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RESEARCH ARTICLE

SOME GEOMETRIC STRUCTURES RELATED TO DESARGUES CONFIGURATION IN PG(2,5)

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Abstract

In this study, we investigate geometric structures related to Desargues configuration in *PG(2,5)*, using an algorithm implemented in C#. We determine different six complete *(11,3)-arcs* containing Desargues configuration, a *(6,2)-arc* not containing Desargues configuration, a complete *(16,4)- arc*, and an affine plane in *PG(2,5)*.

Keywords

Projective plane,
(k,n)-arc,
Complete arc,
Desargues configuration

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1. INTRODUCTION

A set N of points and a set D of subsets of N, referred to lines, make up a projective plane $\mathscr{P}$. Each pair of points is contained in exactly one line, each pair of distinct lines intersects in exactly one point, and there are four points in a position that together define six distinct lines. When considering the incidence relation provided in $\mathscr{P}$, a set B of points and lines that is also a projective plane constitutes a subplane of a projective plane $\mathscr{P}$.

French mathematician Girard Desargues discovered the mathematical statement known as Desargues' theorem in geometry in 1639. Assuming that no two corresponding sides are parallel, the theorem states that if two triangles, $A_1A_2A_3$ and $A_1{}'A_2{}'A_3{}'$, are related to one another in a way that allows them to be viewed perspectively from a single point (i.e., the lines $A_1A_1{}'$, $A_2A_2{}'$, and $A_3A_3{}'$ intersect in one point), then the points of intersection of corresponding sides all lie on one line.

In projective geometry, arcs are very important and have many uses in combinatorics and related domains. A $k{-}arc$ is a set K of k ($k \geq 3$) points in a finite projective plane π (not necessarily Desarguesian) such that no three points of K are collinear (on a line). If the plane π has order p, then $k \geq p + 2$, but the maximum value of k can only be reached if p is even. A $(p + 2)$ $-arc$ is referred to as a hyperoval. Ovals are commonly referred to in the literature, with Hirschfeld being a notable source [1]. A great deal of information has been published about arcs in projective planes, especially about complete (k,2)-arcs with complete quadrangles that generate the Fano plane in the projective planes.

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These are extensively analyzed in [2], [3]. The procedure for identifying and categorizing Fano subplanes in the projective plane, coordinated by elements of a left nearfield of order 9, is described in [4].

Fano configurations in *PG(5,2)* are discovered in [5]. The smallest Cartesian Group methods for categorizing *(k,3)−arcs* in the projective plane of order 9 and order 25 are given in [6], [7]. In [15] by Altıntas, the algorithm (written in C#) is used to examine *(k,2)−arcs* of the projective plane of order 5 coordinatized by elements of *GF(5)* [15].

The main purpose of this study is to investigate some arcs and geometric structures with related to Desargues configuration in PG(2,5). In Section 2, we give some definitions that are important in our study. In Section 3, we constructed the projective plane of order 5 with their lines, points and incidence relation over *GF(5)* and we determine a Desargues configuration. In Section 4, we introduce our algorithm and method to find all the results that related to (*k,n)− arcs*. Here in, different six complete (11,3)-arcs related to Desargues configuration and a (6,2)- arc using the points outside the Desargues configuration are found. And also, the 16 points obtained as a result of applying the algorithm to a (6,2)-arc indicate a (16,4)-arc, and the common points of (16,4)-arc and Desargues configuration specifies an affine plane. In Section 5, we give our results.

2. PRELIMINARIES

This section presents to review some relevant definitions and theorems in projective planes. Here in, some properties of arcs in projective planes are given.

Definition 1. *A (axiomatic) projective plane $\mathscr{P}$ with N being a set of points, D being a set of lines, and $\circ$ being an incidence relation, $\mathscr{P}$ is an incidence structure (N,D,$\circ$) that satisfies the following axioms:*
i) Each pair of distinct points lies on exactly one unique line;
ii) For every pair of distinct lines, there is exactly one unique point where they intersect.;
iii) $\mathscr{P}$ contains a set of four points such that no three of them lie on a single common line.

