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RESEARCH PAPER

Optimizing shellfish aquaculture in nitrogen and fisheries management

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Abstract

Water quality and the invasion of weeds due to nutrient eutrophication have been a concern in major lakes and coastal areas. Scholars have advocated the cultivation of some species of shellfish as a new potential to facilitate the bioremediation of the polluted environment due to excessive nutrients. In this paper, our objective is to determine the optimal area that must be dedicated to shellfish aquaculture relative to the level of nitrogen pollution, other fisheries activities, and the performance of wild catch. The optimal size also depends on the effort outside the water body to control pollution from the point source. We set up transition equations that describe the system's state based on pollution reduction efforts, nitrogen concentration level, and the size of shellfish cultivation. We show that the impact of the nitrogen concentration level in the habitat can be minimized by allocating optimal management efforts to reduce nitrogen waste from the source and setting aside an area for shellfish cultivation. We found the optimal steady-state solutions and analyzed the optimal solutions based on biological and economic parameters.

Keywords: Nitrogen pollution; shellfish aquaculture; space allocation; fisheries management; optimal effort

AMS 2020 Classification: 49N90; 90C90; 91A80; 93C95

1 Introduction

An aquatic ecosystem enriched with nutrients like nitrogen creates conducive conditions for algae and similar species that utilize these nutrients. Nitrogen pollution has been one of the primary threats to coastal water quality [1]. Deterioration in water quality due to excessive nutrient

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loading from diffuse sources has become a major environmental problem in lakes, reservoirs, and coastlines mainly close to urban areas [2–4]. It is common to observe a substantial increase in the algal population on the surface of eutrophic waters (algal bloom). Algal blooms hinder the flow of sunlight and cause a decline in the dissolved oxygen level in the water. This, in turn, causes many marine animals to suffocate and die, creating "dead zones" [5].

The eutrophication of surface waters due to excessive nitrogen upsets the natural balance of aquatic ecosystems [6, 7]. Eutrophication can also seriously affect our ability to use water for recreation, drinking, agriculture, industry, and other purposes [8]. As part of a water quality management program, governmental regulations and marine policies increasingly require nitrogen remediation practices for industrial and residential wastes. The U.S. Environmental Protection Agency (EPA) developed guidelines to identify at-risk surface water bodies and protect them from eutrophication, stating that nitrogen concentrations should not exceed 0.3 mg per liter in streams and rivers and 0.1 mg per liter in lakes and reservoirs [9]. The public has considered eutrophication through waste management as a worthwhile investment for restoring and preserving freshwater lakes and reservoirs while cleaning coastal areas [10].

Shellfish aquaculture reduces coastal eutrophication by assimilating and storing excess nutrients, facilitating the natural denitrification rates when harvested [11, 12]. The optimizing process involves strategically placing shellfish farms in areas with high nutrient levels to leverage their filter-feeding behavior, thereby improving water quality by removing excess nitrogen and phosphorus while producing farmed seafood products, contributing to overall ecosystem health and sustainable fisheries management goals. In [11–13], the authors studied how excess nutrients can be removed from the marine environment when shellfish cultures are harvested. Mykoniatis and Ready in [14] studied the potential contribution of oysters to water quality goals in the fisheries management of Chesapeake Bay, USA. They developed a bioeconomic model of oysters in the bay area that considers the value of oyster harvests, the cost of fishing effort, and the removal of nitrogen from the bay through harvest and denitrification. In [15], they also studied the effects of oyster aquaculture on local water quality. The study investigated how water quality and hydrodynamics varied among farms and inside versus outside the vicinity of caged grow-out areas in southern Chesapeake Bay.

The development of a model that links water body nutrient concentration to its impact on the environment and water quality is a vital component of the management of fisheries in polluted areas. For our study, in the environment under consideration, we assumed nitrogen is the primary source of nutrient enrichment that affects the livelihood of the fish species, degrades the carrying capacity of the habitat, and affects the growth rate of the fish stock. We mainly focus on nutrient reduction efforts from various external sources and the introduction of shellfish cultivation in polluted areas to facilitate nutrient and fisheries management efforts. Since Shellfish store excess nutrients, we assume that the amount of nitrogen removed by shellfish is proportional to the production function or the size of the aquaculture area [11–13]. Moreover, we assume that the positive impact of shellfish on the fish stock can be measured through the growth rate. In this paper, we determine the optimal efforts and the area that must be dedicated to shellfish aquaculture for sustainable environmental and fisheries management. We also attempt to determine the optimal wild fish harvest dependent on the state of the ecosystem.

In the next section, we presented the model and performed a stability analysis of the steady-state solutions. In Section 3, we set up the control version of the model to determine the best trajectory for the actions and recovery rate of the system so that the overall social return from the aquatic ecosystem is maximum. This enables us to determine the optimal harvesting level, shellfish cultivation volume, and the best practice of the management's nitrogen removal rate from the source. In Section 4, we presented the policy implications of the results and the paper's conclusion.

2 Model formulation

In this section, we will set up the dynamic or transition equations of the fish stocks under different scenarios and analyze the systems at the steady state. In Section 2, we present the transition equation of open-access capture fish stock in the presence of nitrogen pollution before the introduction of shellfish aquaculture. In this case, we assume that the main cause of negative externality to the fishing ground is related to the nitrogen concentration level in the water and harvest. Then, in Section 2, we assume the development of shellfish aquaculture and consider the positive impact of shellfish cultivation on the fishing ground that would be reflected through the growth rate of the fish stock. In Section 2, we present a comparative analysis of the long-run behavior of the two systems.

Open access capture fish and nitrogen dynamics

Nitrogen dynamics

We assume that without effective pollution control actions, a large portion of the nitrogen from the vicinity, W, has the potential of reaching water bodies in the area and polluting the environment [14, 16]. Suppose $L(m, W) = (1 - m)W$ is the external rate of nitrogen loading to the water bodies (lakes, reservoirs, or coastal area), where m is a control variable that represents the management's effort to reduce m portion of the nitrogen from the source, where $0 \leq m < 1$. For simplicity, we ignore nitrogen recycling (internal loading) and focus only on elements of nitrogen dynamics because of external sources and internal denitrification [14, 16]. In this case, we describe the dynamics of the nitrogen volume by

$$\frac{dN}{dt} = L(m, W) - \alpha N, \qquad N(0) = N_0 > 0,$$

where α is the natural decay rate.

