Iğdır Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 15(1), 308-315, 2025
Journal of the Institute of Science and Technology, 15(1), 308-315, 2025
ISSN: 2146-0574, eISSN: 2536-4618
Mathematics **DOI: 10.21597/jist.1525499**
Research Article
Received: 01.08.2024 **Accepted: 26.11.2024**

To Cite: Arslantaş, M. (2025). Inverse Nodal Problem for Boundary Value Problem. *Journal of the Institute of Science and Technology*, 15(1), 308-315.

Inverse Nodal Problem for Sturm- Liouville Boundary Value Problem

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Highlights:
- Dirac δ-type potential
- Nodal problem for singular operator
- Discontinuity conditions

Keywords:
- Singular Sturm-Liouville operatör
- Nodal problem
- Inverse problem

ABSTRACT:

Inverse nodal problems has been studied for Sturm-Liouville equations with point δ coaction. First, the eigenvalues of the problem are obtained. Then, the solution of the inverse problem is given by obtaining potential function and the parameters in the boundary conditions with the help of a dense set of nodal points. Lastly, the uniqueness theorem is proven and a constructive procedure for solutions is provided.

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INTRODUCTION

Second order
$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=\lambda y$

the differential equations are called Sturm-Liouville equation, the operators produced by this equation and some different boundary conditions are called Sturm-Liouville operators, and the spectral problems for these operators are called Sturm-Liouville problems.

Boundary value problems for ordinary differential equations date back to the work of Sturm and Liouville in the mid-19th century. In many articles they published, they revealed many features of the boundary value problem. Although the Strum-Liouville theory was initially applied to heat problems, today it is one of the effective methods for investigating many physical problems.

In addition, inverse spectral problems consist of determining the spectral characteristics of the problem. Some spectral problems have an significant place in mathematics and have many practices in basic science (Marchenko, 1977; Levitan, 1984; Poschel & Trubowitz 1987; Freiling & Yurko, 2001; Yurko, 2002; Sadovnichy et al., 2009).

Inverse nodal problem consists in reconstructing the operator from a given dense set of zeros of its eigenfunctions. McLaughlin gave firstly a solution for the inverse nodal problem for the Sturm−Liouville operator (McLaughlin, 1988). Inverse nodal problems were discussed by Hald and McLaughling in their study published in 1989. In this study, inverse nodal problems for regular and singular Sturm-Liouville operators are discussed, uniqueness theorems for their solution and an algorithm for determining the potential are proposed (Hald& McLaughlin, 1989). After this study, many studies have been carried out on the inverse nodal problems for the Sturm-Liouville and Dirac operators (Browne& Sleeman, 1996; Yang, 1997; Hald& McLaughlin, 1998; Law & Yang, 1998; Law et al., 1999; Shieh & Yurko, 2008; Buterin& Shieh, 2009; Yang, 2010; Yang 2010; Chen et al., 2011; Buterin& Shieh, 2012; Guo & Wei, 2013; Yang, 2013; Yang, 2014; Manafov & Kablan, 2015; Wang& Yurko, 2016; Hu et al., 2017; Qin et al., 2019; Xu & Yang, 2019; Wang et al., 2020; Durak, 2022; Çakmak & Keskin, 2023; Amirov et al., 2024; Amirov & Durak 2024).

Let's examine the following Sturm-Liouville problem L:

$$(lu)(x):=-u''(x)+q(x)u(x)=\lambda u,\quad x\in(0,\pi)\setminus\{\xi\} \tag{1}$$

$$U(u):=u'(0)-hu(0)=0, V(u):=u(\pi)=0 \tag{2}$$

$$I(u):=\begin{cases}u(\xi+0)=u(\xi-0)=u(\xi)\\ u'(\xi+0)-u'(\xi-0)=\beta u(\xi)\end{cases} \tag{3}$$

where β is real, $q(x)\in W_2{}^1[0,\pi], \lambda=k^2$ is spectral parameter.

We can also express problem (1)-(3) with the following equation $-u''+(\beta\delta(x-\xi)+q(x))u=\lambda u, x\in(0,\pi)\setminus\{\xi\}$ where $\delta(x)$ is the Dirac function (Manafov, 2015).

On the Hilbert space $L_2([0,\pi])$ consider the linear differential expression $L: u(x)\to -u''(x)+qu(x)$ with a dense domain
$$D(L):=\left\{\begin{matrix}u(x)\in W^{22}\left[(0,\pi)\setminus\{\xi\}\right]\cap W^{20}[(0,\pi)], u'(0)-hu(0)=0,\\ u(\xi+0)=u(\xi-0), u'(\xi+0)-u'(\xi-0)=\beta u(\xi), u(\pi)=0\end{matrix}\right\}.$$

We will take into account the determination of $q(x),\beta$ when the spectral and nodal characteristic are known.

In this study, we get following conclusions of inverse nodal problems.

MATERIALS AND METHODS

Properties of the Spectrum

Let v (x,λ) be solution of equality (1) below the initial conditions $v(0,\lambda)=1, v'(0,\lambda)=h$ and the condition (3).