In $\mathscr{P}$, a closed configuration S is a subset of N ∪ D that remains closed under the intersections of any lines in S and the lines defined by any two distinct points in S. We denote the line through points p and q in $\mathscr{P}$ as $\langle p, q\rangle$.

Definition 2. [9] The se $V(n+1,K)$t is $(n+1)$ −dimensional vector space over the field K and is taken to be the set of vectors $X=\{x_0, x_2, \ldots, x_n\}$, $x_i \in K$. Correspondingly, *PG(n,K)* is *n*-dimensional projective space over K and is the set of elements, called points, P(x) with x ∈ $V(n+1,K)\{0\}$. When $K = GF(q) = F_q$, the finite field of q elements, also called the Galois field of q elements, then $V(n+1,K)$ is written $V(n+1,q)$and *PG(n,K)* is written *PG(n,q).* The order of *PG(n,q)* is q. The number of points in *PG(n,q)* is

$$\theta(n) = \frac{q^{n+1}-1}{q-1}.$$

$(x_1, x_2, \ldots, x_n)$ *represents any point in N, where* $x_1, x_2, \ldots, x_n$ *are non-zero. A triple's nonzero multiples correspond to the same point. Likewise,* $[a_1, a_2, \ldots, a_n]$ *represents any line in D, where* $a_1, a_2, \ldots, a_n$ *are not all 0.* P_2F *represents this projective plane, which is a point-line geometry (N, D, $\circ$) described by F. Incidence Relation:* $\circ$: $(x_1, \ldots, x_n) \circ [a_1, \ldots, a_n] \Leftrightarrow a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = 0$.

Let p be a prime number and r be a positive integer, respectively. The projective plane of order $n = p^r$ *over the finite field* $F = GF(p^r)$*, where* p^r *is the number of elements, is represented by* $P_2F = PG(2, p^r)$ [9].

Theorem 1. (Desargues Theorem) *In the projective plane, let ABC and $A'B'C'$ represent two triangles. If and only if the intersections of the corresponding sides$(AB, A'B'), (BC, B'C'), (CA, C'A')$ lie on a single line, then the lines AA', BB', CC' intersect in a single point.*

Desarguesian projective planes are projective planes which satisfy the Desargues theorem. Every projective plane over a field is Desarguesian.

Definition 4. [10] *In a projective plane, a $(k,n)-arc$ $\mathscr{K}$ is a set of k points such that some line intersects $\mathscr{K}$ in n points, and no line intersects $\mathscr{K}$ in more than n points, where $n \geq 2$.*

Definition 5. [11] *A line l of projective plane is an $\mu-secant$ of $(k,n)-arc$ $\mathscr{K}$ if l intersects $\mathscr{K}$ in μ points. Let τ_i be the total number of i-secants to $\mathscr{K}$. The number of i-secants to $\mathscr{K}$ through a point Q of $\mathscr{P} \setminus \mathscr{K}$ is denoted by σ_i or $\sigma_i(Q)$. Moreover, a point Q of $\mathscr{P} \setminus \mathscr{K}$ is called point of index zero if $\sigma_n(Q) = 0$.*

If there isn't a $(k+1,n)-arc$ that contains a $(k,n)-arc$, then it is complete.

3. PG(2,5) PROJECTIVE PLANE

In this work, we take *PG(2,5)*, which is constructed over *GF(5)* with 31 points and 31 lines under the irreducible polynomial $f(x) = x^3 + 2x^2 + x - 1$, where the elements 0, 1, 2, 3, 4 of *GF(5)*. Every line has six points, and every point has six lines passing through it in *PG(2,5)* [8].

