Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 24.03.2025
DOI : 10.15672/hujms.1525278

Research Article

Unique determination of the initial values of the time-fractional diffusion-wave equation by lateral Cauchy data

Xuyan Jiang, Zhiyuan Li*

School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China

Abstract

This paper focuses on the determination of initial values in fractional wave equations. As is known, there are two initial conditions in fractional wave equations, and we aim to reconstruct these two unknown quantities through the minimum possible lateral Cauchy data. We construct the Liouville theorem on complex plane with the negative real axis removed, which helps us to prove the uniqueness of the inverse problem under consideration. In the final section of this paper, we propose an algorithm that utilizes lateral Cauchy data to recover the two initial values.

Mathematics Subject Classification (2020). 35R30, 26A33, 35R11

Keywords. lateral cauchy data, time fractional, initial values

1. Introduction

Many problems in the fields of science and engineering have encountered anomalous diffusion processes whose mean square displacement behaves like Ct^α, $\alpha \in (1,2)$ as $t \to \infty$. For describing these anomalous diffusion, a time-fractional diffusion-wave equation

$$\partial_t^\alpha u(x,t) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in (0,1) \times (0,T) \tag{1.1}$$

is introduced. Here ∂_t^α is Caputo derivative of order $\alpha \in (1,2)$ and is usually defined by

$$\partial_t^\alpha \phi(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\phi''(\tau)}{(t-\tau)^{\alpha-1}} d\tau, \quad t > 0,$$

where $\Gamma(\cdot)$ is a usual Gamma function. The fractional models account for memory effects and non-Gaussian statistics, which makes it as a powerful tool to describe anomalous phenomena including non-Fickian growth rates, skewness and long-tailed profile,see [1,7], and the references therein.

On the basis of the methods such as the Mittag-Leffler function and Laplace transform, important theoretical results including the unique existence and stability estimates of solutions to fractional diffusion-wave equations were established, see e.g., [4,6,17]. Furthermore, the methods developed in solving the above problem also contribute to the

*Corresponding Author.
Email addresses: 2211400019@nbu.edu.cn (X. Jiang), lizhiyuan@nbu.edu.cn (Z. Li)
Received: 31.07.2024; Accepted: 18.02.2025

subsequent researches. For example, the fractional wave equation exhibits power-law decay (see, e.g., [8,9]. In addition to asymptotic behaviors, unique continuation [13] and the maximum principle [15] have been shown for fractional models.

On the other hand, the fractional wave equation usually contains parameters that cannot be observed directly. For this, we need additional data and rely on inverse problem methodologies. In recent years, significant progress has been made in the field of inverse problems of fractional equation, and while we cannot provide a comprehensive list, the following resources are recommended: For the backward problem, refer to [3,19]. For coefficient inverse problems, see [11,18]. For inverse source problems, consider [2,14,20]. For the uniqueness of fractional order determination, refer to [10,12]. And for the determination of two initial values, see [21], which shows uniqueness using boundary measurements under homogeneous Neumann boundary conditions.

Despite the above achievement, the study of inverse problems for fractional wave equations through lateral Cauchy data still faces many challenges. Therefore, we consider

Problem 1.1. Let $u \in L^\infty(0,T;H^2(0,1))$ be a solution to the problem (1.1). We will discuss whether the two unknowns $u_0 := u(\cdot,0), u_1 := u_t(\cdot,0)$ can be uniquely determined by the lateral Cauchy data $u(0,\cdot)$ and $u_x(0,\cdot)$ in $(0,T)$.

It should be mentioned here that the measurement data is minimum possible. Let us consider the problem:

$$\partial_t^\alpha u(x,t) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in (0,1)\times(0,T).$$

It is easy to verify that $u(x,t) = \sin(\pi x)E_\alpha(-\pi^2 t^\alpha)$ satisfies the aforementioned equation, where

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \ \beta > 0$$

is the twoparameter Mittag-Leffler function defined for $z \in \mathbb{C}$. If $\beta = 1$, we write $E_{\alpha,1} = E_\alpha$ and call it the oneparameter MittagLeffler function. Since

$$\partial_t^\alpha u(x,t) = -\pi^2 \sin(\pi x)E_\alpha(-\pi^2 t^\alpha) = \partial_x^2 u(x,t).$$

Then we can directly verify that

$$\begin{cases} u(x,0) = \sin(\pi x), & u_t(x,0) = 0, \\ u(0,t) = 0, & u_x(0,t) = \pi E_\alpha(-\pi^2 t^\alpha). \end{cases}$$

This example indicates that only measurement data $u(0,t) = 0$ can not derive the $u(x,0) = u_t(x,0) = 0$. So we need the measurement data is minimum possible. If we replace $\sin(\pi x)$ with $\cos(\pi x)$, we can arrive at a similar conclusion.

This is the first time to consider recovering the two initial values of the fractional wave equation by lateral Cauchy data in contrast to [21]. We have the uniqueness result.

Theorem 1.2. *Let $u \in L^\infty(0,T;H^2(0,1))$ be a solution to the equation (1.1). Then we have $u_0(x) = u_1(x) = 0$, $x \in [0,1]$ provided that $u(0,\cdot) = u_x(0,\cdot) \equiv 0$ in $[0,T]$.*

2. Proof of the main theorem

2.1. Initial-boundary value problem

Letting $u_0 \in H^2(0,1)$, $u_1 \in H^1(0,1)$ and $g \in L^\infty(0,T)$, we consider the following initial-boundary value problem

$$\begin{cases} \partial_t^\alpha u(x,t) - \partial_x^2 u(x,t) = 0 & \text{in } (0,1)\times(0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & \text{in } (0,1), \\ u_x(0,\cdot) = 0, \ u_x(1,\cdot) = g & \text{in } (0,T). \end{cases} \tag{2.1}$$

Lemma 2.1 ([16]). *If $0 < \alpha < 2$ is an arbitrary real number, μ is such that $\frac{\alpha\pi}{2} < \mu < \min\{\pi, \pi\alpha\}$, and C are real constants, then*

$$|E_{\alpha,\alpha}(z)| \le \frac{C}{(1+|z|)^2}, \quad (|\arg(z)| \le \mu).$$

We can obtain the estimate for the solution u:

Lemma 2.2. *Let $T > 0$ be a fixed constant, then the problem* (2.1) *admits a unique solution $u \in L^\infty(0,T;H^2(0,1))$. Moreover, there exists constants C and M such that*

$$|u(0,t)| \le Ce^{MT}, \quad \textit{for any } t \in (0,T).$$

Here the constants $C, M > 0$ are independent of T but may depend on α, u_0, u_1.

