmetric non-metric connection.

Theorem 3.7. *Let (M,g_M) and (N,g_N) Riemannian manifolds,*

$$f:(M,g_M)\rightarrow(N,g_N)$$

a Riemannian submersion and $\tilde{R}$,$\acute{R}$ and $\hat{R}$ be Riemannian curvature tensors, $\tilde{S}$,$\acute{S}$ and $\hat{S}$ be Ricci tensors and $\widetilde{\tau}$, τ' and $\widehat{\tau}$ of M, N and the fibre respectively. Then for any $U,V,W,F\in\chi^v(M)$ and $X,Y,Z,H\in\chi^h(M)$, we have the following relations

$$g_M(\tilde{V}(X,Y)Z,H)=g_M(\acute{R}(X,Y)Z,H)-g_M(\mathscr{A}_XH,\mathscr{A}_YZ)+g_M(\mathscr{A}_XZ,\mathscr{A}_YH)+2g_M(\mathscr{A}_XY,\mathscr{A}_ZH)+\frac{1}{2}\eta(\mathscr{A}_YZ)g_M(X,H)-\frac{1}{2}\eta(\mathscr{A}_XZ)g_M(Y,H)$$
$$-\frac{1}{n-2}\{g_M(X,H)[\acute{S}(\acute{Y},\acute{Z})\circ f-\frac{1}{2}\eta(\mathscr{A}_YZ)-\frac{1}{2}\eta(h\tilde{\nabla}_YZ)+\sum_i\{3g_M(\mathscr{A}_{X_i}Y,\mathscr{A}_{X_i}Z)+\frac{1}{2}\eta(\mathscr{A}_{X_i}Z)g_M(Y,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_Y\mathscr{T})_{U_j}Z,U_j)-g_M((\tilde{\nabla}_{U_j}\mathscr{A})_YZ,U_j)+g_M(\mathscr{A}_ZU_j,h\tilde{\nabla}_{U_j}Y)+g_M(\mathscr{T}_{U_j}Z,v\tilde{\nabla}_{U_j}Y)+\frac{1}{2}\eta(Z)g_M(\tilde{\nabla}_{U_j}Y,U_j)$$
$$-g_M(Y,H)[\acute{S}(\acute{X},\acute{Z})\circ f-\frac{1}{2}\eta(\mathscr{A}_XZ)-\frac{1}{2}\eta(h\tilde{\nabla}_XZ)+\sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}Z)+\frac{1}{2}\eta(\mathscr{A}_{X_i}Z)g_M(X,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Z,U_j)-g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XZ,U_j)+g_M(\mathscr{A}_ZU_j,h\tilde{\nabla}_{U_j}X)+g_M(\mathscr{T}_{U_j}Z,v\tilde{\nabla}_{U_j}X)+\frac{1}{2}\eta(Z)g_M(\tilde{\nabla}_{U_j}X,U_j)$$
$$+g_M(Y,Z)[\acute{S}(\acute{X},\acute{H})\circ f-\frac{1}{2}\eta(\mathscr{A}_XH)-\frac{1}{2}\eta(h\tilde{\nabla}_XH)+\sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}H)+\frac{1}{2}\eta(\mathscr{A}_{X_i}H)g_M(X,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}H,U_j)-g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XH,U_j)+g_M(\mathscr{A}_HU_j,h\tilde{\nabla}_{U_j}X)+g_M(\mathscr{T}_{U_j}H,v\tilde{\nabla}_{U_j}X)+\frac{1}{2}\eta(H)g_M(\tilde{\nabla}_{U_j}X,U_j)\}$$
$$-g_M(X,Z)[\acute{S}(\acute{Y},\acute{H})\circ f-\frac{1}{2}\eta(\mathscr{A}_YH)-\frac{1}{2}\eta(h\tilde{\nabla}_YH)+\sum_i\{3g_M(\mathscr{A}_{X_i}Y,\mathscr{A}_{X_i}H)+\frac{1}{2}\eta(\mathscr{A}_{X_i}H)g_M(Y,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_Y\mathscr{T})_{U_j}H,U_j)-g_M((\tilde{\nabla}_{U_j}\mathscr{A})_YH,U_j)+g_M(\mathscr{A}_HU_j,h\tilde{\nabla}_{U_j}Y)+g_M(\mathscr{T}_{U_j}H,v\tilde{\nabla}_{U_j}Y)+\frac{1}{2}\eta(H)g_M(\tilde{\nabla}_{U_j}Y,U_j)$$
$$+\frac{\widetilde{\tau}^M}{(n-1)(n-2)}\{g_M(X,H)g_M(Y,Z)-g_M(Y,H)g_M(X,Z)\},$$

$$g_M(\tilde{V}(X,Y)Z,V)=-2g_M(\mathscr{A}_XY,\mathscr{T}_VZ)+g_M(\tilde{\nabla}_X\mathscr{A}_YZ,V)-g_M(\tilde{\nabla}_Y\mathscr{A}_XZ,V)+g_M(\mathscr{A}_X\acute{\nabla}_YZ,V)-g_M(\mathscr{A}_Y\acute{\nabla}_XZ,V)-\frac{1}{2}\eta(Z)g_M([X,Y],V)$$
$$-\frac{1}{n-2}\{g_M(Y,Z)[g_M(\tilde{\nabla}_V\mathscr{N},X)+\sum_i\{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,V),X_i)-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i)+\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_XX_i,X_i)-\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}X,X_i)$$
$$+g_M(\mathscr{T}_VX_i,[X,X_i]\}-\sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_VU_j,X)\}]-g_M(X,Z)[g_M(\tilde{\nabla}_V\mathscr{N},Y)+\sum_i\{g_M((\tilde{\nabla}_Y\mathscr{A})(X_i,V),X_i)-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V),X_i)$$
$$+\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_YX_i,X_i)-\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}Y,X_i)+g_M(\mathscr{T}_VX_i,[Y,X_i]\}-\sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_VU_j,Y)\}],$$