To align the model to our density-based analysis, we divide both sides of the equation by the total area under consideration, T, and get the concentration of nitrogen equation

$$\frac{dn}{dt} = (1 - m)w - \alpha n, \qquad n(0) = n_0 > 0,$$

where $w = \dfrac{W}{T}$ and $n = \dfrac{N}{T}$ are the amounts of nitrogen loading per unit area and surface densities of nitrogen in the area, respectively.

Capture fishery stock dynamics

We assume nitrogen loading is a source of pollution and causes environmental degradation [1, 17]. It damages the habitat and changes the ecological makeup of the system. The improvement or degradation of the habitat directly affects the carrying capacity of the habitat and the growth rate of fish biomass [18]. In a polluted environment of an aquatic ecosystem, the habitat's carrying capacity and the growth rate of fish stock depend on the pollution level [19, 20]. Like [21], we assume that the open-access carrying capacity and growth rate of fish stock depend on the extent of pollution (i.e., the nitrogen concentration level) [18–22]. Therefore, we describe the transition equation of the stock as

$$\frac{dX}{dt} = r(n)\left(1 - \frac{X}{K(n)}\right)X - H(X, E), \qquad X(0) = X_0 > 0,$$

where $H(X,E)=\sigma EX$ is the harvest rate from the capture fishery for stock size X, effort level, E, and catchability coefficient, σ, and $K(n)=K-\Theta n>0$ and $r(n)=r-\varepsilon_1 n>0$, where Θ measures the impact of nitrogen on the carrying capacity of the habitat, K, r is the intrinsic growth rate, ε_1 is the measure of the impact of nitrogen on the growth rate. If we scale down the above equation by dividing both sides of the equation by the total carrying capacity, K, we get:

$$\frac{dx}{dt}=(r-\varepsilon_1 n)\left(1-\frac{x}{1-\theta n}\right)x-\sigma Ex, \qquad x(0)=x_0>0, \tag{1}$$

where $\theta=\dfrac{\Theta}{K}$, $0<r-\varepsilon_1 n<r$, provided $r-\varepsilon_1 n-\sigma E>0$, and $1-\theta n>0$.

Thus, the dynamic system of equations of density of fish stock and nitrogen concentration is

$$\begin{aligned}\frac{dn}{dt}&=(1-m)w-\alpha n, \qquad n(0)=n_0>0,\\ \frac{dx}{dt}&=(r-\varepsilon_1 n)\left(1-\frac{x}{1-\theta n}\right)x-\sigma Ex, \qquad x(0)=x_0>0.\end{aligned} \tag{2}$$

Steady-state solutions and stability

In the steady state $\dfrac{dn}{dt}=\dfrac{dx}{dt}=0$, this implies the critical points of the above system, Eq. (2), are

$$(n_1^*,x_1^*)=\left(\frac{(1-m)w}{\alpha},0\right),$$

and

$$(n_2^*,x_2^*)=\left(\frac{(1-m)w}{\alpha},\left(1-\frac{\theta(1-m)w}{\alpha}\right)\left(1-\frac{\alpha\sigma E}{\alpha r-\varepsilon_1(1-m)w}\right)\right),$$

provided that $\alpha>\theta(1-m)w$, the decay rate of nitrogen concentration in the lake should be greater than θ times the external nitrogen loading rate to the water body, and $\alpha r-\varepsilon_1(1-m)w-\alpha\sigma E>0$.

To determine the stability of the equilibrium solutions, we first need to derive the Jacobian matrix of the dynamic system in Eq. (2)

$$J(n,x)=\begin{pmatrix}\frac{(r-\varepsilon_1 n)(1-2x-\theta n)-\sigma E(1-\theta n)}{1-\theta n} & -\frac{x\left(rx\theta+\left((1-\theta n)^2-x\right)\varepsilon_1\right)}{(1-\theta n)^2}\\ 0 & -\alpha\end{pmatrix}.$$

Since the eigenvalues of a triangular matrix equal the values on its diagonal, the eigenvalues of the Jacobian matrix are

$$\lambda_1=-\alpha,$$

and

$$\lambda_2=\frac{(1-2x-\theta n)(r-\varepsilon_1 n)-\sigma E(1-\theta n)}{1-\theta n}.$$

The Jacobian matrix at the first critical point, (n_1^*, x_1^*), is

$$J(\cdot) = \begin{pmatrix} r - \varepsilon_1 n - \sigma E & 0 \\ 0 & -\alpha \end{pmatrix}.$$

Observe that $Tr(J) = r - \varepsilon_1 n - \sigma E - \alpha$ and $det(J) = -\alpha(r - \varepsilon_1 n - \sigma E)$. Since $det(J) < 0$, by the Theorem (5.4) in [23], the dynamic system in Eq. (2) is unstable at the critical point

$$(n_1^*, x_1^*) = \left(\frac{(1-m)w}{\alpha}, 0\right).$$

The Jacobian matrix at the second critical point, (n_2^*, x_2^*), is

$$J(\cdot) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & -\alpha \end{pmatrix},$$

where $A_{11} = -(r - \varepsilon_1 n - \sigma E)$, and $A_{12} = -\left(1 - \frac{\sigma E}{r - \varepsilon_1 n}\right)\left[\left(1 - \frac{\sigma E}{r - \varepsilon_1 n}\right)(r\theta - \varepsilon_1) + \varepsilon_1(1 - \theta n)\right]$.

Observe that

$$Tr(J) = -(\alpha + r - \varepsilon_1 n - \sigma E) < 0,$$

and

$$det(J) = \alpha\,(r - \varepsilon_1 n - \sigma E) > 0,$$

at $n = n_2^*$. This implies that the dynamic system Eq. (2) is locally asymptotically stable at the second critical point [23].

Therefore, the stable steady-state nitrogen concentration level is

$$n_s(m) = \frac{(1-m)w}{\alpha}. \tag{3}$$

It is apparent from Eq. (3) that the nitrogen concentration is the ratio of the amount of nitrogen reaching the water bodies to the natural decay rate. The concentration decreases when the rate of nitrogen reduction efforts from the source increases. It also decreases when the natural decay rate increases.