Proof. By the superposition principle, we divide the problem (2.1) into two subproblems, that is $u = w + v$, where w and v are the solution to the following initial-boundary value problems:

$$\begin{cases} \partial_t^\alpha w(x,t) - \partial_x^2 w(x,t) = 0 & \text{in } (0,1)\times(0,T), \\ w(x,0) = u_0(x), \quad w_t(x,0) = u_1(x) & \text{in } (0,1), \\ w_x(0,\cdot) = 0, w_x(1,\cdot) = 0 & \text{in } (0,T), \end{cases} \tag{2.2}$$

and

$$\begin{cases} \partial_t^\alpha v(x,t) - \partial_x^2 v(x,t) = 0 & \text{in } (0,1)\times(0,T), \\ v(x,0) = 0, \quad v_t(x,0) = 0 & \text{in } (0,1), \\ v_x(0,\cdot) = 0, v_x(1,\cdot) = g & \text{in } (0,T) \end{cases} \tag{2.3}$$

respectively. For (2.2), by an argument similar to [2] we can get $w(x,t)$ is of exponential growth with respect to time, that is, $\|w(\cdot,t)\|_{H^1(0,1)} \le Ce^{MT}$ for any t in $(0,T)$. The constant C only depends on u_0, u_1.

Next we prove that there holds a similar estimate for the solution v in (2.3). For this, we set $\{\lambda_n, \varphi_n\}_{n=1}^\infty$ are Neumann eigensystems of the elliptic operator $-\partial_x^2$. It is well known that $0 = \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ and $\lambda_n \to \infty$ as $n \to \infty$, and that the sequence $\{\varphi_n\}_{n=1}^\infty$ is orthonormal basis in $L^2(0,1)$. Then taking the scalar product of (2.3) with φ_n and integrating by parts, we arrive at

$$\begin{aligned} &\partial_t^\alpha \int_0^1 v(x,t)\varphi_n(x)dx - \int_0^1 v(x,t)\partial_x^2\varphi_n(x)dx \\ =&\varphi_n(1)g(t) - \partial_x\varphi_n(1)h_1(t) + \partial_x\varphi_n(0)h_0(t) = \varphi_n(1)g(t). \end{aligned}$$

Setting $H(t) := \varphi_n(1)g(t)$, from [5], it follows that $v_n(t) := \int_0^1 v(x,t)\varphi_n(x)dx$ solve the following initial value problem for the ordinary differential equation

$$\begin{cases} \partial_t^\alpha v_n(t) + \lambda_n v_n(t) = H(t) & \text{in } (0,T) \\ v_n(0) = \frac{d}{dt}v_n(0) = 0. \end{cases}$$

By the use of the Mittag-Leffler function, it is not difficult to see that

$$v_n(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n(t-s)^\alpha\right) H(s)ds,$$

from which we can get the solution representation formula

$$v(\cdot,t) = \sum_{n=1}^\infty v_n(t)\varphi_n = \sum_{n=1}^\infty \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n(t-s)^\alpha\right) H(s)ds \right) \varphi_n.$$

Now we introduce the Neumann eigensystem $\{\mu_n, \varphi_n\}$ of the elliptic operator $A_1 := -\partial_x^2 + 1$. In fact, one can see that $\mu_n = \lambda_n + 1$. We then define the Banach space $D(A_1^\gamma)$ for $\gamma > 0$ by

$$D(A_1^\gamma) := \left\{ \varphi \in L^2(0,1); \sum_{n=1}^\infty \mu_n^{2\gamma}(\varphi,\varphi_n)^2 < \infty \right\}$$

with the norm:

$$\|\varphi\|_{D(A_1^{\gamma})} := \left\{ \sum_{n=1}^{\infty} \mu_n^{2\gamma} (\varphi, \varphi_n)^2 \right\}^{\frac{1}{2}}.$$

We use $\|\cdot\|_{H^1(0,1)}$ to denote the $H^2(0,1)$ norm for short if no conflict occurs, and we have the relation between the two norms on $H^1(0,1)$ and $D(A_1^{\frac{1}{2}})$: $\|\varphi\|_{H^1(0,1)} \le C\|\varphi\|_{D(A_1^{\frac{1}{2}})}$. Consequently, by a direct calculation, we arrive at the inequality

$$\|v(\cdot,t)\|^2_{H^1(0,1)} \le \sum_{n=1}^{\infty} \lambda_n \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n (t-s)^{\alpha}\right) H(s) ds \right)^2.$$

From the assumption $g \in L^{\infty}(0,T)$, we know $H(t) \in L^{\infty}(0,T)$, which further implies that

$$\begin{aligned} \|v(\cdot,t)\|^2_{H^1(0,1)} &\le C \sum_{n=1}^{\infty} \lambda_n \left(\int_0^t (t-s)^{\alpha-1} \left| E_{\alpha,\alpha}\left(-\lambda_n (t-s)^{\alpha}\right) \right| ds \right)^2 \\ &= C \sum_{n=1}^{\infty} \lambda_n \left(\int_0^t s^{\alpha-1} \left| E_{\alpha,\alpha}\left(-\lambda_n s^{\alpha}\right) \right| ds \right)^2. \end{aligned}$$

Now in view of the estimate for the Mittag-Leffler functions in Lemma 2.1, we have

$$\left| E_{\alpha,\alpha}\left(-\lambda_n s^{\alpha}\right) \right| \le \frac{C}{(1+\lambda_n s^{\alpha})^2},$$

which implies that

$$\|v(\cdot,t)\|^2_{H^1(0,1)} \le C \sum_{n=1}^{\infty} \left(\int_0^t s^{\alpha-1} \frac{\lambda_n^{\frac{1}{2}}}{(1+\lambda_n s^{\alpha})^2} ds \right)^2. \tag{2.4}$$

By directly calculating the integral on the right side of equation (2.4), we can obtain:

$$\begin{aligned} \|v(\cdot,t)\|^2_{H^1(0,1)} &\le C \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n^2 + \lambda_n^3 t^{\alpha}} - \frac{1}{\lambda_n^2} \right)^2 \\ &= C \sum_{n=1}^{\infty} \left(\frac{t^{\alpha}}{\lambda_n + \lambda_n^2 t^{\alpha}} \right)^2 < C \sum_{n=1}^{\infty} \left(\frac{T^{\alpha}}{\lambda_n} \right)^2 = CT^{2\alpha} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}. \end{aligned}$$