$$g_M(\tilde{V}(X,Y)V,W)=\frac{1}{2}\left[g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),W)-g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),W)-\eta(V)g_M(\mathscr{A}_XY,W)\right]$$
$$+g_M(\tilde{\nabla}_Xv\tilde{\nabla}_YV,W)-g_M(\tilde{\nabla}_Yv\tilde{\nabla}_XV,W)-g_M(\tilde{\nabla}_{[X,Y]}V,W),$$

$$g_M(\tilde{V}(X,V)Y,W)=g_M((\tilde{\nabla}_X\mathscr{T})_VY,W)-g_M((\tilde{\nabla}_V\mathscr{A})_XY,W)+g_M(\mathscr{A}_YW,h\tilde{\nabla}_VX)+g_M(\mathscr{T}_WY,v\tilde{\nabla}_VX)+\frac{1}{2}\eta(Y)g_M(\tilde{\nabla}_VX,W)$$
$$-\frac{1}{2}\eta(h\tilde{\nabla}_XY)g_M(V,W)-\frac{1}{n-2}\{-g(V,W)[\acute{S}(\acute{X},\acute{Y})\circ f-\frac{1}{2}\eta(\mathscr{A}_XY)-\frac{1}{2}\eta(h\tilde{\nabla}_XY)+\sum_i\{3g_M(\mathscr{A}_{X_i}X,\mathscr{A}_{X_i}Y)+$$
$$\frac{1}{2}\eta(\mathscr{A}_{X_i}Y)g_M(X,X_i)\}+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Y,U_j)-g_M((\tilde{\nabla}_{U_j}\mathscr{A})_XY,U_j)+g_M(\mathscr{A}_YU_j,h\tilde{\nabla}_{U_j}X)+g_M(\mathscr{T}_{U_j}Y,v\tilde{\nabla}_{U_j}X)$$
$$+\frac{1}{2}\eta(Y)g_M(\tilde{\nabla}_{U_j}X,U_j)]-g(X,Y)[\hat{S}(V,W)+g_M(\mathscr{N},\mathscr{T}_VW)-\frac{1}{2}\eta(\mathscr{T}_VW)+\sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_VX_i,W)$$
$$+g_M(\mathscr{A}_{X_i}W,h\tilde{\nabla}_VX_i)+g_M(\mathscr{T}_WX_i,v\tilde{\nabla}_VX_i)+\frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_VX_i,W)-\frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\}$$
$$-\sum_j\{g_M(\mathscr{T}_VU_j,\mathscr{T}_WU_j)-\frac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(V,U_j)\}]\}-\frac{\widetilde{\tau}^M}{(n-1)(n-2)}\{g_M(X,Y)g_M(V,W)\},$$

$$g_M(\tilde{V}(U,V)W,X)=g_M((\tilde{\nabla}_U\mathscr{T})_VW,X)-g_M((\tilde{\nabla}_V\mathscr{T})_UW,X)-\frac{1}{n-2}\{g_M(V,W)[g_M(\tilde{\nabla}_U\mathscr{N},X)$$
$$+\sum_i\{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,U),X_i)-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,U),X_i)+\frac{1}{2}\eta(U)g_M(\tilde{\nabla}_XX_i,X_i)-\frac{1}{2}\eta(U)g_M(\tilde{\nabla}_{Xi}X,X_i)+g_M(\mathscr{T}_UX_i,[X,X_i]\}$$
$$-\sum_j\{g_M((\tilde{\nabla}_{Uj}\mathscr{T})_UU_j,X)\}]-g_M(U,W)[g_M(\tilde{\nabla}_X\mathscr{N},V)+\sum_i\{g_M((\tilde{\nabla}_V\mathscr{A})(X_i,X),X_i)-g_M((\tilde{\nabla}_{X_i}\mathscr{A})(V,X),X_i)$$
$$+\frac{1}{2}\eta(X)g_M(\tilde{\nabla}_VX_i,X_i)-\frac{1}{2}\eta(X)g_M(\tilde{\nabla}_{X_i}V,X_i)+g_M(\mathscr{T}_XX_i,[V,X_i]\}-\sum_j\{g_M((\tilde{\nabla}_{Uj}\mathscr{T})_XU_J,V)\}]\},$$

$$g_M(\tilde{V}(U,V)W,F) = g_M(\hat{R}(U,V)W,F) - g_M(\mathscr{T}_U F, \mathscr{T}_V W) + g_M(\mathscr{T}_V F, \mathscr{T}_U W) + \tfrac{1}{2}\eta(\mathscr{T}_V W)g_M(U,F) - \tfrac{1}{2}\eta(\mathscr{T}_U W)g_M(V,F)$$
$$-\tfrac{1}{n-2}\{g_M(U,F)[\hat{S}(V,W) + g_M(\mathscr{N}, \mathscr{T}_V W) - \tfrac{1}{2}\eta(\mathscr{T}_V W)$$
$$+\sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i, W) + g_M(\mathscr{A}_{X_i}W, h\tilde{\nabla}_V X_i) + g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_V X_i)$$
$$+\tfrac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_V X_i, W) - \tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j \{g_M(\mathscr{T}_V U_j, \mathscr{T}_W U_j)$$
$$-\tfrac{1}{2}\eta(\mathscr{T}_{V_j}W)g_M(V,U_j)\} - g_M(V,F)[\hat{S}(U,W) + g_M(\mathscr{N}, \mathscr{T}_U W) - \tfrac{1}{2}\eta(\mathscr{T}_U W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_U X_i, W) + g_M(\mathscr{A}_{X_i}W, h\tilde{\nabla}_U X_i)$$
$$+g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_U X_i) + \tfrac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_U X_i, W) - \tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,W)\} - \sum_j \{g_M(\mathscr{T}_U U_j, \mathscr{T}_W U_j)$$
$$-\tfrac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(U,W_j)\}] + g_M(V,W)[\hat{S}(U,F) + g_M(\mathscr{N}, \mathscr{T}_U F) - \tfrac{1}{2}\eta(\mathscr{T}_U F)$$
$$+\sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_U X_i, F) + g_M(\mathscr{A}_{X_i}F, h\tilde{\nabla}_U X_i) + g_M(\mathscr{T}_F X_i, v\tilde{\nabla}_U X_i) + \tfrac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_U X_i, F) - \tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,F)\} - \sum_j \{g_M(\mathscr{T}_U U_j, \mathscr{T}_F U_j)$$
$$-\tfrac{1}{2}\eta(\mathscr{T}_{U_j}F)g_M(U,F_j)\}] - g_M(U,W)[\hat{S}(V,F) + g_M(\mathscr{N}, \mathscr{T}_V F) - \tfrac{1}{2}\eta(\mathscr{T}_V F) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i, F) + g_M(\mathscr{A}_{X_i}F, h\tilde{\nabla}_V X_i) + g_M(\mathscr{T}_F X_i, v\tilde{\nabla}_V X_i)$$
$$+\tfrac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_V X_i, F) - \tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,F)\} - \sum_j \{g_M(\mathscr{T}_V V_j, \mathscr{T}_F V_j) - \tfrac{1}{2}\eta(\mathscr{T}_{V_j}F)g_M(V,F_j)\}]\}$$
$$+\tfrac{\tilde{\tau}^M}{(n-1)(n-2)}[g_M(U,F)g_M(V,W) - g_M(V,F)g_M(U,W)],$$