The projective plane *PG(2,5)* has a point set *N* that is $N = \{N_i | \ i = 1, 2, \ \ldots, \ 31\}$, where

$N_1 = (0,0,1), N_2 = (1,1,1), N_3 = (1,2,2), N_4 = (1,4,2), N_5 = (1,4,3), N_6 = (1,3,4), N_7 = (1,0,3), N_8 = (1,3,1),$
$N_9 = (1,2,4), N_{10} = (1,0,4), N_{11} = (1,0,1), N_{12} = (1,2,1), N_{13} = (1,2,3), N_{14} = (1,3,0), N_{15} = (0,1,3),$
$N_{16} = (1,1,3), N_{17} = (1,3,3), N_{18} = (1,3,2), N_{19} = (1,4,0), N_{20} = (0,1,4), N_{21} = (1,1,0), N_{22} = (0,1,1),$
$N_{23} = (1,1,2), N_{24} = (1,4,4), N_{25} = (1,0,2), N_{26} = (1,4,1), N_{27} = (1,2,0), N_{28} = (0,1,2), N_{29} = (1,1,4),$
$N_{30} = (1,0,0), N_{31} = (0,1,0).$

Table 1 provides the *PG(2,5)* incident relation table. Each row represents points located on the line $D_i, i = 1, 2, \ldots, 31$.

Table 1. The lines of *PG(2,5)*

D_1	N_2	N_3	N_{17}	N_{22}	N_{24}	N_{30}
D_2	N_3	N_4	N_{18}	N_{23}	N_{25}	N_{31}
D_3	N_4	N_5	N_{19}	N_{24}	N_{26}	N_1
D_4	N_5	N_6	N_{20}	N_{25}	N_{27}	N_2
D_5	N_6	N_7	N_{21}	N_{26}	N_{28}	N_3
D_6	N_7	N_8	N_{22}	N_{27}	N_{29}	N_4
D_7	N_8	N_9	N_{23}	N_{28}	N_{30}	N_5
D_8	N_9	N_{10}	N_{24}	N_{29}	N_{31}	N_6
D_9	N_{10}	N_{11}	N_{25}	N_{30}	N_1	N_7
D_{10}	N_{11}	N_{12}	N_{26}	N_{31}	N_2	N_8
D_{11}	N_{12}	N_{13}	N_{27}	N_1	N_3	N_9
D_{12}	N_{13}	N_{14}	N_{28}	N_2	N_4	N_{10}
D_{13}	N_{14}	N_{15}	N_{29}	N_3	N_5	N_{11}
D_{14}	N_{15}	N_{16}	N_{30}	N_4	N_6	N_{12}
D_{15}	N_{16}	N_{17}	N_{31}	N_5	N_7	N_{13}
D_{16}	N_{17}	N_{18}	N_1	N_6	N_8	N_{14}

D_{17}	N_{18}	N_{19}	N_2	N_7	N_9	N_{15}
D_{18}	N_{19}	N_{20}	N_3	N_8	N_{10}	N_{16}
D_{19}	N_{20}	N_{21}	N_4	N_9	N_{11}	N_{17}
D_{20}	N_{21}	N_{22}	N_5	N_{10}	N_{12}	N_{18}
D_{21}	N_{22}	N_{23}	N_6	N_{11}	N_{13}	N_{19}
D_{22}	N_{23}	N_{24}	N_7	N_{12}	N_{14}	N_{20}
D_{23}	N_{24}	N_{25}	N_8	N_{13}	N_{15}	N_{21}
D_{24}	N_{25}	N_{26}	N_9	N_{14}	N_{16}	N_{22}
D_{25}	N_{26}	N_{27}	N_{10}	N_{15}	N_{17}	N_{23}
D_{26}	N_{27}	N_{28}	N_{11}	N_{16}	N_{18}	N_{24}
D_{27}	N_{28}	N_{29}	N_{12}	N_{17}	N_{19}	N_{25}
D_{28}	N_{29}	N_{30}	N_{13}	N_{18}	N_{20}	N_{26}
D_{29}	N_{30}	N_{31}	N_{14}	N_{19}	N_{21}	N_{27}
D_{30}	N_{31}	N_1	N_{15}	N_{20}	N_{22}	N_{28}
D_{31}	N_1	N_2	N_{16}	N_{21}	N_{23}	N_{29}