Obviously, by noting $\lambda_n = \pi^2 n^2$, the series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}$ is convergent, that is, there exists a constant $C > 0$ such that

$$\|v(\cdot,t)\|_{H^1(0,1)} < CT^{\alpha}.$$

The Rellich-Kondrachov Theorem states that the space $H^1(0,1)$ can be imbedded into the space $C[0,1]$, therefore, there exists a constant C independent of v such that

$$\|v(\cdot,t)\|_{C[0,1]} < CT^{\alpha} \|v(\cdot,t)\|_{H^1(0,1)}.$$

Collecting all the above estimates, we can easily obtain:

$$|u(\cdot,t)| < |w(\cdot,t)| + |v(\cdot,t)| < Ce^{MT}.$$

Here the constant C and M are independent of T. We finish the proof of the lemma. □

2.2. Uniqueness for inverse problem

Letting $u \in L^\infty(0,T;H^2(0,1))$ be a solution to the equation (1.1), we set $u_0(x) := u(x,0)$, $u_1(x) = u_t(x,0)$ and $g(t) = u_x(1,t)$. Moreover, we extend the function g smoothly to the interval $[0,+\infty)$ by letting $g \equiv 0$ outside of $(0,T+1)$, and by $\tilde{g}$ we denote the extension, and by U we denote the solution to the following auxiliary system

$$\begin{cases} \partial_t^\alpha U - \partial_x^2 U = 0 & \text{in } (0,1)\times(0,+\infty), \\ U(\cdot,0) = u_0,\ U_t(\cdot,0) = u_1 & \text{in } (0,1), \\ U_x(0,\cdot) = 0,\ U_x(1,\cdot) = \tilde{g} & \text{in } (0,+\infty). \end{cases}$$

Taking Laplace transform

$$\mathcal{L}\{f(t)\}(s) = \widehat{f}(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

with respect to t on both sides of the above equation. From the Laplace transform formula for the Caputo fractional derivative:

$$\mathcal{L}\left\{\partial_t^\alpha f\right\}(s) = s^\alpha \widehat{f}(s) - \sum_{k=0}^{n-1} f^{(k)}(0)s^{\alpha-1-k}, n-1<\alpha\le n, n\in\mathbb{N}.$$

We have:

$$\begin{cases} -\partial_x^2\widehat{U}(x;s) + s^\alpha \widehat{U}(x;s) = s^{\alpha-1}u_0(x) + s^{\alpha-2}u_1(x), \quad x\in(0,1), s>0, \\ \widehat{U}_x(0;s) = 0,\ \widehat{U}_x(1;s) = \int_0^{T+1} \tilde{g}(t)e^{-st}dt, \quad s>0. \end{cases}$$

Lemma 2.3. *For any $\xi \in \mathbb{R}$ and $S > 0$, the Laplace transform $\widehat{U}(x;s)$ admits the following identity:*

$$\begin{aligned} &\int_0^1 -\partial_x^2\widehat{U}(x;s)\cos((1-x)i\xi)dx \\ &= -\widehat{U}_x(1;s) - \xi^2\int_0^1 \widehat{U}(x;s)\cos((1-x)i\xi)dx - i\xi\sin(i\xi)\widehat{U}(0;s). \end{aligned}$$

Proof. This can be directly done by integration by parts. □

Now taking scalar product with the above equation and $\cos((1-x)i\xi)$ on $x \in [0,1]$ yields

$$\begin{aligned} &(s^\alpha - \xi^2)\int_0^1 \widehat{U}(x;s)\cos((1-x)i\xi)dx - i\xi\sin(i\xi)\widehat{U}(0;s) - \widehat{U}_x(1;s) \\ =&s^{\alpha-1}\int_0^1 u_0(x)\cos((1-x)i\xi)dx + s^{\alpha-2}\int_0^1 u_1(x)\cos((1-x)i\xi)dx. \end{aligned}$$

In view of the fact

$$\cos((1-x)i\xi) = \frac{e^{(1-x)\xi} + e^{(x-1)\xi}}{2},$$

the above equality can be rephrased as

$$\begin{aligned} &\frac{1}{2}s^{\alpha-1}\int_0^1 u_0(x)e^{(1-x)\xi}dx + \frac{1}{2}s^{\alpha-2}\int_0^1 u_1(x)e^{(1-x)\xi}dx \\ =&(s^\alpha-\xi^2)\int_0^1 \widehat{U}(x;s)\cos((1-x)i\xi)dx - i\xi\sin(i\xi)\widehat{U}(0;s) - \widehat{U}_x(1;s) \\ &-\frac{s^{\alpha-1}}{2}\int_0^1 u_0(x)e^{(x-1)\xi}dx - \frac{s^{\alpha-2}}{2}\int_0^1 u_1(x)e^{((x-1)\xi)}dx. \end{aligned}$$

Letting $\xi^2 = s^\alpha$, that is $\xi = s^{\frac{\alpha}{2}}$, we see that

$$\begin{aligned}
&\frac{1}{2}s^{\alpha-1}\int_0^1 u_0(x)e^{(1-x)s^{\frac{\alpha}{2}}}dx + \frac{1}{2}s^{\alpha-2}\int_0^1 u_1(x)e^{(1-x)s^{\frac{\alpha}{2}}}dx \\
=& - is^{\frac{\alpha}{2}}\sin\left(is^{\frac{\alpha}{2}}\right)\widehat{U}(0;s) - \int_0^T g(t)e^{-st}dt \\
&- \frac{s^{\alpha-1}}{2}\int_0^1 u_0(x)e^{(x-1)s^{\frac{\alpha}{2}}}dx - \frac{s^{\alpha-2}}{2}\int_0^1 u_1(x)e^{(x-1)s^{\frac{\alpha}{2}}}dx.
\end{aligned} \tag{2.5}$$

Based on identity (2.5), we can finish the proof of the main result.