$$g_M(\tilde{V}(X,Y)V,H) = g_M((\tilde{\nabla}_X A)(Y,V),H) - g_M((\tilde{\nabla}_Y A)(X,V),H) + \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_X Y,H) - \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X,H)$$
$$+g_M(T_V H,[X,Y]) - \tfrac{1}{n-2}\{g_M(X,H)[g_M(\tilde{\nabla}_V \mathscr{N},Y) + \sum_i \{g_M((\tilde{\nabla}_Y A)(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}A)(Y,V),X_i)$$
$$+\tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X_i,X_i) - \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}Y,X_i) + g_M(T_V X_i,[Y,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}T)_V U_j,Y)\} - g_M(Y,H)[g_M(\tilde{\nabla}_V \mathscr{N},X)$$
$$+\sum_i \{g_M((\tilde{\nabla}_X A)(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}A)(X,V),X_i) + \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_X X_i,X_i) - \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}X,X_i) + g_M(T_V X_i,[X,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}T)_V U_j,X)\}],$$

where

$$\widetilde{\tau} = \widehat{\tau} + \tau' \circ f - \sum_i \eta(h\widetilde{\nabla}_{X_i}X_i) + \sum_{i,j}\{2g_M((\widetilde{\nabla}_{X_i}\mathscr{T})_{U_j}X_i,U_j) + 2g_M(\mathscr{A}_{X_i}U_j, h\nabla_{U_j}X_i)$$
$$+2g_M(\mathscr{T}_{U_j}X_i, v\widetilde{\nabla}_{U_j}X_i) + \eta(X_i)g_M(\widetilde{\nabla}_{U_j}X_i,U_j) + 3g_M(\mathscr{A}_{X_i}U_j, \mathscr{A}_{X_i}U_j)\} - \eta(\mathscr{N}).$$

Proof. We can easily show by making an inner production between $\tilde{V}$ and H and using teorem 2.14 in 2.22. We can do other proofs similarly. ☐

Corollary 3.8. *Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion, where (M, g_M) and (N, g_N) are Riemannian manifolds. If the Riemannian submersion has total umblical fibres ($\mathscr{N} = 0$), then the weyl conformal curvature tensor is given by*

$$g_M(\tilde{V}(X,Y)Z,V) = -2g_M(\mathscr{A}_X Y, \mathscr{T}_V Z) + g_M(\tilde{\nabla}_X \mathscr{A}_Y Z,V) - g_M(\tilde{\nabla}_Y \mathscr{A}_X Z,V) + g_M(\mathscr{A}_X \acute{\nabla}_Y Z,V) - g_M(\mathscr{A}_Y \acute{\nabla}_X Z,V) - \tfrac{1}{2}\eta(Z)g_M([X,Y],V)$$
$$+\sum_i \{g_M((\tilde{\nabla}_X \mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i) + \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_X X_i,X_i) - \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}X,X_i) + g_M(\mathscr{T}_V X_i,[X,X_i]\}$$
$$-\sum_j \{g_M((\tilde{\nabla}_{U_i}\mathscr{T})_V U_j,X)\}] - g_M(X,Z)[\sum_i \{g_M((\tilde{\nabla}_Y \mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V),X_i)$$
$$+\tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X_i,X_i) - \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}Y,X_i) + g_M(\mathscr{T}_V X_i,[Y,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j,Y)\}],$$

$$g_M(\tilde{V}(X,V)Y,W) = g_M((\tilde{\nabla}_X \mathscr{T})_V Y,W) - g_M((\tilde{\nabla}_V \mathscr{A})_X Y,W) + g_M(\mathscr{A}_Y W, h\tilde{\nabla}_V X) + g_M(\mathscr{T}_W Y, v\tilde{\nabla}_V X) + \tfrac{1}{2}\eta(Y)g_M(\tilde{\nabla}_V X,W) - \tfrac{1}{2}\eta(h\tilde{\nabla}_X Y)g_M(V,W)$$
$$-\tfrac{1}{n-2}\{-g(V,W)[\acute{S}(\acute{X},\acute{Y}) \circ f - \tfrac{1}{2}\eta(\mathscr{A}_X Y) - \tfrac{1}{2}\eta(h\tilde{\nabla}_X Y) + \sum_i \{3g_M(\mathscr{A}_{X_i}X, \mathscr{A}_{X_i}Y) + \tfrac{1}{2}\eta(\mathscr{A}_{X_i}Y)g_M(X,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_X \mathscr{T})_{U_j}Y,U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_X Y,U_j) + g_M(\mathscr{A}_Y U_j, h\tilde{\nabla}_{U_j}X) + g_M(\mathscr{T}_{U_j}Y, v\tilde{\nabla}_{U_j}X) + \tfrac{1}{2}\eta(Y)g_M(\tilde{\nabla}_{U_j}X,U_j)]$$
$$-g(X,Y)[\hat{S}(V,W) - \tfrac{1}{2}\eta(T_V W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i,W) + g_M(\mathscr{A}_{X_i}W, h\tilde{\nabla}_V X_i) + g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_V X_i) + \tfrac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_V X_i,W)$$
$$-\tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j \{g_M(\mathscr{T}_V U_j, \mathscr{T}_W U_j) - \tfrac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(V,U_j)\}] + \tfrac{\tilde{\tau}^M}{(n-1)(n-2)}\{g_M(X,Y)g_M(V,W)\},$$