Since *PG(2,5)* is a plane over a field, any two triangles that are perspective from a center are also perspective from an axis. For example, let triangles $A_1A_2A_3$ *and* $A_1'A_2'A_3'$ be taken perspective from a point center point N_1 such that $A_1 = N_3$, $A_2 = N_2$, $A_3 = N_6$ and $A_1' = N_9$, $A_2' = N_{23}$, $A_3' = N_{14}$. Thus we have intersections points with the lines of these triangles:

$$A_1A_2 \cap A_1'A_2' = N_{30}$$
$$A_1A_3 \cap A_1'A_3' = N_{26}$$
$$A_2A_3 \cap A_2'A_3' = N_{20}$$

It is seen that N_{20}, N_{26}, N_{30} intersection points are collinear and their perspective axis the line D_{28}.

4. THE ALGORITHM USING DESARGUES CONFIGURATION TO CONSTRUCT *(k,n)*-ARCS IN PG(2,5)

In this section, we illustrate the algorithm that used to construct $(\boldsymbol{k},\boldsymbol{n})$ *−arcs* in *PG(2,5)*, and determined $(\boldsymbol{k},\mathbf{3})$ *−arcs* up to no-secant, bisecant or secant distrubitions.

4. 1. Method: Finding $(\boldsymbol{k},\boldsymbol{n})$ *−arcs*

Step 1: Finding the points and lines of *PG(2,5)* by irreducible polynomial.
In this case, we get 31 points and 31 lines as shown in Table 1.
Step 2: Consider a Desargues Configuration in *PG(2,5)*, Then, there is ten points and in this Desargues configuration forms $(10,3)$ *−arc* in *PG(2,5) which is not complete.* For example,

$$\mathcal{A} = \{N_1, N_2, N_3, N_6, N_9, N_{14}, N_{20}, N_{23}, N_{26}, N_{30}\}$$

Step 3: There are 6 points not on the Desargues configuration. Adding these points to $\mathcal{A}$ gives us six different complete $(11,3)$ *−arcs* in *PG(2,5).*

If we take into consideration the points of the Desargues configuration $\mathcal{A} = \{N_1, N_2, N_3, N_6, N_9, N_{14}, N_{20}, N_{23}, N_{26}, N_{30}\}$, it is $(10,3)$ *−arc* but not complete arc. In this projective plane, the points of this $(10,3)$ *−arc* are deleted, then $N_4, N_{10}, N_{11}, N_{15}, N_{19}, N_{31}$ points are remained. Six lines pass through each of these points. Five of them intersect the Desargues configuration at two points, and one is the remaining line, which does not intersect it at all. See Table 2.

Table 2. Desargues Configuration and remaining points incidence relations

Point	Secant lines	Not intersect
N_4	$D_2, D_3, D_{12}, D_{14}, D_{19}$	D_6
N_{10}	$D_8, D_9, D_{12}, D_{18}, D_{25}$	D_{20}
N_{11}	$D_9, D_{10}, D_{13}, D_{19}, D_{21}$	D_{26}
N_{15}	$D_{13}, D_{14}, D_{17}, D_{25}, D_{30}$	D_{23}
N_{19}	$D_3, D_{17}, D_{18}, D_{21}, D_{29}$	D_{27}
N_{31}	$D_2, D_8, D_{10}, D_{29}, D_{30}$	D_{15}

Step 4: Adding the remaining points $\{N_4, N_{10}, N_{11}, N_{15}, N_{19}, N_{31}\}$ to $(10,3)-arc$ will give us $(11,3)-arcs$.