Proof of Theorem 1.2. On the basis of the above formula (2.5) and noting $u_0, u_1 \in L^2(0,1)$, we see that

$$\left|\int_0^1 u_0(x)e^{(x-1)s^{\frac{\alpha}{2}}}dx\right| \le \|u_0\|_{L^1(0,1)}, \quad \left|\int_0^1 u_1(x)e^{(x-1)s^{\frac{\alpha}{2}}}dx\right| \le \|u_1\|_{L^1(0,1)},$$

which implies

$$\begin{aligned}
J :=& \left|s^{\alpha-1}\int_0^1 u_0(x)e^{(1-x)s^{\frac{\alpha}{2}}}dx + s^{\alpha-2}\int_0^1 u_1(x)e^{(1-x)s^{\frac{\alpha}{2}}}dx\right| \\
\le& 2s^{\frac{\alpha}{2}}e^{s^{\frac{\alpha}{2}}}\left|\int_0^\infty U(0,t)e^{-st}dt\right| + 2\left|\int_0^{T+1}\tilde{g}(t)e^{-st}dt\right| + s^{\alpha-1}\|u_0\|_{L^1(0,1)} + s^{\alpha-2}\|u_1\|_{L^1(0,1)}.
\end{aligned}$$

Here the inequality is due to the Euler formula

$$\sin(is^{\frac{\alpha}{2}}) = \frac{e^{-s^{\frac{\alpha}{2}}} - e^{s^{\frac{\alpha}{2}}}}{2i}.$$

From the above settings, we see that $U = u$ in $[0,1]\times[0,T]$, which combined with $u(0,t) = 0$ for $t \in [0,T]$ implies

$$J \le s^{\alpha-1}\|u_0\|_{L^1(0,1)} + s^{\alpha-2}\|u_1\|_{L^1(0,1)} + s^{\frac{\alpha}{2}}e^{s^{\frac{\alpha}{2}}}\left|\int_T^\infty U(0,t)e^{-st}dt\right| + 2\left|\int_0^{T+1}\tilde{g}(t)e^{-st}dt\right|.$$

From the choice of g, it follows that

$$\left|\int_0^{T+1}\tilde{g}(t)e^{-st}dt\right| \le \|\tilde{g}\|_{L^\infty(0,T+1)}\int_0^{T+1}e^{-st}dt \le \|\tilde{g}\|_{L^\infty(0,T+1)}s^{-1}, \quad s > 0.$$

Moreover, we can conclude from Lemma 2.2 that $|U(0,t)| \le Ce^{Mt}$, where M is a sufficiently, large constant therefore

$$\int_T^{+\infty}|U(0,t)|e^{-st}dt \le \int_T^M Ce^{(M-s)t}dt = \frac{Ce^{MT}}{s-M}e^{-sT}, \quad s > 2M.$$

On the basis of the above calculation, we further arrive at the estimate

$$J \le s^{\alpha-1}\|u_0\|_{L^1(0,1)} + s^{\alpha-2}\|u_1\|_{L^1(0,1)} + Cs^{\frac{\alpha}{2}}e^{s^{\frac{\alpha}{2}}}\frac{e^{MT}}{s-M}e^{-sT} + 2s^{-1}\|\tilde{g}\|_{L^\infty(0,T+1)}, \quad s > 2M,$$

which implies $J \le Cs^{\alpha-1}$, $s >> 1$ by noting $\alpha \in (1,2)$. We finally obtain

$$\left|\int_0^1 (u_0(x) + s^{-1}u_1(x))e^{(1-x)s^{\frac{\alpha}{2}}}dx\right| \le C, \quad s >> 1. \tag{2.6}$$

Changing the variable by taking $z = s^{\frac{\alpha}{2}}$, then (2.6) implies that

$$\left|\int_0^1 (z^{\frac{2}{\alpha}}u_0(x) + u_1(x))e^{(1-x)z}dx\right| \le Cz^{\frac{2}{\alpha}}, \quad z >> 1.$$

We set

$$F(z) := \int_0^1 (z^{-\frac{2}{\alpha}}u_0(x) + z^{-\frac{4}{\alpha}}u_1(x))e^{(1-x)z}dx. \tag{2.7}$$

Because this function is multi-valued on the complex plane, one need to cut off the negative real axis to get a single valued branch. We will show the above function is identically zero on the complex plane cutting off the negative real axis. In the case of entire functions, Liouville's theorem is a typical tool to demonstrate the aforementioned fact. Consequently, we intend to establish Liouville's theorem for analytic functions on complex plane cutting off the negative real axis. For this, the Cauchy Integral Formula yields

$$F(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{F(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i}\int_{\Gamma_{\pm}\cup\Gamma_{\varepsilon}\cup l_{\pm}\cup l} \frac{F(\xi)}{\xi - z} d\xi,$$

where l_+, l_- are lines with fixed imaginary parts $c, -c$ respectively, l is line with fixed real part $\Re z = c$ ($c > 0$ is a constant). Γ_+, Γ_- are the curves with argument angles of θ and $-\theta$ respectively ($0 < \theta < \pi$). Γ_ε is an arc: $\{|z| = \varepsilon,\ \arg z \in (-\theta, \theta)\}$. The constant c is sufficiently large so that z is within the area enclosed by these curves.

Since $F(z)$ is independent of c, we have $F(z) \to 0$ on $l_\pm$ as $c \to +\infty$, which implies

$$F(z) = \frac{1}{2\pi i}\left(\int_{\Gamma_+} + \int_{\Gamma_-} + \int_{l} + \int_{\Gamma_\varepsilon}\right) \frac{F(\xi)}{\xi - z} d\xi.$$

If $z \in \Gamma_\varepsilon$, we have the series expansion

$$\frac{F(\xi)}{\xi - z} = \frac{1}{z}\cdot\frac{F(\xi)}{\frac{\xi}{z} - 1} = \frac{-F(\xi)}{z}\sum_{n=0}^{\infty}\frac{\xi^n}{z^n}$$

and then we have

$$F(z) = \sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}} + \frac{1}{2\pi i}\left(\int_{\Gamma_+} + \int_{\Gamma_-} + \int_{l}\right)\frac{F(\xi)}{\xi - z} d\xi, \quad a_n := -\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} F(\xi)\cdot\xi^n d\xi.$$

Here we call the above identity as the Cauchy Integral Formula on complex plane with the negative real axis removed. Now fixing a point $\xi_0 \in \Gamma_+$, we claim that $\lim_{z\to\xi_0} F(z) = 0$. If not, there is a neighborhood $U(\xi_0)$ of ξ_0 such that $|F(z)| > c_0 > 0$ for $z \in U(\xi_0)$, hence that

$$\int_{\Gamma_+\cap U(\xi_0)} \frac{F(\xi)}{\xi - z} d\xi \to \infty, \quad \text{as } z \to \xi_0.$$

On the other hand, it is obvious that

$$\int_{\Gamma_+\setminus U(\xi_0)} \frac{F(\xi)}{\xi - z} d\xi, \ \int_{\Gamma_-} \frac{F(\xi)}{\xi - z} d\xi, \ \int_{l} \frac{F(\xi)}{\xi - z} d\xi \text{ are bounded for } z \in U(\xi_0).$$

Then $F(z)$ is unbounded in $U(\xi_0)$, which contradicts to $|F(z)| \le C|z|^{-\frac{2}{\alpha}}$. Thus we must have $lim_{z\to\xi_0} F(z) = 0$. Since ξ_0 is arbitrarily fixed, we derive that $F = 0$ on Γ_+. Similarly, $F = 0$ on Γ_- and l. Finally, we obtain $F(z) = \sum_{n=0}^{\infty} a_n z^{-n-1}$.