And the equilibrium stock size, x_s, as a function of wild catch effort, E, and m is

$$x_s(E, m) = \left(1 - \frac{\theta(1-m)w}{\alpha}\right)\left(1 - \frac{\alpha\sigma E}{\alpha r - \varepsilon_1(1-m)w}\right) = (1 - \theta n_s(m))\left(1 - \frac{\sigma E}{r - \varepsilon_1 n_s(m)}\right). \tag{4}$$

The above equation, Eq. (4), implies that capture fish effort and nitrogen concentration negatively impact the stock of captured fish. However, the allocation of management's effort for nitrogen reduction from the source positively impacts the steady state stock size. The corresponding steady-state or sustainable yield

$$h_s(E,m) = \sigma E(1-\theta n_s(m))\left(1-\frac{\sigma E}{r-\varepsilon_1 n_s(m)}\right). \tag{5}$$

The maximum sustainable yield (MSY) is attained at the effort level E such that $\frac{\partial h_s(E,m)}{\partial E} = 0$. That is at

$$E_{MSY}(m;r,\sigma) = \frac{1}{2\sigma}(r-\varepsilon_1 n_s(m)). \tag{6}$$

From Eq. (6), the critical effort level is negatively impacted by the nitrogen concentration; the higher the impact, the lower the effort. Then, the value of MSY, h_{MSY}, at this effort can be found by substituting $E = E_{MSY}$ into Eq. (5)

$$h_{MSY}(m) = \frac{(1-\theta n_s(m))(1-\varepsilon_1 n_s(m))}{4},$$

and the biomass level at the MSY is

$$x_{MSY}(m) = \frac{(1-\theta n_s(m))}{2}.$$

The steady-state net profit, $P_s(E,m) = ph_s(E,m) - cE$, where p is a competitive market price and c is the cost per unit effort per unit area assumed to be constants, attains its maximum at an effort level E such that

$$\frac{\partial P_s(E,m)}{\partial E} = 0.$$

That is, the sustainable effort that maximizes profit (MSP),

$$E_{MSP}(m;r,\sigma,c,p) = \frac{1}{2\sigma}(r-\varepsilon_1 n_s(m))\left(1-\frac{c}{p\sigma(1-\theta n_s(m))}\right), \tag{7}$$

is positive provided that $c < p\sigma(1-\theta n_s(m))$. Observe that $E_{MSP} < E_{MSY}$. By plugging Eq. (7), with $E = E_{MSP}$, into Eq. (5), the harvest level that maximizes revenue, h_{MSP}, is

$$h_{MSP}(m) = \frac{(1-\theta n_s(m))(1-\varepsilon_1 n_s(m))}{4}\left(1-\frac{c^2}{p^2\sigma^2(1-\theta n_s(m))^2}\right),$$

and the corresponding biomass level is

$$x_{MSP}(m) = \frac{(1-\theta n_s(m))}{2}\left(1+\frac{c}{p\sigma(1-\theta n_s(m))}\right).$$

From Eq. (3), the nitrogen level, $n_s(m)$, decreases with the increment of management's effort, m. The reduction of nitrogen concentration positively impacts the growth rate of the stock, $r-\varepsilon_1 n_s(m)$, and the carrying capacity, $1-\theta n_s(m)$, which in turn results in relaxing the restrictions on the optimal effort (i.e., management's effort allocation is directly related to the effort level that maximizes the profit). For example, we numerically solve the system for $\theta = 0.10$ and $\varepsilon_1 = 0.085$, and find the stock size and the corresponding effort that maximizes revenue, $x_{MSP} = 0.574167$ and $E_{MSP} = 45.3789$, respectively. Moreover, if we assume that the nitrogen concentration level degrades the environment more than expected, say $\theta = 0.11$ and $\varepsilon_1 = 0.086$, the stock size decreases to $x_{MSP} = 0.571166$, and the optimal catch effort reduces to $E_{MSP} = 45.2804$. The effort level is inversely related to the measures of the impact of nitrogen concentration on the carrying capacity, θ, and on the intrinsic growth rate, ε_1. Moreover, the effort level is determined not only by biological and impact factors but also by the profitability of the shellfish aquaculture, which depends on the cost of production and market price relation, $c < p\sigma(1 - \theta n_s(m))$. This condition is hard to achieve, especially at the beginning of the investment. Therefore, private investors would participate in the remediation effort if we subsidized them until their revenue becomes larger than the cost of production. This helps to maintain a sustainable environment and supply of farmed shellfish and, at the same time, supports the economy by creating jobs.

Even though the degradation of the aquatic ecosystem and the decline in water quality from excessive nitrogen can primarily be minimized by reducing external nitrogen loading, it is important to implement an effort-limiting policy that helps the recovery of the environment. In Figure 1, we sketched the effort that maximizes the rent and the corresponding stock size. Both decrease with the increase in pollution.

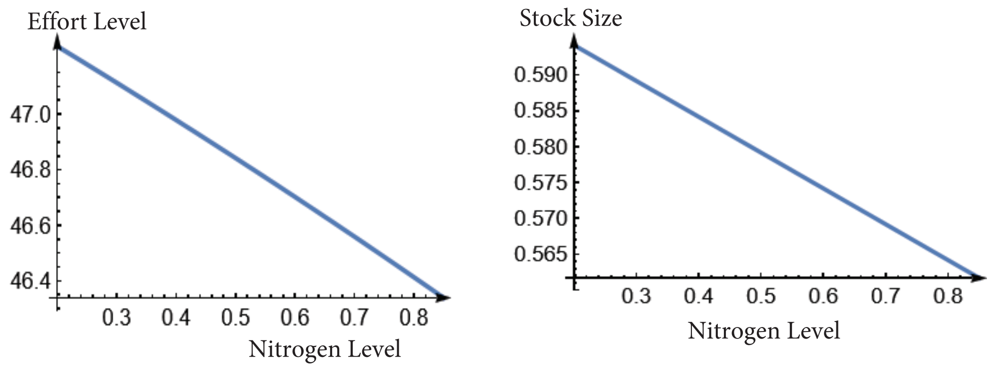

Figure 1. Effort level, E_{MSP}, and the stock size, x_{MSP}, as a function of nitrogen pollution level

Capture fish, nitrogen concentration, and shellfish aquaculture

Regardless of policies and regulatory measures intended to reduce external nitrogen loading, more nitrogen loading from a non-point source is inevitable [1]. Shellfish aquaculture has been considered an alternative to reduce the nitrogen level in water bodies because a significant amount of nitrogen is embedded in shellfish meat and shells. Shellfish also reduce the nitrogen through its denitrification process [14, 24].