Now we take the set $\mathcal{A} = \{N_1, N_2, N_3, N_6, N_9, N_{14}, N_{20}, N_{23}, N_{26}, N_{30}\}$ such that these points construct Desargues Configuration. We provide the following algorithm (written in C#) to find complete *(11,3)-arcs* of *PG(2,5)*:

Steps of Algorithm

$A \leftarrow Read(Excel\ File)$
$B \leftarrow Read(Text\ File)$
$C \leftarrow A$
while s(C)>0
  $B_i \leftarrow input(b), \{b|b \in C, b \notin B, i = s(B) + 1\}$
  j=1
  while $j \leq s(B)$
    for k=(j+1) to s(B)
      $m \leftarrow the\ index\ of\ row\ on\ B_j, B_k$
      $D \leftarrow A_{mn};\ \{A_{mn}|A_{mn} \neq B_j, A_{mn} \neq B_k, n = 1,\ldots,10\}$
      Remove a from A; $\{a|a \in A, a \in D\}$
      $C \leftarrow c;\ \{c|c \in A, c \notin C\}$
    end for
    j=j+1
  end while
end while

Since *PG(2,5)* is Desarguesian, choosing a specific Desargues configuration in the proofs provided throughout the paper does not affect the generality.

Theorem 1. Let $\mathcal{A}$ be Desargues configuration in *PG(2,5)*. If the given algorithm is applied to the points of Desargues configuration to find (k,3)- *arcs*, there is a (6,2)- *arc* constructing with the remaining points.

Proof. Let $\mathcal{A} = \{N_1, N_2, N_3, N_6, N_9, N_{14}, N_{20}, N_{23}, N_{26}, N_{30}\}$ be Desargues configuration in *PG(2,5)*. If we apply the algorithm to $\mathcal{A}$, the points $N_4, N_{10}, N_{11}, N_{15}, N_{19}$, and N_{31} are remained in *PG(2,5)*. Since all points of a projective plane lie on a pencil of lines through a single point, then the points of a Desargues configuration lie on a pencil of lines through a point. In this case, each line through any point outside the configuration contains two points of the Desargues configuration, while one line not contain any points of the Desargues configuration. In projective plane, there is only one line passing through any two points. So, the line passing through any two of the remaining points does not intersect the Desargues configuration. Hence, the remaining point set $\{N_4, N_{10}, N_{11}, N_{15}, N_{19}, N_{31}\}$ is a (6,2) arc.

From this point on, we will denote the remaining points set $\{N_4, N_{10}, N_{11}, N_{15}, N_{19}, N_{31}\}$ by C.

Theorem 2. There are six different complete (11,3) −*arcs* containing a Desargues configuration in PG(2,5).

Proof. Let $\mathcal{A}$ be a Desargues configuration. It is obtained (6,2*)-arc* by applying the algorithm to $\mathcal{A}$ from Theorem 1. Every point of this (6,2*)-arc* is on six lines such that five of them intersect Desargues configuration $\mathcal{A}$ in two points and one of them don't have common point with $\mathcal{A}$. If each of these points are added to $\mathcal{A}$ that is (10,3)-*arc*, it is obtained six different complete (11,3)-*arc*.

Example Let be Desargues configuration $\mathcal{A}$, following the implementation of this algorithm, all complete (11,3)−*arcs* are given in Table 3.

Table 3. Complete (11,3)-arcs

Name of the arc	Completion of (10,3) Arc $\mathcal{A}$	Complete (11,3)-arcs
$\mathcal{A}_1$	$\mathcal{A} \cup \{N_4\}$	$\{N_1, N_2, N_3, N_4, N_6, N_9, N_{14}, N_{20}, N_{23}, N_{26}, N_{30}\}$
$\mathcal{A}_2$	$\mathcal{A} \cup \{N_{10}\}$	$\{N_1, N_2, N_3\ N_6, N_9, N_{10}, N_{14}, N_{20}, N_{23}, N_{26}, N_{30}\}$
$\mathcal{A}_3$	$\mathcal{A} \cup \{N_{11}\}$	$\{N_1, N_2, N_3, N_6, N_9, N_{11}, N_{14}, N_{20}, N_{23}, N_{26}, N_{30}\}$
$\mathcal{A}_4$	$\mathcal{A} \cup \{N_{15}\}$	$\{N_1, N_2, N_3, N_6, N_9, N_{14}, N_{15}, N_{20}, N_{23}, N_{26}, N_{30}\}$
$\mathcal{A}_5$	$\mathcal{A} \cup \{N_{19}\}$	$\{N_1, N_2, N_3, N_6, N_9, N_{14}, N_{19}, N_{20}, N_{23}, N_{26}, N_{30}\}$
$\mathcal{A}_6$	$\mathcal{A} \cup \{N_{31}\}$	$\{N_1, N_2, N_3, N_6, N_9, N_{14}, N_{20}, N_{23}, N_{26}, N_{30}, N_{31}\}$