Theorem 1. Solutions v (x,λ) of problems (1)-(3) have the following asymptotic expressions as $|\lambda|\to\infty$:

$$v(x,\lambda)=\cos kx+\left(h+\frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin kx}{k}+o\left(\frac{e^{\tau x}}{k}\right),\ x\in[0,\xi), \tag{4}$$

$$v(x,\lambda)=\left(1-\frac{\beta}{2k}\sin 2k\xi-\frac{\beta}{4k}(1+\cos 2k\xi)\int_\xi^x q(t)dt\right)\cos kx+\left(\frac{\beta}{2}(1+\cos 2k\xi)+\frac{1}{k}\left(h+\frac{1}{2}\int_0^x q(t)dt\right)\right)\sin kx+o\left(\frac{e^{\tau x}}{k}\right), x\in(\xi,\pi]. \tag{5}$$

Proof. The solution of equation (1) that satisfies the $v(0,\lambda)=1, v'(0,\lambda)=h$ conditions is obtained as follows:

$$v(x,\lambda)=\cos kx+\frac{h}{k}\sin kx+\frac{1}{k}\int_0^x \sin k(x-t)q(t)v(t,\lambda)dt,\qquad x\in[0,\xi).$$

From here, if the (3) discontinuity conditions are applied to the solution above; we obtain the solution in the second interval as follows.

$$v(x,\lambda)=\left(1-\frac{\beta}{k}\sin k\xi\cos k\xi\right)\cos kx+\left(\beta\cos^2 k\xi+\frac{h}{k}\right)\sin kx+\frac{\beta h}{k^2}\sin k\xi\sin k(x-\xi)$$
$$+\frac{1}{k}\int_0^\xi \sin k(x-t)q(t)v(t,\lambda)dt$$
$$+\frac{\beta}{k^2}\int_0^\xi \sin k(\xi-t)\sin k(x-\xi)q(t)v(t,\lambda)dt+\frac{1}{k}\int_\xi^x \sin k(x-t)q(t)v(t,\lambda)dt,\ x\in(\xi,\pi].$$

If the solutions $v(x,\lambda)$ are written back into the integrals in the expressions of the above solutions, (4) and (5) equations are obtained.

Then $U(v)=0$. Denote $\omega(\lambda)=-V(v)=-v(\pi,\lambda)$. $\omega(\lambda)$ is the characteristic function of the problem L.

Let's define the following function

$$\omega_0(\lambda):=\cos k\pi+\frac{\beta}{2}(1+\cos 2k\xi)\sin k\pi=0. \tag{6}$$

Since $\omega_0(\lambda)$ is an entire function, its zeros are simple and real. Therefore, using the Zhdanovich article (Zhdanovich, 1960), we obtain the zeros of $\omega_0(\lambda)$ as follows:

$$k_n^0=\frac{n\pi}{\pi+\xi}+\eta_n.$$

$\{\eta_n\}$ is bounded sequence.

Lemma 1. (Amirov et al., 2024) The zeros of $\omega(\lambda)$ are as follows:

$$k_n=k_n^0+o(1).$$

Using Lemma 1, we can get following result.

Lemma 2. The following equations are valid for the zeros of $\omega(\lambda)$:

$$k_n=k_n^0+\varepsilon_n \tag{7}$$

where $\varepsilon_n \to 0, n \to \infty$.

Proof. If the expression $k_n = k_n^0 + \varepsilon_n$ is substituted in (5), from equation $\omega(\lambda_n) = 0$ we obtain;

$$\varepsilon_n = \frac{d_n}{k_n^0} + o\left(\frac{1}{k_n^0}\right)$$

Where

$$d_n = \frac{2\beta sin2k_n^0\xi cosk_n^0\pi + \beta(1 + cos2k_n^0\xi)cosk_n^0\pi - 4h\left(1 + \frac{1}{2}\int_0^\pi q(t)dt\right)sink_n^0\pi}{4(-\pi sin\, k_n^0\pi + \frac{\beta\pi}{2}(1 + cos2k_n^0\xi)cosk_n^0\pi - \xi\beta sin2k_n^0\xi sink_n^0\pi)}.$$

RESULTS AND DISCUSSION

Let's write the eigenfunctions of the boundary value problem (1)-(3) in the form $u_n(x) = v(x, \lambda_n)$. $v(x, \lambda_n)$ are real- valued functions. If (7) is written into (4) and (5); we get the asymptotic formulaes for $|n| \to +\infty$, uniformly in x:

$$v(x, \lambda_n) = cosk_n^0x - \left(\varepsilon_n x - \frac{1}{k_n^0}\left(h + \frac{1}{2}\int_0^x q(t)dt\right)\right)sink_n^0x + o\left(\frac{e^{\tau x}}{k_n^0}\right), x \in [0, \xi), \quad (8)$$

$$v(x, \lambda_n) = \left(1 - \frac{\beta}{2k_n^0}sin2k_n^0\xi - \frac{\beta}{4k_n^0}(1 + cos2k_n^0\xi)\int_\xi^x q(t)dt + \frac{\beta}{2}\varepsilon_n x(1 + cos2k_n^0\xi)\right)cosk_n^0x + \left(\frac{\beta}{2}(1 + cos2k_n^0\xi - 2\varepsilon_n\xi sin2k_n^0\xi) + \frac{1}{k_n^0}\left(h + \frac{1}{2}\int_0^x q(t)dt\right) - \varepsilon_n x\right)sink_n^0x + o\left(\frac{e^{\tau x}}{k_n^0}\right), x \in (\xi, \pi]. \quad (9)$$