Shellfish aquaculture dynamics

Suppose A is the size of the area dedicated to shellfish aquaculture production from the total polluted area under consideration, T. Like [25], we assume the aquaculture expansion rate depends on the relative size of the aquaculture, a, and the magnitude of nitrogen pollution, n,

$$\frac{da}{dt} = \left(v - \rho\frac{a}{n}\right), \qquad a(0) = a_0 \geq 0,$$

where $a = \dfrac{A}{T}$ is the portion of the area dedicated to aquaculture, v is exogenous control variable determined by the management and ρ is a conversion factor provided $\rho a \leq vn$.

Nitrogen dynamics

Let the amount of nitrogen removed by the shellfish production be proportional to the size of the aquaculture area [21]. Hence, let βan be the reduction rate of the nitrogen concentration due to shellfish, where β is a conversion constant. Let α be the natural rate of nitrogen decay through processes other than the amount removed by shellfish cultivation [14, 16]. Following [14, 16], we extend the dynamic equation of nitrogen from section 2.1 as

$$\frac{dn}{dt} = (1-m)w - \beta an - \alpha n, \qquad n(0) = n_0 > 0.$$

Capture fishery stock dynamics

As mentioned earlier, nitrogen contributes to environmental degradation and reduces the carrying capacity of the habitat as well as the growth rate of the fish stock [18–20]. We assume that its impact depends on the concentration level of nitrogen in the environment, n. Although aquaculture development takes away fishing areas and has some negative externalities on the environment, shellfish farming has a net positive effect. It helps filter the nitrogen and clean the water, improving the stock's growth rate. Therefore, we assume that the per-capita growth rate of the fishery decreases with nitrogen concentration level and increases with shellfish aquaculture size, a. Hence, we set up the transition equation of the stock as

$$\frac{dX}{dt} = r(n,a)\left(1 - \frac{X}{K(n)}\right)X - H(X,E), \qquad X(0) = X_0 > 0,$$

where $H(X,E) = \sigma EX$ is the harvest rate from the capture fishery from a multi-species stock size X, E is the effort level and σ is the catchability coefficient, $r(n,a) = r - \varepsilon_1 n + \varepsilon_2 a$, and $K(n) = K - \Theta n > 0$, where Θ is the measures of the nitrogen level on the environment and ε_2 is the measure of the impact of aquaculture on the growth rate.

If we scale down the above equation by dividing both sides of the equation by the total carrying capacity, K, we get:

$$\frac{dx}{dt} = (r - \varepsilon_1 n + \varepsilon_2 a)\left(1 - \frac{x}{1-\theta n}\right)x - \sigma Ex, \qquad x(0) = x_0 > 0,$$

where $\theta = \dfrac{\Theta}{K}$, provided $0 < r - \varepsilon_1 n + \varepsilon_2 a \leq r$, $r - \varepsilon_1 n + \varepsilon_2 - \sigma E > 0$, and $1 - \theta n > 0$.

The equations aligned for fish stock, nitrogen concentration, and shellfish aquaculture are

$$\begin{aligned}
\frac{dx}{dt} &= (r - \varepsilon_1 n + \varepsilon_2 a)\left(1 - \frac{x}{1-\theta n}\right)x - \sigma Ex, \qquad x(0) = x_0 > 0,\\
\frac{dn}{dt} &= (1-m)w - \beta an - \alpha n, \qquad n(0) = n_0 > 0,\\
\frac{da}{dt} &= \left(v - \rho\frac{a}{n}\right), \qquad a(0) = a_0 \geq 0.
\end{aligned} \tag{8}$$

Stability of the steady-state solutions

In steady state $\frac{dx}{dt} = \frac{dn}{dt} = \frac{da}{dt} = 0$, implies the critical points of the above system are

$$(x_1^*, n_1^*, a_1^*) = \left(0, \frac{S(v,m) - \alpha\rho}{2v\beta}, \frac{1}{\beta}\left(\frac{2vw\beta(1-m)}{S(v,m) - \alpha\rho} - \alpha\right)\right),$$

and

(x_2^*, n_2^*, a_2^*), where

$$x_2^* = \left(1 - \frac{\theta(S(v,m) - \alpha\rho)}{2v\beta}\right)\left(1 - \frac{E\sigma}{r - \varepsilon_1\left(\frac{S(v,m) - \alpha\rho}{2v\beta}\right) + \varepsilon_2\frac{1}{\beta}\left(\frac{2vw\beta(1-m)}{S(m,v) - \alpha\rho} - \alpha\right)}\right),$$

$$n_2^* = \frac{S(v,m) - \alpha\rho}{2v\beta}, \text{ and } a_2^* = \frac{1}{\beta}\left(\frac{2vw\beta(1-m)}{S(v,m) - \alpha\rho} - \alpha\right),$$

and $S(v,m) = \sqrt{\alpha^2\rho^2 + 4\rho\beta(1-m)wv}$.

To determine the stability of the equilibrium, we derive the Jacobian matrix of the system

$$J(\cdot) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & -(\alpha + \beta a) & -\beta n \\ 0 & \frac{\rho a}{n^2} & -\frac{\rho}{n} \end{pmatrix},$$

where

$$\begin{aligned} A_{11} &= \frac{(1 - \theta\,\mathrm{n} - 2x)(r - \varepsilon_1 n + \varepsilon_2 a)}{1 - \theta \mathrm{n}} - \sigma E, \\ A_{12} &= \frac{x\left(-\theta x(r + \varepsilon_2 a) + \varepsilon_1\left(x - (1-\theta n)^2\right)\right)}{(1-\theta n)^2}, \\ A_{13} &= \frac{\varepsilon_2 x(1 - x - \theta n)}{1 - \theta n}. \end{aligned}$$

Then, we evaluate the eigenvalues of the Jacobian matrix at each critical point. The eigenvalues at the first critical point, (x_1^*, n_1^*, a_1^*), are

$$\lambda_1^1 = \frac{-(\alpha n + \beta a n + \rho) - \sqrt{(\alpha n + \beta a n + \rho)^2 - 4n(\alpha\rho + 2\beta\rho a)}}{2n},$$

$$\lambda_2^1 = \frac{-(\alpha n + \beta a n + \rho) + \sqrt{(\alpha n + \beta a n + \rho)^2 - 4n(\alpha\rho + 2\beta\rho a)}}{2n},$$

and

$$\lambda_3^1 = r - \varepsilon_1 n + \varepsilon_2 a - \sigma E.$$

Since $r - \varepsilon_1 n + \varepsilon_2 a - \sigma E$ has to be positive (otherwise the stock become extinct), the trivial equilibrium is not stable [26]. And the eigenvalues of the Jacobian matrix at the second critical point, (x_2^*, n_2^*, a_2^*), are

$$\lambda_1^2 = \frac{-(\alpha n + \beta a n + \rho) - \sqrt{(\alpha n + \beta a n + \rho)^2 - 4n(\alpha\rho + 2\beta\rho a)}}{2n},$$

$$\lambda_2^2 = \frac{-(\alpha n + \beta a n + \rho) + \sqrt{(\alpha n + \beta a n + \rho)^2 - 4n(\alpha\rho + 2\beta\rho a)}}{2n},$$

and

$$\lambda_3^2 = -(r - \varepsilon_1 n + \varepsilon_2 a - \sigma E).$$

In this case, all of the eigenvalues are negative, implying the dynamic system is stable at the critical point (x_2^*, n_2^*, a_2^*) [26].