$$g_M(\tilde{V}(U,V)W,X) = g_M((\tilde{\nabla}_U \mathscr{T})_V W,X) - g_M((\tilde{\nabla}_V \mathscr{T})_U W,X) - \tfrac{1}{n-2}\{g(V,W)[\sum_i \{g_M((\tilde{\nabla}_X \mathscr{A})(X_i,U),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,U),X_i)$$
$$+\tfrac{1}{2}\eta(U)g_M(\tilde{\nabla}_X X_i,X_i) - \tfrac{1}{2}\eta(U)g_M(\tilde{\nabla}_{Xi}X,X_i) + g_M(\mathscr{T}_U X_i,[X,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_U U_j,X)\}]$$
$$-g_M(U,W)[\sum_i \{g_M((\tilde{\nabla}_X \mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i) + \tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_X X_i,X_i)$$
$$-\tfrac{1}{2}\eta(V)g_M(\tilde{\nabla}_{Xi}X,X_i) + g_M(\mathscr{T}_V X_i,[X,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j,X)\}]\},$$

$$g_M(\tilde{V}(U,V)W,F) = g_M(\hat{R}(U,V)W,F) - g_M(\mathscr{T}_U F, \mathscr{T}_V W) + g_M(\mathscr{T}_V F, \mathscr{T}_U W) + \frac{1}{2}\eta(\mathscr{T}_V W)g_M(U,F) - \frac{1}{2}\eta(\mathscr{T}_U W)g_M(V,F)$$
$$-\frac{1}{n-2}\{g_M(U,F)[\hat{S}(V,W) - \frac{1}{2}\eta(\mathscr{T}_V W) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i, W) + g_M(\mathscr{A}_{X_i}W, h\tilde{\nabla}_V X_i) + g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_V X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_V X_i, W)$$
$$-\frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j\{g_M(\mathscr{T}_V U_j, \mathscr{T}_W U_j) - \frac{1}{2}\eta(\mathscr{T}_{V_j}W)g_M(V,U_j)\} - g_M(V,F)[\hat{S}(U,W) - \frac{1}{2}\eta(\mathscr{T}_U W)$$
$$+\sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_U X_i, W) + g_M(\mathscr{A}_{X_i}W, h\tilde{\nabla}_U X_i) + g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_U X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_U X_i, W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,W)\}$$
$$-\sum_j\{g_M(\mathscr{T}_U U_j, \mathscr{T}_W U_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(U,W_j)\}] + g_M(V,W)[\hat{S}(U,F) - \frac{1}{2}\eta(\mathscr{T}_U F) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_U X_i, F)$$
$$+g_M(\mathscr{A}_{X_i}F, h\tilde{\nabla}_U X_i) + g_M(\mathscr{T}_F X_i, v\tilde{\nabla}_U X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_U X_i, F) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,F)\} - \sum_j\{g_M(\mathscr{T}_U U_j, \mathscr{T}_F U_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}F)g_M(U,F_j)\}]$$
$$-g_M(U,W)[\hat{S}(V,F) - \frac{1}{2}\eta(\mathscr{T}_V F) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i, F) + g_M(\mathscr{A}_{X_i}F, h\tilde{\nabla}_V X_i) + g_M(\mathscr{T}_F X_i, v\tilde{\nabla}_V X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_V X_i, F)$$
$$-\frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,F)\} - \sum_j\{g_M(\mathscr{T}_V V_j, \mathscr{T}_F V_j) - \frac{1}{2}\eta(\mathscr{T}_{V_j}F)g_M(V,F_j)\}]\} + \frac{\tilde{\tau}^M}{(n-1)(n-2)}[g_M(U,F)g_M(V,W) - g_M(V,F)g_M(U,W)],$$

$$g_M(\tilde{V}(X,Y)V,H) = g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),H) - g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),H) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_X Y,H) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X,H) + g_M(\mathscr{T}_V H,[X,Y])$$
$$-\frac{1}{n-2}\{g_M(X,H) + \sum_i\{g_M((\tilde{\nabla}_Y\mathscr{A})(Y_i,V),Y_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V),X_i) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X_i,X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}Y,X_i) + g_M(\mathscr{T}_V X_i,[Y,X_i]\}$$
$$-\sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j, Y)\} - g_M(Y,H) + \sum_i\{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_X X_i,X_i)$$
$$-\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}X,X_i) + g_M(\mathscr{T}_V X_i,[X,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j, X)\}].$$

3.5. M-projective curvature tensor

In this section we give curvature relations of M-projective curvature tensor in Riemannian submersions endowed with a new type of semi-symmetric non-metric connection.