Noting that $1 < \frac{2}{\alpha} < 2$ and $2 < \frac{4}{\alpha} < 4$ from $1 < \alpha < 2$, we have in the case of $n > 3$ that

$$\int_{\Gamma_\varepsilon}\int_0^1 \left(\xi^{n-\frac{2}{\alpha}} u_0(x) + \xi^{n-\frac{4}{\alpha}} u_1(x)\right) e^{(1-x)\xi} dx d\xi \to 0, \quad \text{as } \varepsilon \to 0$$

in view of $|F(z)| \le C|z|^{-\frac{2}{\alpha}}$, hence that $a_n = 0$, $n > 3$, which implies that

$$F(z) = \sum_{n=0}^{3} a_n z^{-n-1}. \tag{2.8}$$

Moreover, by (2.7), we can see that $z^4 F(z) \to 0$ as $\Re z \to -\infty$. Therefore, a_n, $n \le 3$ must be zero, that is $F \equiv 0$.

By considering $\cos((\varepsilon - x)i\xi)$, $x \in (0, \varepsilon)$ instead of $\cos((1 - x)i\xi)$, we repeat the above argument and we can prove that for any $\varepsilon \in (0, 1)$ the following complex valued function

$$F_\varepsilon(z) := \int_0^\varepsilon (z^{-\frac{2}{\alpha}} u_0(x) + z^{-\frac{4}{\alpha}} u_1(x)) e^{(\varepsilon - x)z} dx$$

is identically vanished on $\mathbb{C}$ cutting off the negative axis. Multiplying both sides of this equation by $e^{(1-\varepsilon)z}$ yields

$$\int_0^\varepsilon (z^{-\frac{2}{\alpha}}u_0(x) + z^{-\frac{4}{\alpha}}u_1(x))e^{(1-x)z}dx \equiv 0.$$

Taking derivative with respect to ε on both sides of the above equation yields

$$(z^{\frac{2}{\alpha}}u_0(\varepsilon) + u_1(\varepsilon))e^{(1-\varepsilon)z} = 0,\ \varepsilon \in (0,1),\ z \in \mathbb{C}.$$

We finally obtain $u_0(\varepsilon) = u_1(\varepsilon) = 0$ for $\varepsilon \in (0,1)$, which finishes the proof. □

Theorem 2.4. *Let $T > 0$ be a fixed constant and $u \in L^\infty(0,T;H^2(0,1))$ be a solution to the fractional diffusion-wave equation* (1.1)*. Then we have*

$$u(x,t) = 0,\ (x,t) \text{ in } [0,1]\times[0,T]$$

provided that $u \equiv 0$ in $I \times [0,T]$, where I is a nonempty open sub interval of $(0,1)$.

Proof. Firstly, it is easy to use similar argument in proving Theorem 1.2 to derive that $\tilde{g}(t) \equiv 0, t > 0$, then $g \equiv 0$ in $(0,T)$. From the uniqueness of the IBVP we see that $U \equiv 0$. Consider that U is an extension of u, namely $u \equiv 0$ in $(0,1)\times(0,T)$. Setting $I = (x_0,x_1)$ with $(x_0,x_1) \subset [0,1]$. By $u|_{I\times(0,T)}$, we have $u(x_0,\cdot) = u_x(x_0,\cdot) = 0$ and $u(x_1,\cdot) = u_x(x_1,\cdot) = 0$ in $(0,T)$. Changing independent variables $x \to x_0 - x$ and $x \to x - x_1$ in the intervals $(0,x_0)$ and $(x_1,1)$ respectively, and applying the above conclusion in Theorem 1.2, we obtain $u \equiv 0$ in $(0,x_0)\times(0,T)$ and $(x_1,1)\times(0,T)$. We then complete the proof of the theorem .

□

2.3. Numerical simulation

In this section, we aim to utilize a new numerical method to determine the two initial values of the time-fractional wave equation using lateral Cauchy data $u_x(0,t)$. For general lateral Cauchy problem

$$\begin{cases} \partial_t^\alpha u(x,t) - \partial_x^2 u(x,t) = 0 & \text{in } (0,1)\times(0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } (0,1), \\ u(0,\cdot) = g_1(t), u_x(0,\cdot) = g_2(t) & \text{in } (0,T), \end{cases}$$

due to the lack of information on the right boundary of the equation, conventional numerical methods cannot be applied here. To obtain information about $u(x,t)$ on the right boundary, we propose using the cut-off function $\chi(x) \in C^\infty[0,1]$:

$$\chi(x) = \begin{cases} 1, & 0 < x < 1-\varepsilon, \\ \in (0,1], & 1-\varepsilon < x < 1, \\ 0, & x = 1, \end{cases}$$

to construct the necessary boundary data by multiplying it with $u(x,t)$ where $\varepsilon > 0$ is an arbitrarily small fixed constant. We set $w(x,t) = \chi(x)u(x,t)$, and it can be easily verified that

$$\begin{cases} \partial_t^\alpha w(x,t) - \partial_x^2 w(x,t) + 2\frac{\chi'}{\chi}\partial_x w(x,t) + \frac{\chi\chi'' - 2(\chi')^2}{\chi^2}w(x,t) = 0 & \text{in } (0,1)\times(0,T), \\ w(x,0) = \chi(x)u_0(x), \quad w_t(x,0) = \chi(x)u_1(x) & \text{in } (0,1), \\ w(0,t) = g_1(t), \quad w(1,t) = 0 & \text{in } (0,T). \end{cases} \tag{2.9}$$

In this way, we obtain the right boundary information. Next, we describe the algorithmic details. First, we define a forward operator

$$\mathcal{F} : (a(x), b(x)) \to w_x(0,t;a,b),$$

where $a(x) = \chi(x)u_0(x), b(x) = \chi(x)u_1(x)$ and $w_x(0,t;a,b)$ is the solution of (2.9) with the initial values $a(x)$, $b(x)$. Consequently, the inverse problem is transformed into solving the following abstract operator equation

$$\mathcal{F}(a,b) = w_x(0,t;a,b) \triangleq h(t).$$

It is well known that inverse problems are usually ill-posed. Therefore, in this paper, we employ the Tikhonov regularization method. Unlike the conjugate gradient method, this paper employs the Levenberg-Marquardt method to find approximate solutions for $u_0(x)$ and $u_1(x)$ on $(0,1-\varepsilon)$, i.e., $a(x), b(x)$ on $(0,1-\varepsilon)$. Let a^k, b^k be the k-th approximation of a, b, then it follows from the Fréchet derivative that

$$\mathcal{F}(a,b) \approx \mathcal{F}\left(a^k,b^k\right) + \mathcal{F}'_a\left(a^k,b^k\right)\left(a-a^k\right) + \mathcal{F}'_b\left(a^k,b^k\right)\left(b-b^k\right).$$