Therefore, the stable steady-state nitrogen concentration and shellfish aquaculture size in terms of the control variables, m and v, are

$$n_s(m,v) = \frac{\sqrt{4\beta\rho(1-m)wv + \alpha^2\rho^2} - \alpha\rho}{2\beta v} = \frac{(1-m)w}{\alpha + \beta a_s(m,v)}, \tag{9}$$

and

$$a_s(m,v) = \frac{1}{\beta}\left(\frac{2vw\beta(1-m)}{\sqrt{4\beta\rho(1-m)wv + \alpha^2\rho^2} - \alpha\rho} - \alpha\right) = \frac{1}{\beta}\left(\frac{(1-m)w}{n_s(m,v)} - \alpha\right). \tag{10}$$

From Eq. (9), the nitrogen concentration decreases when the production of shellfish or external effort increases. We can also observe from Eq. (10) that shellfish aquaculture size is positive whenever $(1-m)w \geq \alpha n_s(m,v)$.

Then the steady-state stock of fish, x_s, can be written in terms of aquaculture and nitrogen concentration (both depend on m and v) as

$$x_s(m,v,E) = (1 - \theta n_s(m,v))\left(1 - \frac{\sigma E}{r - \varepsilon_1 n_s(m,v) + \varepsilon_2 a_s(m,v)}\right). \tag{11}$$

From Eq. (11) the equilibrium stock size decreases as the nitrogen concentration or fishing effort increases. However, it increases when the production of shellfish aquaculture or the impact factor, ε_2, on the habitat increases. This implies that even though aquaculture takes away the fishing area and creates pressure in the open-access fishing ground, shellfish aquaculture has a net positive impact on the stock. The corresponding sustainable yield is given by

$$h_s(m,v,E) = \sigma E(1 - \theta n_s(m,v))\left(1 - \frac{\sigma E}{r - \varepsilon_1 n_s(m,v) + \varepsilon_2 a_s(m,v)}\right),$$

where $n_s(m,v)$ and $a_s(m,v)$ are give by Eqs. (9) and (10).

The maximum sustainable yield (MSY), the largest yield or catch that can be potentially taken from the stock, is attained at effort level E such that $\frac{\partial h_s}{\partial E}(E, v, m) = 0$. Solving $\frac{\partial h_s}{\partial E}(E, m, v) = 0$ for E, we find the maximum sustainable effort,

$$E_{MSY}(m, v) = \frac{1}{2\sigma}(r - \varepsilon_1 n_s(m, v) + \varepsilon_2 a_s(m, v)). \tag{12}$$

Eq. (12) reveals that if the size of aquaculture farming is fixed and the nitrogen concentration increases, the maximum sustainable effort must be reduced.
We can also solve the equation

$$\frac{\partial(ph_s(m, v, E) - cE)}{\partial E} = 0,$$

to find the effort level that maximizes the net economic rent from wild catch,

$$E_{MSY}(m, v) = \frac{1}{2\sigma}(r - \varepsilon_1 n_s(m, v) + \varepsilon_2 a_s(m, v)) \left(1 - \frac{c}{\sigma p\,(1 - \theta n_s(m, v))}\right). \tag{13}$$

From Eqs. (12) and (13), the optimal effort level is less than the maximum sustainable effort provided $c < \sigma p(1 - \theta n_s)$. Moreover, in Eq. (13), when excessive nitrogen concentration in the aquatic ecosystem, n_s, is reduced, the stock in the fishing ground, x_s, gets better. The increment of fish stock enables us to relax restrictions imposed on the effort, E_{MSY}, and harvest more fish. Observe that E_{MSY} is directly related to the size of the aquaculture. The development of shellfish aquaculture must be encouraged, besides controlling nitrogen loading from the source when the aquatic ecosystem becomes excessively enriched with nitrogen.

Numerical solutions and sensitivity analysis

In this section, we investigate the impact of introducing the shellfish farm in a polluted environment and perform a sensitivity analysis. Like [27], we develop a deterministic bioeconomic model that describes the transition dynamics and interrelationships of the systems using parameters. Then, sensitivity analysis of the optimal solutions is investigated by assigning different values for significant biological parameters and performance measures (in Table 2, the changes in parameter values are highlighted in bold). Some of the values are taken from [28], and we choose reasonable values for the other parameters based on the conditions we impose on the model. Using values from Table 1, we numerically solve for the optimal solutions and report the results in Table 2.

Table 1. Parameters and their values used for the stability of the system

Parameter	Description	Value	Unit
r	Growth rate parameter	1.8	1/year
σ	Catchability coefficient per unit effort	0.015	1/vessel/year
α	The natural decay rate of nitrogen	0.6, 0.65	1/year
β	Conversion factor	3, 4	1/year
ρ	Conversion factor	1	1/year
θ	Measure of the impact of nitrogen on the environment	0.1, 0.11	1/year
ε_1	Measure of the impact of nitrogen on the growth rate	0.085	1/year
ε_2	Measure of the impact of shellfish on the growth rate	.097	1/year
p	Unit price of wild catch in US dollars	15	1/US$

Table 2. Optimal steady-state solutions when nitrogen negatively impacts the environment and shellfish farming benefits the habitat

θ	ε_1	ε_2	β	α	m	a_s	n_s	E_{MSP}	x_{MSP}	h_{MSP}
0.10	0.085	0	0	0.6	0.7	0	0.6000	45.3789	0.574167	0.390826
0.10	0.085	0.097	3	0.6	0.7	0.310366	0.237175	47.4459	0.592308	0.421539
0.10	0.085	0.097	**4**	0.6	0.7	0.27389	0.212319	47.4524	0.593551	0.422481
0.10	0.085	0.097	3	**0.65**	0.7	0.269038	0.208556	47.4534	0.593739	0.422624
0.10	0.085	0.097	3	0.6	**0.75**	0.244961	0.189892	47.4581	0.594672	0.4233