$$g_M(\tilde{W}(X,Y)Z,H) = g_M(\acute{R}(X,Y)Z,H) - g_M(\mathscr{A}_X H, \mathscr{A}_Y Z) + g_M(\mathscr{A}_X Z, \mathscr{A}_Y H) + 2g_M(\mathscr{A}_X Y, \mathscr{A}_Z H) + \frac{1}{2}\eta(\mathscr{A}_Y Z)g_M(X,H) - \frac{1}{2}\eta(\mathscr{A}_X Z)g_M(Y,H)$$
$$-\frac{1}{2(n-1)}\{g(X,H)[\acute{S}(\acute{Y},\acute{Z})\circ f - \frac{1}{2}\eta(\mathscr{A}_Y Z) - \frac{1}{2}\eta(h\tilde{\nabla}_Y Z) + \sum_i\{3g_M(\mathscr{A}_{X_i}Y, \mathscr{A}_{X_i}Z) + \frac{1}{2}\eta(\mathscr{A}_{X_i}Z)g_M(Y,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_Y\mathscr{T})_{U_j}Z, U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_Y Z, U_j) + g_M(\mathscr{A}_Z U_j, h\tilde{\nabla}_{U_j}Y) + g_M(\mathscr{T}_{U_j}Z, v\tilde{\nabla}_{U_j}Y) + \frac{1}{2}\eta(Z)g_M(\tilde{\nabla}_{U_j}Y, U_j)]$$
$$-g_M(Y,H)[\acute{S}(\acute{X},\acute{Z})\circ f - \frac{1}{2}\eta(\mathscr{A}_X Z) - \frac{1}{2}\eta(h\tilde{\nabla}_X Z) + \sum_i\{3g_M(\mathscr{A}_{X_i}X, \mathscr{A}_{X_i}Z) + \frac{1}{2}\eta(\mathscr{A}_{X_i}Z)g_M(X,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Z, U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_X Z, U_j) + g_M(\mathscr{A}_Z U_j, h\tilde{\nabla}_{U_j}X)$$
$$+g_M(\mathscr{T}_{U_j}Z, v\tilde{\nabla}_{U_j}X) + \frac{1}{2}\eta(Z)g_M(\tilde{\nabla}_{U_j}X, U_j)] + g_M(Y,Z)[\acute{S}(\acute{X},\acute{H})\circ f - \frac{1}{2}\eta(\mathscr{A}_X H) - \frac{1}{2}\eta(h\tilde{\nabla}_X H)$$
$$+\sum_i\{3g_M(\mathscr{A}_{X_i}X, \mathscr{A}_{X_i}H) + \frac{1}{2}\eta(\mathscr{A}_{X_i}H)g_M(X,X_i)\} + \sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}H, U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_X H, U_j) + g_M(\mathscr{A}_H U_j, h\tilde{\nabla}_{U_j}X)$$
$$+g_M(\mathscr{T}_{U_j}H, v\tilde{\nabla}_{U_j}X) + \frac{1}{2}\eta(H)g_M(\tilde{\nabla}_{U_j}X, U_j)] - g_M(X,Z)[\acute{S}(\acute{Y},\acute{H})\circ f - \frac{1}{2}\eta(\mathscr{A}_Y H) - \frac{1}{2}\eta(h\tilde{\nabla}_Y H)$$
$$+\sum_i\{3g_M(\mathscr{A}_{X_i}Y, \mathscr{A}_{X_i}H) + \frac{1}{2}\eta(\mathscr{A}_{X_i}H)g_M(Y,X_i)\} + \sum_j g_M((\tilde{\nabla}_Y\mathscr{T})_{U_j}H, U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_Y H, U_j) + g_M(\mathscr{A}_H U_j, h\tilde{\nabla}_{U_j}Y)$$
$$+g_M(\mathscr{T}_{U_j}H, v\tilde{\nabla}_{U_j}Y) + \frac{1}{2}\eta(H)g_M(\tilde{\nabla}_{U_j}Y, U_j)]$$

$$g_M(\tilde{W}(X,Y)Z,V) = -2g_M(\mathscr{A}_X Y, \mathscr{T}_V Z) + g_M(\tilde{\nabla}_X\mathscr{A}_Y Z, V) - g_M(\tilde{\nabla}_Y\mathscr{A}_X Z, V) + g_M(\mathscr{A}_X\acute{\nabla}_Y Z, V) - g_M(\mathscr{A}_Y\acute{\nabla}_X Z, V) - \frac{1}{2}\eta(Z)g_M([X,Y],V)$$
$$-\frac{1}{2(n-1)}\{g_M(Y,Z)[g_M(\tilde{\nabla}_V\mathscr{N}, X) + \sum_i\{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i)$$
$$+\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_X X_i, X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}X, X_i) + g_M(\mathscr{T}_V X_i,[X,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j, X)\}]$$
$$-g_M(X,Z)[g_M(\tilde{\nabla}_V\mathscr{N}, Y) + \sum_i\{g_M((\tilde{\nabla}_Y\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V),X_i)$$
$$+\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X_i, X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}Y, X_i) + g_M(\mathscr{T}_V X_i,[Y,X_i]\} - \sum_j\{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j, Y)\}]$$

$$g_M(\tilde{W}(X,Y)V,W) = \frac{1}{2}\left[g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),W) - g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),W) - \eta(V)g_M(\mathscr{A}_X Y,W)\right]$$
$$+g_M(\tilde{\nabla}_X v\tilde{\nabla}_Y V,W) - g_M(\tilde{\nabla}_Y v\tilde{\nabla}_X V,W) - g_M(\tilde{\nabla}_{[X,Y]}V,W),$$

$$g(\tilde{W}(X,V)Y,W) = g_M((\tilde{\nabla}_X\mathscr{T})_V Y,W) - g_M((\tilde{\nabla}_V\mathscr{A})_X Y,W) + g_M(\mathscr{A}_Y W, h\tilde{\nabla}_V X) + g_M(\mathscr{T}_W Y, v\tilde{\nabla}_V X) + \frac{1}{2}\eta(Y)g_M(\tilde{\nabla}_V X,W)$$
$$-\frac{1}{2}\eta(h\tilde{\nabla}_X Y)g_M(V,W) - \frac{1}{2(n-1)}\{-g(V,W)[\acute{S}(\acute{X},\acute{Y})\circ f - \frac{1}{2}\eta(\mathscr{A}_X Y) - \frac{1}{2}\eta(h\tilde{\nabla}_X Y) + \sum_i\{3g_M(\mathscr{A}_{X_i}X, \mathscr{A}_{X_i}Y) + \frac{1}{2}\eta(\mathscr{A}_{X_i}Y)g_M(X,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_X\mathscr{T})_{U_j}Y, U_j) - g_M((\tilde{\nabla}_{U_j}\mathscr{A})_X Y, U_j) + g_M(\mathscr{A}_Y U_j, h\tilde{\nabla}_{U_j}X) + g_M(\mathscr{T}_{U_j}Y, v\tilde{\nabla}_{U_j}X) + \frac{1}{2}\eta(Y)g_M(\tilde{\nabla}_{U_j}X, U_j)]$$
$$-g_M(X,Y)[\hat{S}(V,W) + g_M(\mathscr{N}, \mathscr{T}_V W) - \frac{1}{2}\eta(\mathscr{T}_V W) + \sum_i\{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i, W) + g_M(\mathscr{A}_{X_i}W, h\tilde{\nabla}_V X_i)$$
$$+g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_V X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_V X_i, W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j\{g_M(\mathscr{T}_V U_j, \mathscr{T}_W U_j) - \frac{1}{2}\eta(\mathscr{T}_{U_j}W)g_M(V,U_j)\}]\},$$