Theorem 3. Let $P_1, P_2, \ldots, P_6$, $i = 1,2, \ldots ,6$ be the vertices of 6-gon determining by (6,2)-*arc* C. If the algorithm is applied to the points on the diagonals $P_i P_{i+3}$, *i=1,2,3* of 6-*gon*, the remaining point set forms a complete *(16,4)*-ar*c*.

Proof. We consider $P_1, P_2, \ldots, P_6$, $i = 1,2, \ldots ,6$ as the vertices of 6-gon determining by (6,2)-*arc* C, there are the remaining 16 points in *PG(2,5).* It is easily seen that the remaining 16 points construct complete (*16,4)-arc* by applying algorithm.

From this point onward, this complete *(16,4)-arc* will be denoted by D.

Theorem 4. There is an affine plane determined by the common points of *(16,4)-arc* D and Desargues configuration $\mathcal{A}$ in *PG(2,5).*

Proof. Since *PG(2,5)* is Desarguesian, it applies to all Desargues configurations. For example, we consider Desargues configuration $\mathcal{A} = \{N_1, N_2, N_3, N_6, N_9, N_{14}, N_{20}, N_{23}, N_{26}, N_{30}\}$ and the complete *(16,4)-arc* $D = \{N_1, N_5, N_7, N_{9,}, N_{13}, N_{14}, N_{17}, N_{18}, N_{21}, N_{22}, N_{23}, N_{24}, N_{25}, N_{27}, N_{28}, N_{29}\}$ from Theorem 3. From this, the intersection set of $\mathcal{A}$ and D is $\{N_1, N_9, N_{14}, N_{23}\}$. If we define the set of points as $\{N_1, N_9, N_{14}, N_{23}\}$, the set of lines as $\{N_1N_9, N_9N_{23}, N_1N_{14}, N_{14}N_{23}, N_1N_{23}, N_9N_{14}\}$, and the incidence relation $\circ$ as every point being on the line that contains it, the geometric structure *(N,D,*$\circ$*)* is an affine plane.

6. CONCLUSION

In this work, it is determined that some arcs and geometric structures in *PG(2,5)* by giving the algorithm implemented in C#. The following conclusions are found:

1. In *PG(2,5),*there is exactly one *(6,2)-arc* that does not contain any common points with the *(10,3)*-arc determined by each Desargues configuration.
2. In *PG(2,5),* there are six different complete *(11,3)-arcs* containing each Desargues configuration.

3. In *PG(2,5)*, by applying the algorithm to *(6,2)-arcs* that do not intersect with the Desargues configuration, a *(16,4)-arc* is obtained.
4. An affine plane can be defined with the intersection points of the obtained *(16,4)-arc* D and the Desargues configuration $\mathcal{A}$.

These findings provide significant insights into the relationships between Desargues configurations and arcs, enhancing our understanding of their roles in *PG(2,5).*

CONFLICT OF INTEREST

The author(s) stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Elif Altıntaş Kahriman: Methodology, Formal analysis, Investigation, Resources, Writing – Original Draft, Writing – Review & Editing, Visualization**, Ayşe Bayar:** Conceptualization, Methodology, Validation, Formal analysis, Resources, Writing – Review & Editing, Visualization

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