From the steady-state optimal numerical solutions summarized in Table 2:

i) Before the introduction of shellfish aquaculture (when $a = \varepsilon_2 = \beta = 0$), if we manage to reduce the external nitrogen loading by 70%, the nitrogen concentration level in the water would be 0.6 mg.

ii) After the introduction of shellfish aquaculture (when $\varepsilon_2 = 0.097$, $\theta = 0.1$, and $\beta = 3$), the concentration reduces from 0.6 to 0.237175 mg as long as we dedicate 0.31 of the area to the sector and keep the external effort level at $m = 0.70$.

iii) When the conversion constant that determines the reduction rate of nitrogen concentration due to shellfish cultivation, β, increases from 3 to 4, the nitrogen concentration level decreases from .2372 to .2086.

iv) When the natural decay rate, α, increases from 0.6 to 0.65, like the above case, nitrogen concentration decreases while the fish stock size increases.

In general, the development of shellfish aquaculture helps the recovery of the aquatic ecosystem and restores the fish stock. The increment of the fish stock enables us to relax the restriction on the optimal effort. As the rate of nitrogen reduction due to shellfish farming or natural denitrification increases, the optimal aquaculture size reduces, and the fish stock size and the optimal yield increase. In Figure 2, we displayed the state of the stock before and after the introduction of shellfish farming in the environment at any impact level of nitrogen in the area, θ. It shows that the introduction of aquaculture slows the decline of the environment and fish stock.

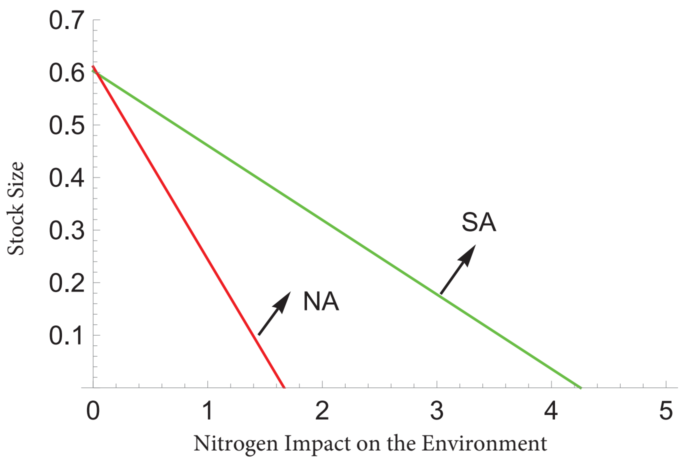

Figure 2. Comparison of stock size of fish (in tonnes), x, at each nitrogen concentration impact level $(1/year)$, θ, before the development of shellfish aquaculture (NA) and after shellfish aquaculture development (SA) for $\sigma = 0.015$, $r = 1.8$, $\alpha = .6$, $\beta = 3$, $m = .7$, $w = 1.2$, and $\theta = .10$

3 Optimized management strategy

In this section, we will consider a control version of the dynamic equations in the previous section, where we determine the asymptotic behavior of the trajectory of the control variables E, m, and v that maximizes the long-run overall benefit from the system. Recent research suggests that improving the quality of water indirectly benefits the community around the coastal line–for example, it appreciates home values near the water bodies [29]. In our optimization model, we only include the economic contributions of shellfish through the supply chain, even though sometimes the net profit may not be positive and needs some type of government subsidy. The primary focus of our research agenda is on the ecological or environmental benefits of shellfish aquaculture.

Let the net profit from capture fishery be $p_1\sigma Ex - cE$, where p_1 is the market price and c is the cost per unit effort per unit area. Suppose the production function for aquaculture is $Z(A) = P_a A$, where P_a is per unit area production of farmed fish in kilograms. We can rewrite the production function in terms of $a = \dfrac{A}{T}$ as $Z(a) = TP_a a$, where T is the surface area of the water under consideration. Let $z(a) = \dfrac{Z(a)}{T}$ and the cost of production is quadratic $c_a(P_a)^2 a$, where c_a is a cost parameter [28]. Then, the net profit from aquaculture is $p_2 Z(a) - C_1(a,v)$, where $C_1(a,v) = c_1\left(v - \rho\frac{a}{n}\right) + c_a(P_a)^2 a$ is a one-time cost of acquiring an extra unit of aquaculture area plus the operation cost for the aquaculture, and p_2 is the competitive market price of a shellfish and c_1 is one time per unit per area leasing cost. If $C_2(m) = c_2 m$ is the cost of removing the pollutant at the source, where c_2 is per unit cost, and all future costs and benefits are discounted at a positive social discount rate of δ, the optimization problem is

$$\max J(E,m,v) = \max_{E,m,v} \int_0^\infty [(p_1\sigma x - c)E + p_2 z(a) - C_1(a,v) - C_2(m)]e^{-\delta t} dt,$$

subject to the dynamic equations in Eq. (8). Thus we seek an optimal control, E^*, v^* , and m^*, and the corresponding states, x^*, a^*, and n^*, that simultaneously solves the equations in Eq. (8) such that

$$J(E^*, m^*, v^*) = \max\{J(E,v,m)|\,(E,v,m) \in U\},$$

where the control set U is compact $U = \{(E(t), v(t), m(t))|\, 0 \le E(t) \le E_{\max},\ 0 \le m(t) \le m_{\max}, t \in [0,\infty)\}$.

The current-value Hamiltonian corresponding to our problem is

$$\begin{aligned}\mathcal{H}(x,n,a,E,m,v,\lambda_x,\lambda_a,\lambda_n) \quad &= \quad B(x,a,E,v,n,m) + \lambda_x\left[(r - \varepsilon_1 n + \varepsilon_2 a)\left(1 - \frac{x}{1-\theta n}\right)x - \sigma E x\right] \\ &+ \quad \lambda_a\left(v - \rho\frac{a}{n}\right) + \lambda_n\Big[(1-m)w - \beta a n - \alpha n\Big],\end{aligned}$$

where $B(x,a,E,v,n,m) = (p_1\sigma x - c)E + p_2 z(a) - C_1(a,v) - C_2(m)$ is the net profit at time t, and λ_x, λ_a and λ_n are the shadow values of the stock, aquaculture, and nitrogen concentration level, respectively.