$$g_M(\tilde{W}(U,V)W,X) = g_M((\tilde{\nabla}_U \mathcal{T})_V W,X) - g_M((\tilde{\nabla}_V \mathcal{T})_U W,X) - \frac{1}{2(n-1)}\{g_M(V,W)[g_M(\tilde{\nabla}_X \mathcal{N},U) + \sum_i \{g_M((\tilde{\nabla}_U \mathcal{A})(X_i,X),X_i)$$
$$-g_M((\tilde{\nabla}_{X_i}\mathcal{A})(U,X),X_i) + \frac{1}{2}\eta(X)g_M(\tilde{\nabla}_U X_i,X_i) - \frac{1}{2}\eta(X)g_M(\tilde{\nabla}_{X_i}U,X_i) + g_M(\mathcal{T}_X X_i,[U,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathcal{T})_X U_j,U)\}]$$
$$-g_M(U,W)[g_M(\tilde{\nabla}_X \mathcal{N},V) + \sum_i \{g_M((\tilde{\nabla}_V \mathcal{A})(X_i,X),X_i) - g_M((\tilde{\nabla}_{X_i}\mathcal{A})(V,X),X_i)$$
$$+\frac{1}{2}\eta(X)g_M(\tilde{\nabla}_V X_i,X_i) - \frac{1}{2}\eta(X)g_M(\tilde{\nabla}_{X_i}V,X_i) + g_M(\mathcal{T}_X X_i,[V,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathcal{T})_X U_j,V)\}]\}$$

$$g_M(\tilde{W}(U,V)W,F) = g_M(\hat{R}(U,V)W,F) - g_M(\mathcal{T}_U F,\mathcal{T}_V W) + g_M(\mathcal{T}_V F,\mathcal{T}_U W) + \frac{1}{2}\eta(\mathcal{T}_V W)g_M(U,F) - \frac{1}{2}\eta(\mathcal{T}_U W)g_M(V,F)$$
$$-\frac{1}{2(n-1)}\{g(U,F)[\hat{S}(V,W) + g_M(\mathcal{N},\mathcal{T}_V W) - \frac{1}{2}\eta(\mathcal{T}_V W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathcal{T})_V X_i,W) + g_M(\mathcal{A}_{X_i}W,h\tilde{\nabla}_V X_i)$$
$$+g_M(\mathcal{T}_W X_i,v\tilde{\nabla}_V X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_V X_i,W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j \{g_M(\mathcal{T}_V U_j,\mathcal{T}_W U_j) - \frac{1}{2}\eta(\mathcal{T}_{U_j}W)g_M(V,U_j)\}]$$
$$-g_M(V,F)[\hat{S}(U,W) + g_M(\mathcal{N},\mathcal{T}_U W) - \frac{1}{2}\eta(\mathcal{T}_U W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathcal{T})_U X_i,W) + g_M(\mathcal{A}_{X_i}W,h\tilde{\nabla}_U X_i)$$
$$+g_M(\mathcal{T}_W X_i,v\tilde{\nabla}_U X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_U X_i,W) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,W)\} - \sum_j \{g_M(\mathcal{T}_U U_j,\mathcal{T}_W U_j) - \frac{1}{2}\eta(\mathcal{T}_{U_j}W)g_M(U,U_j)\}]$$
$$+g(V,W)[\hat{S}(U,F) + g_M(\mathcal{N},\mathcal{T}_U F) - \frac{1}{2}\eta(\mathcal{T}_U F) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathcal{T})_U X_i,F) + g_M(\mathcal{A}_{X_i}F,h\tilde{\nabla}_U X_i)$$
$$+g_M(\mathcal{T}_F X_i,v\tilde{\nabla}_U X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_U X_i,F) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,F)\} - \sum_j \{g_M(\mathcal{T}_U U_j,\mathcal{T}_F U_j) - \frac{1}{2}\eta(\mathcal{T}_{U_j}F)g_M(U,F_j)\}]$$
$$-g_M(U,W)[\hat{S}(V,F) + g_M(\mathcal{N},\mathcal{T}_V F) - \frac{1}{2}\eta(\mathcal{T}_V F) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathcal{T})_V X_i,F) + g_M(\mathcal{A}_{X_i}F,h\tilde{\nabla}_V X_i)$$
$$+g_M(\mathcal{T}_F X_i,v\tilde{\nabla}_V X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_V X_i,F) - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,F)\} - \sum_j \{g_M(\mathcal{T}_V U_j,\mathcal{T}_F U_j) - \frac{1}{2}\eta(\mathcal{T}_{U_j}F)g_M(V,U_j)\}]\},$$

$$g_M(\tilde{W}(X,Y)V,H) = g_M((\tilde{\nabla}_X \mathcal{A})(Y,V),H) - g_M((\tilde{\nabla}_Y \mathcal{A})(X,V),H) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_X Y,H) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X,H) + g_M(\mathcal{T}_V H,[X,Y])$$
$$-\frac{1}{2(n-1)}\{g(X,H)[g_M(\tilde{\nabla}_V \mathcal{N},Y) + \sum_i \{g_M((\tilde{\nabla}_Y \mathcal{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathcal{A})(Y,V),X_i) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X_i,X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}Y,X_i)$$
$$+g_M(\mathcal{T}_V X_i,[Y,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathcal{T})_V U_j,Y)\} - g_M(Y,H)[g_M(\tilde{\nabla}_V \mathcal{N},X) + \sum_i \{g_M((\tilde{\nabla}_X \mathcal{A})(X_i,V),X_i)$$
$$-g_M((\tilde{\nabla}_{X_i}\mathcal{A})(X,V),X_i) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_X X_i,X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}X,X_i) + g_M(\mathcal{T}_V X_i,[X,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathcal{T})_V U_j,X)\}]\}$$