Therefore, the problem reduces to solving

$$\begin{aligned}\min J(\delta a^k,\delta b^k) = &\frac{1}{2}\left\|\mathcal{F}'_a(a^k,b^k)\delta a^k + \mathcal{F}'_a(a^k,b^k)\delta b^k - \left(h^\delta - \mathcal{F}(a^k,b^k)\right)\right\|^2_{L^2(0,T)} \\ &+ \frac{\lambda_k}{2}\left\|\delta a^k\right\|^2_{L^2(0,1)} + \frac{\mu_k}{2}\left\|\delta b^k\right\|^2_{L^2(0,1)},\end{aligned} \tag{2.10}$$

where $\delta a^k = a^{k+1} - a^k$ and $\delta b^k = b^{k+1} - b^k$, $\lambda_k, \mu_k > 0$ are regularization parameters, h^δ is the noisy functions of h.

Suppose that $\{\varphi_n(x), n = 1,2,\dots,\infty\}$ and $\{\psi_n(x), n = 1,2,\dots,\infty\}$ are two sets of basis functions in $L^2(0,1)$. Let

$$a^k(x) \approx \sum_{n=1}^N a_n^k \varphi_n(x) \quad \text{and} \quad b^k(x) \approx \sum_{n=1}^N b_n^k \psi_n(x)$$

where $N \in \mathbb{N}$ and $a_n^k, b_n^k, n = 1,2,\dots,N$ are the expansion coefficients. we set

$$\Phi^N = \operatorname{span}\{\varphi_1,\varphi_2,\dots,\varphi_N\}, \quad \Psi^N = \operatorname{span}\{\psi_1,\psi_2,\dots,\psi_N\}$$

and two $N-$dimensional vectors $\boldsymbol{a^k} = (a_1^k, a_2^k, \dots, a_N^k)$, $\boldsymbol{b^k} = (b_1^k, b_2^k, \dots, b_N^k) \in \mathbb{R}^N$. We aim to approximate a^k and b^k using $\boldsymbol{a}^k$ and $\boldsymbol{b}^k$. We combine $\boldsymbol{a}^k$ and $\boldsymbol{b}^k$ into a single vector, denoted as $\boldsymbol{p}^k = (\boldsymbol{a}^k, \boldsymbol{b}^k)$. Discretize the spatial domain into $M+1$ points: $0 = t_0 < t_1 < \cdots < t_M = T$. An effective approach to numerically solving (2.10) is to address the following minimization problem

$$\min_{\delta \mathbf{p}^k \in \mathbb{R}^N}\left\{\left\|\delta \mathbf{p}^k F^T_{\mathbf{p}^k} - (W-U)\right\|^2_{L^2(0,T)} + \delta\mathbf{p}^k D_k (\delta \mathbf{p}^k)^T\right\}, \tag{2.11}$$

where $\delta \boldsymbol{p}^k = \boldsymbol{p}^{k+1} - \boldsymbol{p}^k$,

$$F_{\mathbf{p}^k} = (f_{ij})_{M\times 2N}, \quad f_{ij} = \frac{w_x\left(0,t_i;\left(p_1^k,\dots,p_j^k+\tau,\dots,p_{2N}^k\right)\right) - w_x\left(0,t_i;\boldsymbol{p}^k\right)}{\tau},$$

τ denotes the numerical differential step, and

$$U = \left(w_x\left(0,t_1;\boldsymbol{p}^k\right), w_x\left(0,t_2;\boldsymbol{p}^k\right),\dots,w_x\left(0,t_M;\boldsymbol{p}^k\right)\right),$$

$$W = \left(h^\delta(t_1), h^\delta(t_2),\dots,h^\delta(t_M)\right),$$

and $D_k = \operatorname{diag}((\lambda_k(\varphi_i,\varphi_j)_{L^2})_{N\times N}, (\mu_k(\psi_i,\psi_j)_{L^2})_{N\times N})$.

It is easy to verify that problem (2.11) is equivalent to the following normal equation

$$\left(D_k + F^T_{\boldsymbol{p}^k}F_{\boldsymbol{p}^k}\right)\delta\boldsymbol{p}^k = F^T_{\boldsymbol{p}^k}\left(W^T - U^T\right).$$

Thus we have

$$\delta\boldsymbol{p}^k = \left(D_k + F^T_{\boldsymbol{p}^k}F_{\boldsymbol{p}^k}\right)^{-1}\left(F^T_{\boldsymbol{p}^k}\left(W^T - U^T\right)\right).$$

Next, we present two specific examples. First of all, we need some necessary clarifications. The noisy data is generated by adding a random perturbation, that is

$$h^\delta = h + \epsilon h \cdot (2 \cdot \mathrm{rand}(\mathrm{size}(h)) - 1).$$

The corresponding noise level is calculated by $\delta = \left\| h^\delta - h \right\|_{L^2(0,1)}$. In order to demonstrate the accuracy of the numerical solution, we compute the error using

$$e_k^{(a)} = \frac{\left\| a^k(x) - a(x) \right\|_{L^2(0,1-\varepsilon)}}{\|a(x)\|_{L^2(0,1-\varepsilon)}}, \quad e_k^{(b)} = \frac{\left\| b^k(x) - b(x) \right\|_{L^2(0,1-\varepsilon)}}{\|b(x)\|_{L^2(0,1-\varepsilon)}}.$$

Regarding the selection of two regularization parameters, we consider the following form

$$\lambda_{k+1} = \lambda_0 r^k, \quad k = 0, 1, \ldots$$

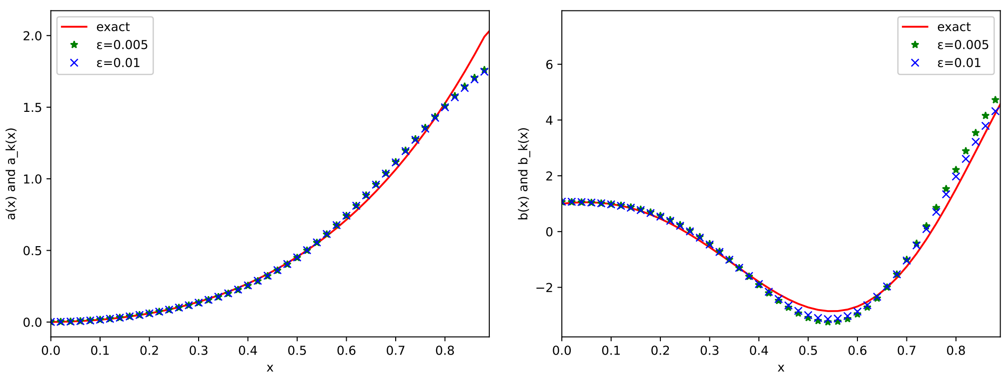
for some $\lambda_0 > 0$ and $0 < r < 1$.