The optimal control shall be a combination of bang-bang and singular control since the problem under consideration is a linear control problem. Our study focuses on the singular control and the associated optimal singular solutions. The optimality condition for the control variables is to satisfy

$$\frac{\partial \mathcal{H}(\cdot)}{\partial v} = -c_1 + \lambda_a, \quad \frac{\partial \mathcal{H}(\cdot)}{\partial E} = -c + \sigma x(p_1 - \lambda_x), \text{ and } \frac{\partial \mathcal{H}(\cdot)}{\partial m} = -c_3 - w, \tag{14}$$

where the switching functions are

$$\Psi_1(t) = -c_1 + \lambda_a, \ \Psi_2(t) = -c + \sigma x(p - \lambda_x), \text{ and } \Psi_3(t) = -c_3 - w\lambda_n.$$

It is known that in the case of a singular solution, we have

$$\Psi_1(t) = 0, \Psi_2(t) = 0, \text{ and } \Psi_3(t) = 0.$$

That is

$$-c_1 + \lambda_a = 0, \ -c + \sigma x(p - \lambda_x) = 0, \text{ and } -c_3 - w\lambda_n = 0. \tag{15}$$

The adjoint equations corresponding to the states are

$$\begin{aligned}
\frac{d\lambda_a}{dt} - \delta\lambda_a &= -\frac{\partial \mathcal{H}(\cdot)}{\partial a} = -P_a(p_2 - c_a P_a) + \beta\lambda_n n - x\varepsilon_2\left(1 - \frac{x}{1-\theta n}\right)\lambda_x - \frac{(c_1 - \lambda_a)\rho}{n}, \\
\frac{d\lambda_x}{dt} - \delta\lambda_x &= -\frac{\partial \mathcal{H}(\cdot)}{\partial x} = -\frac{(r - \varepsilon_1 n + \varepsilon_2 a)(1 - 2x - \theta n)\lambda_x}{1 - \theta n} - \sigma E(p_1 - \lambda_x), \\
\frac{d\lambda_n}{dt} - \delta\lambda_n &= -\frac{\partial \mathcal{H}(\cdot)}{\partial n} = (\alpha + \beta a)\lambda_n + \frac{x\left(\theta x(r + a\varepsilon_2) + \varepsilon_1\left(-x + (-1+\theta n)^2\right)\right)\lambda_x}{(-1+\theta n)^2} + \frac{c_1\rho a}{n^2} - \frac{\rho\lambda_a a}{n^2}.
\end{aligned} \tag{16}$$

In the steady state, $\frac{dx}{dt} = \frac{dn}{dt} = \frac{da}{dt} = \frac{d\lambda_x}{dt} = \frac{d\lambda_n}{dt} = \frac{d\lambda_a}{dt} = 0$, implying the following equations

$$\begin{aligned}
&(1-m)w - \beta a n - \alpha n = 0, \\
&\left(v - \rho\frac{a}{n}\right) = 0, \\
&(r - \varepsilon_1 n + \varepsilon_2 a)\left(1 - \frac{x}{1-\theta n}\right)x - \sigma E x = 0, \\
&\delta\lambda_x = \frac{(r - \varepsilon_1 n + \varepsilon_2 a)(1 - 2x - \theta n)\lambda_x}{1-\theta n} + E(p_1 - \lambda_x)\sigma, \\
&\delta\lambda_a = P_a(p_2 - c_a P_a) - \beta\lambda_n n + \varepsilon_2 x\left(1 - \frac{x}{1-\theta n}\right)\lambda_x + \frac{(c_1 - \lambda_a)\rho}{n}, \\
&\delta\lambda_n = -(\alpha + \beta a)\lambda_n - \frac{x\left(x(r + a\varepsilon_2)\theta + \varepsilon_1\left(-x + (-1+\theta n)^2\right)\right)\lambda_x}{(-1+\theta n)^2} - \frac{a c_1\rho}{n^2} + \frac{a\lambda_a\rho}{n^2}.
\end{aligned} \tag{17}$$

The interior optimal solutions for the state and control variables can be found by solving the system of equations in Eq. (15) and Eq. (17).

Numerical solutions and sensitivity analysis

In this section, we find numerical solutions to the above system, Eq. (15) and Eq. (17), by specifying aquaculture's per-unit production and operation costs, unit market price, and assigning appropriate parameter values given in Table 1 and Table 3. Sensitivity analysis is also performed for the conversion factors, the measures of the negative impacts of nitrogen, and the positive benefits of shellfish aquaculture. The results are summarized in Table 4.

Table 3. Parameters and their values used for numerical solutions

Parameter	Description	Value	Unit
P_a	Aquaculture production per unit square meter in kg	0.446	m^2/Kg
c_a	Cost parameter for aquaculture	0.1682	$1/US\$$
c_1	Cost of acquiring a square meter of aquaculture area in US dollars	1	$1/m^2/US\$$
c_2	The cost of removing the pollutant at the source in US dollars	0.135	1/US$
c	Cost per unit effort per unit carrying capacity for wild-catch	0.05	1/US$
δ	Positive social discount rate	0.05	1/year
θ	Measure of the impact of nitrogen on the environment	0.1, 0.11	1/vessel/year
ε_1	Measure of the impact of nitrogen on the growth rate	0.085, 0.086	1/year
ε_2	Measure of the impact of shellfish on the growth rate	0.097, 0.098	1/year
p_1	Unit price of wild catch in US dollars	15	1/US$
p_2	Unit price of farmed shellfish in US dollars	3	1/US$

Table 4. Steady-state optimal solutions for the state and control variables

ε_1	ε_2	θ	E^*	x^*	n^*	a^*	m^*	h^*
0.085	0.097	0.1	48.5255	0.584869	0.211729	0.273105	0.701388	0.425716
0.086	0.097	0.1	48.523	0.584869	0.211729	0.274299	0.700546	0.425694
0.085	**0.098**	0.1	48.5373	0.584931	0.210493	0.273202	0.703063	0.4258647
0.085	0.097	**0.11**	48.5804	0.583832	0.212129	0.306619	0.678127	0.425069

The numerical solutions in Table 4 show that

- As the impact of nitrogen concentration level on the habitat or the growth rate, θ or ε_1, increases, the optimal stock size, x^*, decreases, and consequently optimal harvest, h^*, declines. This shows a need to increase the optimal size of shellfish aquaculture since a significant amount of nitrogen can be removed through shellfish harvest and denitrification. This, in turn, allows us to reduce external effort.
- If the effectiveness of shellfish cultivation on the environment, ε_2, increases, we can observe that the nitrogen concentration level tends to approach a lower level. As a result of this, we can relax the optimal effort limits on the fishing ground.