Proof. We can easily show by making an inner production between $\tilde{W}$ and H and using teorem 2.14 in 2.23. We can do other proofs similarly. □

Corollary 3.9. *Let $f:(M,g_M)$ and (N,g_N) be a Riemannian submersion, where (M,g_M) and (N,g_N) are Riemannian manifolds. If the Riemannian submersion has total umblical fibres ($\mathcal{N}=0$), then the $M-$projective curvature tensor is given by*

$$g(\tilde{W}(X,Y)Z,V) = -2g_M(\mathcal{A}_X Y,\mathcal{T}_V Z) + g_M(\tilde{\nabla}_X \mathcal{A}_Y Z,V) - g_M(\tilde{\nabla}_Y \mathcal{A}_X Z,V) + g_M(\mathcal{A}_X \acute{\nabla}_Y Z,V) - g_M(\mathcal{A}_Y \acute{\nabla}_X Z,V) - \frac{1}{2}\eta(Z)g_M([X,Y],V)$$
$$-\frac{1}{2(n-1)}\{g_M(Y,Z)[\sum_i \{g_M((\tilde{\nabla}_X \mathcal{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathcal{A})(X,V),X_i) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_X X_i,X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}X,X_i) + g_M(\mathcal{T}_V X_i,[X,X_i]\}$$
$$-\sum_j \{g_M((\tilde{\nabla}_{U_j}\mathcal{T})_V U_j,X)\}] - g_M(X,Z)[\sum_i \{g_M((\tilde{\nabla}_Y \mathcal{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathcal{A})(Y,V),X_i)$$
$$+\frac{1}{2}\eta(V)g_M(\tilde{\nabla}_Y X_i,X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}Y,X_i) + g_M(\mathcal{T}_V X_i,[Y,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathcal{T})_V U_j,Y)\}]$$

$$g_M(\tilde{W}(X,V)Y,W) = g_M((\tilde{\nabla}_X \mathcal{T})_V Y,W) - g_M((\tilde{\nabla}_V \mathcal{A})_X Y,W) + g_M(\mathcal{A}_Y W,h\tilde{\nabla}_V X) + g_M(\mathcal{T}_W Y,v\tilde{\nabla}_V X) + \frac{1}{2}\eta(Y)g_M(\tilde{\nabla}_V X,W)$$
$$-\frac{1}{2}\eta(h\tilde{\nabla}_X Y)g_M(V,W) - \frac{1}{2(n-1)}\{-g_M(V,W)[\acute{S}(\acute{X},\acute{Y})\circ f - \frac{1}{2}\eta(\mathcal{A}_X Y) - \frac{1}{2}\eta(h\tilde{\nabla}_X Y) + \sum_i \{3g_M(\mathcal{A}_{X_i}X,\mathcal{A}_{X_i}Y) + \frac{1}{2}\eta(\mathcal{A}_{X_i}Y)g_M(X,X_i)\}$$
$$+\sum_j g_M((\tilde{\nabla}_X \mathcal{T})_{U_j}Y,U_j) - g_M((\tilde{\nabla}_{U_j}\mathcal{A})_X Y,U_j) + g_M(\mathcal{A}_Y U_j,h\tilde{\nabla}_{U_j}X) + g_M(\mathcal{T}_{U_j}Y,v\tilde{\nabla}_{U_j}X) + \frac{1}{2}\eta(Y)g_M(\tilde{\nabla}_{U_j}X,U_j)]$$
$$-g_M(X,Y)[\hat{S}(V,W) - \frac{1}{2}\eta(\mathcal{T}_V W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathcal{T})_V X_i,W) + g_M(\mathcal{A}_{X_i}W,h\tilde{\nabla}_V X_i) + g_M(\mathcal{T}_W X_i,v\tilde{\nabla}_V X_i) + \frac{1}{2}\eta(X_i)g_M(\tilde{\nabla}_V X_i,W)$$
$$-\frac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i)g_M(V,W)\} - \sum_j \{g_M(\mathcal{T}_V U_j,\mathcal{T}_W U_j) - \frac{1}{2}\eta(\mathcal{T}_{U_j}W)g_M(V,U_j)\}]\},$$

$$g_M(\tilde{W}(U,V)W,X) = g_M((\tilde{\nabla}_U \mathcal{T})_V W,X) - g_M((\tilde{\nabla}_V \mathcal{T})_U W,X) - \frac{1}{2(n-1)}\{g_M(V,W)[\sum_i \{g_M((\tilde{\nabla}_X \mathcal{A})(X_i,U),X_i) - g_M((\tilde{\nabla}_{X_i}\mathcal{A})(X,U),X_i)$$
$$+\frac{1}{2}\eta(U)g_M(\tilde{\nabla}_X X_i,X_i) - \frac{1}{2}\eta(U)g_M(\tilde{\nabla}_{X_i}X,X_i) + g_M(\mathcal{T}_U X_i,[X,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathcal{T})_U U_j,X)\}]$$
$$-g_M(U,W)[(\sum_i \{g_M((\tilde{\nabla}_X \mathcal{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathcal{A})(X,V),X_i) + \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_X X_i,X_i) - \frac{1}{2}\eta(V)g_M(\tilde{\nabla}_{X_i}X,X_i) + g_M(\mathcal{T}_V X_i,[X,X_i]\}$$
$$-\sum_j \{g_M((\tilde{\nabla}_{U_j}\mathcal{T})_V U_j,X)\}]\}$$