Without loss of generality, we assume $T = 1$ and $g_1(t) = 0$. We set $\varepsilon = 0.1$ and utilize the cut-off function to extract the portion of u within the interval $(0, 0.9)$. That is, we define the cut-off function as follows

$$\chi(x) = \begin{cases} 1, & 0 < x < 0.9, \\ \exp\left(\frac{(x-1+0.1)^2}{(x-1+0.1)^2 - 0.1^2}\right) & 0.9 < x < 1, \\ 0, & x = 1. \end{cases}$$

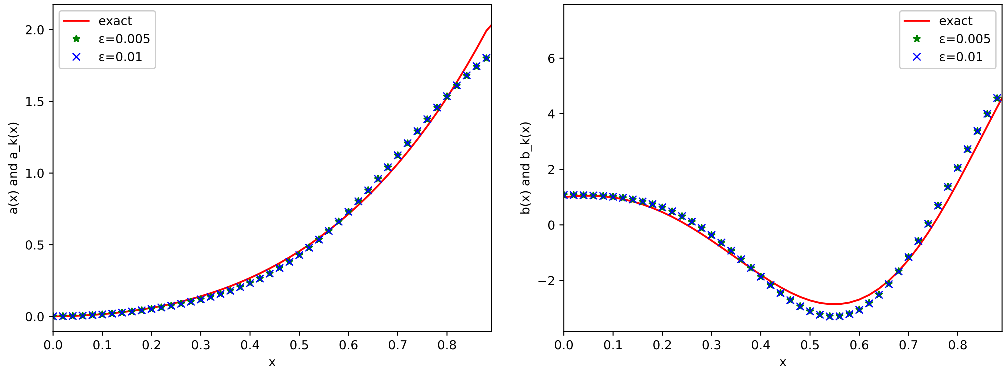
Then we can know that $w(x,t) = u(x,t)$ in $(0, 0.9)$. We solve the direct problem (2.9) by finite difference method. We take the grid size for time and space variable in the finite difference algorithm are $\Delta t = \frac{1}{50}, \Delta x = \frac{1}{50}$. The term $w_x(0,t)$ is represented by $(w_1^j - w_0^j)/dx$, where $w_i^j \approx w(x_i, t_j)$ denotes the approximate values of w at the grid points.

Example 2.5. Let $a(x) = \chi(x)x^2 e^x$ and $b(x) = \chi(x)e^{2x}\cos(2\pi x)$. We consider noise levels $\epsilon = 0.005$ and $\epsilon = 0.01$. The order α are set to 1.4 and 1.8. The initial guess is given by $a^0 = b^0 = (0, \ldots, 0)$.



(a) $a_k(x)$ **(b)** $b_k(x)$

Figure 1. The approximate solutions for Example 2.5 for $\alpha = 1.4$



(a) $a_k(x)$ **(b)** $b_k(x)$

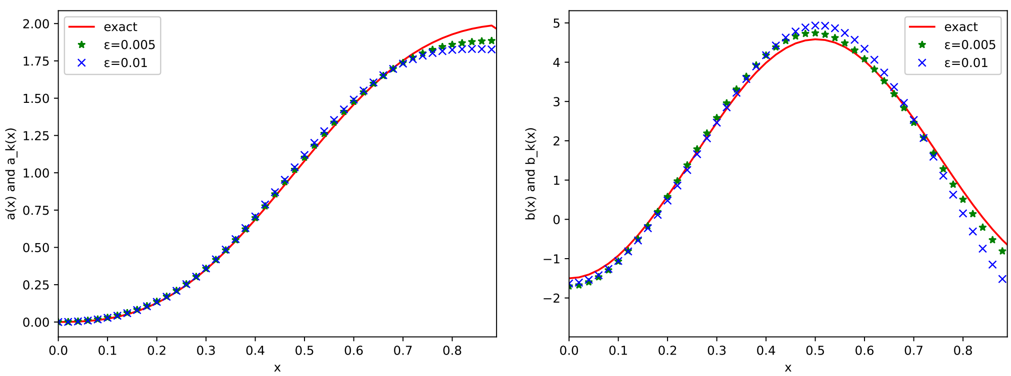
Figure 2. The approximate solutions for Example 2.5 for $\alpha = 1.8$

Figure 1 illustrates the inversion of two initial values for Example 2.5 under the condition $\alpha = 1.4$. Figure 2 shows the inversion of two initial values for Example 1 under the condition $\alpha = 1.8$. By observing Figures 1 and 2, it is evident that the value of α affects the results of our inversion for the two initial values when the regularization parameter is fixed. Specifically, as α increases within the range of $(1, 2)$, the error associated with the reconstruction decreases correspondingly.

Table 1. The error (e_k^a, e_k^b) for Example 2.5

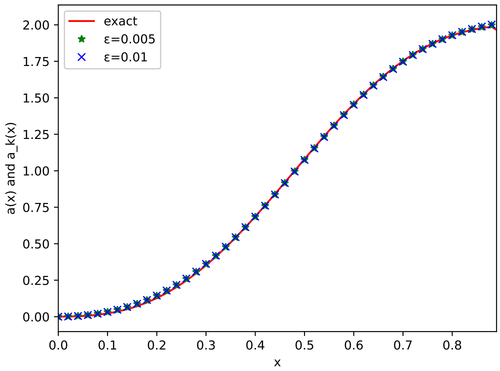
$\epsilon, (e_k^a, e_k^b), \alpha$	**1.4**	**1.8**
0.005	(0.0620,0.1685)	(0.0554,0.1470)
0.1	(0.0650,0.1100)	(0.0551,0.1496)

Example 2.6. Let $a(x) = \chi(x)(2x - \sin(2\pi x)/\pi)$ and $b(x) = \chi(x)(-3\cos(2\pi x) - \frac{x^3}{3} + \frac{x^2}{2} + \frac{3}{2})$. We consider noise levels $\epsilon = 0.005$ and $\epsilon = 0.01$. The order α are set to 1.4 and 1.8. The initial guess is given by $a^0 = b^0 = (0, \ldots, 0)$.