Like the previous section, the control version reflects the positive contributions of shellfish aquaculture toward protecting aquatic ecosystems from eutrophication while increasing the supply of shellfish to the market and improving capture fish.

Transition dynamics

Following the computation of the optimal steady-state solution of the state, co-state, and control variables, it is natural to determine the trajectories of the states toward close-to-equilibrium solutions over a finite but large time. Because of the non-linear nature of the functional forms of the equations used in the dynamic analysis and the number of equations, it is not easy to find analytic solutions. Therefore, we use the fourth-order Runge–Kutta forward-backward sweep numerical method to solve the system of Eqs. (8) and (16). First, we approximate the state equations in Eq. (8), by first-order forward difference, and the corresponding co-state equations in Eq. (16), by first-order backward difference equations. Then by substituting the values of the parameters given in Table 3, using initial values, $x(0) = 0.27$, $n(0) = 0.6$, and $a(0) = 0$, and a guess for optimal control, say the steady-state values, we solve the state equations forward for the discrete-time interval of $[0, t_f]$ partitioned into k parts using a time step h such that $t_f = kh$. Then, using the state values at t_f and the transversality condition at t_f, we find the values of the co-state at t_f and solve the co-state equations backward. After each forward-backward computation, we update the control values using the state and co-state values and repeat the process until the control values become sufficiently close. The accuracy or convergence of the iterative method is based on Hackbusch [30]. Figure 3 displays the trajectories of the numerical solutions generated using Hermite interpolation of order 3. Note that for all combinations of the conversion factors, decay rates, and other parameter values given in Table 3, the trend is the same (i.e., as the aquaculture increases, the nitrogen level decreases while the fish stock increases) except they converge to different terminal points.

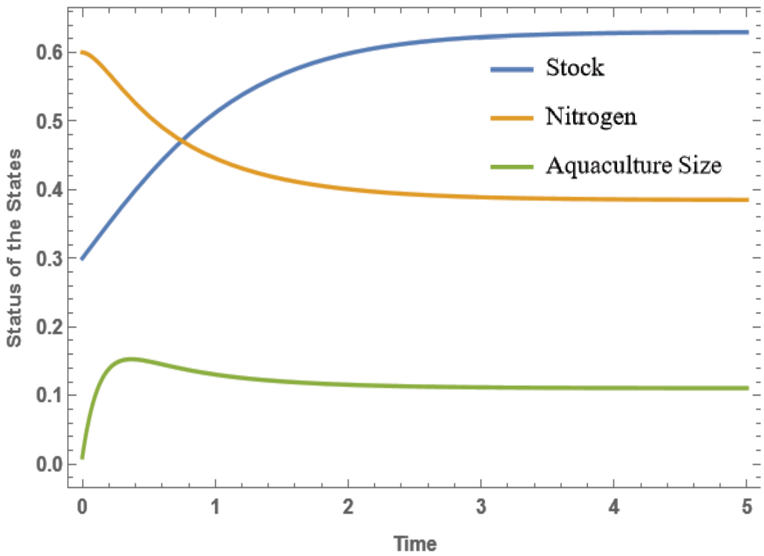

Figure 3. The trajectories of the optimal fish stock size and nitrogen level relative to the aquaculture expansion rate

4 Conclusion

This study assumes that shellfish aquaculture helps remediate nitrogen in an eutrophic aquatic ecosystem. We also attempted to determine the optimal sizes of shellfish aquaculture and the optimal capture fish effort before and after introducing shellfish aquaculture into the system. This ties in with our finding of the optimal nitrogen reduction effort needed to sustain the ecosystem. We reviewed various models related to our research and set up our model based on recent developments in the field and our intended research objectives. The system was analyzed in two scenarios to evaluate the effects of shellfish aquaculture.

In the first scenario, we defined the transition equation of open-access capture fish stock in an environment polluted by excessive nitrogen before the introduction of shellfish aquaculture. Excessive nitrogen contributes to environmental degradation, impacts the carrying capacity, and affects the growth rate of the fish stock. Our analysis shows a positive impact of nitrogen pollution reduction practices on the stock size and sustainable harvest. In addition, we show that the effort level that maximizes profit and optimal effort is inversely related to nitrogen concentration.

In the second scenario, we extended the model by considering the development of shellfish aquaculture and including the positive impacts of shellfish cultivation on fishing grounds, the environment, and fish stock. We assume that the effect on the fish stock is measured through growth rate, and the amount of nitrogen removed by shellfish production is proportional to the size of the aquaculture area. Even though aquaculture takes away the fishing area and creates pressure in the open access fishing ground, we show that the development of shellfish aquaculture helps the recovery of polluted aquatic ecosystems and restocks the fishing ground. In this case, we can increase the optimal effort and harvest more captured shellfish, reducing the scarcity of shellfish in the market.

Eutrophication management and control is based primarily on the restriction of nutrient inputs in bodies of water and nutrient reduction strategies at the point source. Then, it requires collaborating with fisheries managers to integrate shellfish aquaculture into broader ecosystem management plans. This approach can limit nitrogen loading and remove excessive nitrogen through shellfish harvest. This study compares the impact of nitrogen eutrophication on the fish stock before and after the development of shellfish cultivation. We show that shellfish aquaculture can be considered as an alternative option to reduce nitrogen accumulation in aquatic ecosystems. It supports the economy by creating jobs and supplying more farmed shellfish. It also indirectly improves the performance of wild catches.

Declarations

Use of AI tools

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

No Data associated with the manuscript.

Ethical approval (optional)

The authors state that the research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The authors declare that they have no conflict of interest.

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Author’s contributions

T.B.G.: Conceptualization, Methodology, Investigation, Formal analysis, Writing – original draft, Writing – Reviewing and Editing. W.T.B.: Conceptualization, Methodology, Formal analysis, Supervision, Writing – Reviewing and Editing. T.G.A.: Conceptualization, Supervision, Writing – Reviewing and editing. S.D.Z.: Conceptualization, Supervision, Writing – Reviewing and editing. All authors discussed the results and agreed to publish the manuscript.

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