$$
\begin{aligned}
g_M(\tilde{W}(U,V)W,F) &= g_M(\hat{R}(U,V)W,F) - g_M(\mathscr{T}_U F, \mathscr{T}_V W) + g_M(\mathscr{T}_V F, \mathscr{T}_U W) + \tfrac{1}{2}\eta(\mathscr{T}_V W) g_M(U,F) - \tfrac{1}{2}\eta(\mathscr{T}_U W) g_M(V,F) \\
&- \tfrac{1}{2(n-1)}\{g(U,F)[\hat{S}(V,W) - \tfrac{1}{2}\eta(\mathscr{T}_V W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i, W) + g_M(\mathscr{A}_{X_i} W, h\tilde{\nabla}_V X_i) \\
&+ g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_V X_i) + \tfrac{1}{2}\eta(X_i) g_M(\tilde{\nabla}_V X_i, W) - \tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i) g_M(V,W)\} - \sum_j \{g_M(\mathscr{T}_V U_j, \mathscr{T}_W U_j) - \tfrac{1}{2}\eta(\mathscr{T}_{U_j} W) g_M(V,U_j)\}] \\
&- g_M(V,F)[\hat{S}(U,W) - \tfrac{1}{2}\eta(\mathscr{T}_U W) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_U X_i, W) + g_M(\mathscr{A}_{X_i} W, h\tilde{\nabla}_U X_i) \\
&+ g_M(\mathscr{T}_W X_i, v\tilde{\nabla}_U X_i) + \tfrac{1}{2}\eta(X_i) g_M(\tilde{\nabla}_U X_i, W) - \tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i) g_M(U,W)\} - \sum_j \{g_M(\mathscr{T}_U U_j, \mathscr{T}_W U_j) - \tfrac{1}{2}\eta(\mathscr{T}_{U_j} W) g_M(U,U_j)\}] \\
&+ g_M(V,W)[\hat{S}(U,F) - \tfrac{1}{2}\eta(\mathscr{T}_U F) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_U X_i, F) + g_M(\mathscr{A}_{X_i} F, h\tilde{\nabla}_U X_i) \\
&+ g_M(\mathscr{T}_F X_i, v\tilde{\nabla}_U X_i) + \tfrac{1}{2}\eta(X_i) g_M(\tilde{\nabla}_U X_i, F) - \tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i) g_M(U,F)\} - \sum_j \{g_M(\mathscr{T}_U U_j, \mathscr{T}_F U_j) - \tfrac{1}{2}\eta(\mathscr{T}_{U_j} F) g_M(U,F_j)\}] \\
&- g_M(U,W)[\hat{S}(V,F) - \tfrac{1}{2}\eta(\mathscr{T}_V F) + \sum_i \{g_M((\tilde{\nabla}_{X_i}\mathscr{T})_V X_i, F) + g_M(\mathscr{A}_{X_i} F, h\tilde{\nabla}_V X_i) \\
&+ g_M(\mathscr{T}_F X_i, v\tilde{\nabla}_V X_i) + \tfrac{1}{2}\eta(X_i) g_M(\tilde{\nabla}_V X_i, F) - \tfrac{1}{2}\eta(h\tilde{\nabla}_{X_i}X_i) g_M(V,F)\} - \sum_j \{g_M(\mathscr{T}_V U_j, \mathscr{T}_F U_j) - \tfrac{1}{2}\eta(\mathscr{T}_{U_j} F) g_M(V,U_j)\}]\},
\end{aligned}
$$

$$
\begin{aligned}
g_M(\tilde{W}(X,Y)V,H) &= g_M((\tilde{\nabla}_X\mathscr{A})(Y,V),H) - g_M((\tilde{\nabla}_Y\mathscr{A})(X,V),H) + \tfrac{1}{2}\eta(V) g_M(\tilde{\nabla}_X Y, H) - \tfrac{1}{2}\eta(V) g_M(\tilde{\nabla}_Y X, H) + g_M(\mathscr{T}_V H, [X,Y]) \\
&- \tfrac{1}{2(n-1)}\{g_M(X,H)[\sum_i \{g_M((\tilde{\nabla}_Y\mathscr{A})(X_i,V),X_i) - g_M((\tilde{\nabla}_{X_i}\mathscr{A})(Y,V),X_i) + \tfrac{1}{2}\eta(V) g_M(\tilde{\nabla}_Y X_i, X_i) - \tfrac{1}{2}\eta(V) g_M(\tilde{\nabla}_{X i} Y, X_i) \\
&+ g_M(\mathscr{T}_V X_i, [Y,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j, Y)\} - g_M(Y,H)[\sum_i \{g_M((\tilde{\nabla}_X\mathscr{A})(X_i,V),X_i) \\
&- g_M((\tilde{\nabla}_{X_i}\mathscr{A})(X,V),X_i) + \tfrac{1}{2}\eta(V) g_M(\tilde{\nabla}_X X_i, X_i) - \tfrac{1}{2}\eta(V) g_M(\tilde{\nabla}_{X i} X, X_i) + g_M(\mathscr{T}_V X_i, [X,X_i]\} - \sum_j \{g_M((\tilde{\nabla}_{U_j}\mathscr{T})_V U_j, X)\}]\}
\end{aligned}
$$

4. Conclusion

The paper's primary contribution lies in its detailed calculations and analysis of these curvature tensors within the context of Riemannian submersions, particularly under a new type of semi-symmetric metric non- metric connection. It sheds light on how these tensors behave in the presence of total umbilic fibers, providing a comprehensive understanding of their characteristics under this specific geometric condition. The results of our study have significant implications for both mathematics and theoretical physics. In the field of mathematics, understanding the behavior of curvature tensors under different types of connections and geometric conditions enhances our knowledge of differential geometry and its applications. This could potentially lead to new developments in geometric analysis and the theory of submersions, impacting areas such as geometric topology and the study of fiber bundles.

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