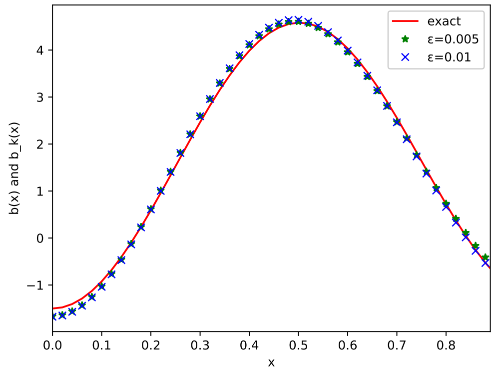


(a) $a_k(x)$ **(b)** $b_k(x)$

Figure 3. The approximate solutions for Example 2.6 for $\alpha = 1.4$



(a) $a_k(x)$



(b) $b_k(x)$

Figure 4. The approximate solutions for Example 2.6 for $\alpha = 1.8$

Figure 3 showcases the inversion of two initial values for Example 2.6, utilizing our method with α set to 1.4. In contrast, Figure 4 presents the inversion of the same two initial values for Example 2.6, but this time with α adjusted to 1.8, again using our method. In Example 2.6, we have discovered a consistent conclusion with that of Example 2.5: as the value of α increases, the accuracy of the inversion results improves. Moreover, both examples indicate that the inversion of the first initial value, $a(x)$, consistently yields more precise results than that of $b(x)$. We hypothesize that the errors incurred during the inversion of $a(x)$ may accumulate through the iterative process, subsequently affecting the accuracy of the inversion of $b(x)$.

Table 2. The error (e_k^a, e_k^b) for Example 2.6

$\epsilon, (e_k^a, e_k^b), \alpha$	**1.4**	**1.8**
0.005	(0.0291,0.0552)	(0.0063,0.0339)
0.1	(0.0449,0.1212)	(0.0074,0.0357)

Overall, these two examples indicate that our method can achieve satisfactory inversion results for the initial values over most of the domain, even in the absence of right boundary conditions.

Acknowledgements

The authors thank National Natural Science Foundation of China (no. 12271277) and Ningbo Youth Leading Talent Project (no. 2024QL045). This work is partly supported by the Open Research Fund of Key Laboratory of Nonlinear Analysis & Applications (Central China Normal University), Ministry of Education, China.

Data availability. No data was used for the research described in the article.

References

[1] D.A. Benson, S.W. Wheatcraft and M.M. Meerschaert, *Application of a fractional advection-dispersion equation*, Water ResourRes. **36** (6), 1403-1412, 2000.

[2] X. Cheng and Z. Li, *Uniqueness and stability for inverse source problem for fractional diffusion-wave equations*, J. Inverse Ill-Posed Probl. **31** (6), 885-904, 2023.

[3] G. Floridia and M. Yamamoto, Backward problems in time for fractional diffusion-wave equation, *Inverse Probl.* **36** (12) 125016, 2020.

[4] R. Gorenflo, Y. Luchko and M. Yamamoto, *Time-fractional diffusion equation in the fractional Sobolev spaces*, Fract. Calc. Appl. Anal. **18**, 799-820, 2015.

[5] Y. Kian, Z. Li, Y. Liu and M. Yamamoto, *The uniqueness of inverse problems for a fractional equation with a single measurement*, Math. Ann. **380** (3), 1465-1495, 2021.

[6] A. Kubica, K. Ryszewska and M. Yamamoto, *Time-fractional differential equations: A theoretical introduction*, Springer, Singapore, 2020.

[7] M. Levy and B. Berkowitz, *Measurement and analysis of non-Fickian dispersion in heterogeneous porous media*, J. Contam. Hydrol. **64** (3-4), 203-226, 2003.

[8] Z. Li, X. Huang and Y. Liu, *Initial-boundary value problems for coupled systems of time-fractional diffusion equations*, Fract. Calc. Appl. Anal. **26** (2), 533-566, 2023.

[9] Z. Li, Y. Liu and M. Yamamoto, *Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients*, Appl. Math. Comput. **257**, 381-397, 2015.

[10] Z. Li, Y. Liu and M. Yamamoto, *Inverse problems of determining parameters of the fractional partial differential equations*, In: Kochubei A and Luchko Y, editors. *Volume 2 Fractional Differential Equations*, 431-442. De Gruyter, Berlin, Boston, 2019.

[11] Z. Li and M. Yamamoto, *Inverse problems of determining coefficients of the fractional partial differential equations*, Handbook of fractional calculus with applications. Vol. 2, 443-464, 2019.

[12] G. Li, D. Zhang, X. Jia and M. Yamamoto, *Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation*, Inverse Probl. **29** (6), 065014, 2013.

[13] C.L. Lin and G. Nakamura, *Unique continuation property for anomalous slow diffusion equation*, Commun. Partial Differ. Equ. **41** (5), 749-758, 2016.

[14] Y. Liu, Z. Li and M. Yamamoto, *Inverse problems of determining sources of the fractional partial differential equations*, Handbook of fractional calculus with applications. Vol. 2, 411-430, 2019.

[15] Y. Luchko, *Maximum principle for the generalized time-fractional diffusion equation*, J. Math. Anal. Appl. **351** (1), 218-223, 2009.

[16] I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.

[17] K. Sakamoto and M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl. **382** (1), 426-447, 2011.

[18] T. Wei and X. Yan, *Uniqueness for identifying a space-dependent zeroth-order coefficient in a time-fractional diffusion-wave equation from a single boundary point measurement*, Appl. Math. Lett. **112**, 106814, 2021.

[19] T. Wei and Y. Zhang, *The backward problem for a time-fractional diffusion-wave equation in a bounded domain*, Comput. Math. Appl. **75** (10), 3632-3648, 2018.

[20] M. Yamamoto, *Uniqueness for inverse source problems for fractional diffusion-wave equations by data during not acting time*, Inverse Probl. **39** (2), 024004, 2023.

[21] Y. Zhang, T. Wei and Y.X. Zhang, *Simultaneous inversion of two initial values for a time-fractional diffusion-wave equation*, Numer. Methods Partial Differ. Equ. **37** (1), 24-43, 2021.