From oscillation theorem it is clear the eigenfunction $v(x, \lambda_n)$has exactly $|n|$ (simple) zeros in the interval $(0, \pi)$:

$$0 < x_{n_j}^1 < \cdots < x_{n_j}^k < \xi < x_{n_j}^{k+1} < \cdots < x_{n_j}^{n-1} < \pi.$$

Theorem 2: We get the following asymptotic expressions for nodal points as $|n| \to +\infty$ uniformly in $j \in \mathbb{Z}$:

$$x_n^j = \frac{\left(j - \frac{1}{2}\right)\pi}{k_n^0} + \frac{\varepsilon_n x_n^j}{k_n^0} - \frac{1}{(k_n^0)^2}\left(h + \frac{1}{2}\int_0^{x_n^j} q(t)dt\right) + o\left(\frac{1}{(k_n^0)^2}\right), x_n^j \in [0, \xi), \quad (10)$$

$$x_n^j = \frac{\left(j - \frac{1}{2}\right)\pi}{k_n^0} + \frac{1}{k_n^0}arctan\left(\frac{\beta}{2}(1 + cos2k_n^0\xi)\right) + \frac{4A_n(x_n^j)}{(k_n^0)^2(4 + \beta^2(1 + cos2k_n^0\xi)^2)} + o\left(\frac{1}{(k_n^0)^2}\right),$$

$$x_n^j \in (\xi, \pi], \quad (11)$$

where

$$A_n(x) = h + \frac{1}{2}\int_0^x q(t)dt - \xi\beta d_n sin2k_n^0\xi - xd_n + \frac{\beta^2}{4}(1 + cos2k_n^0\xi)\left(sin2k_n^0\xi + \frac{1}{2}(1 + cos2k_n^0\xi)\int_\xi^x q(t)dt\right) - xd_n(1 + cos2k_n^0\xi) + o\left(\frac{1}{k_n^0}\right).$$

Proof: According to the definition of the nodal point, if equations (8) and (9) are equal to zero, the following equations are obtained.

$$tan\left(k_n^0x + \frac{\pi}{2}\right) = -\varepsilon_n x + \frac{1}{k_n^0}\left(h + \frac{1}{2}\int_0^x q(t)dt\right) + o\left(\frac{1}{k_n^0}\right), x \in [0, \xi),$$

$$tan\left(k_n^0x+\frac{\pi}{2}\right) = \frac{\beta}{2}(1+cos2k_n^0\xi) + \frac{1}{k_n^0}\left\{h+\frac{1}{2}\int_0^x q(t)dt - \beta\xi d_n sin2k_n^0\xi - xd_n + \frac{\beta^2}{4}(1+cos2k_n^0\xi)\left(sin2k_n^0\xi+\frac{1}{2}(1+cos2k_n^0\xi)\int_\xi^x q(t)dt - xd_n(1+cos2k_n^0\xi)\right)\right\} + o\left(\frac{1}{k_n^0}\right), x\in(\xi,\pi].$$

From this equations;

$$k_n^0x_n^j+\frac{\pi}{2}=j\pi+\varepsilon_nx_n^j-\frac{1}{k_n^0}\left(h+\frac{1}{2}\int_0^{x_n^j}q(t)dt\right)+o\left(\frac{1}{k_n^0}\right), x_n^j\in[0,\xi),$$

$$k_n^0x_n^j+\frac{\pi}{2}=j\pi+arctan\left(\frac{\beta}{2}(1+cos2k_n^0\xi)\right)+\frac{4A_n(x_n^j)}{k_n^0(4+\beta^2(1+cos2k_n^0\xi)^2)}+o\left(\frac{1}{k_n^0}\right),$$
$$x_n^j\in(\xi,\pi]$$

are obtained. Then, equations (10) and (11) are obtained.

It is clear from the expression of $\{\lambda_n^0\}_{n\geq1}$ that $\{\eta_n\}_{n\geq1}$ is a real sequence. Since $sup_n|\eta_n|\leq M<+\infty$, let's choose subsequence $\{n_t\}_{t\geq0}\subset\mathbb{N}$ as $\lim_{t\to\infty}\eta_{n_t}=\eta_0<+\infty$ .Let's define the set $\Re=\{\xi:\xi=\frac{p_0}{q_0}\pi,\ p_0<q_0,\ p_0,q_0\in\mathbb{N}\}$. It is clear that the set $\Re$ is dense in the range $(0,\pi)$ and consists of irrational numbers in the form $\xi\pi,\xi\in(0,1)\cap\mathbb{Q}$, in this range.

Let's take any points $\xi\in\Re\subset(0,\pi)$ and choose the sequence $\{n_t\}_{t\geq0}$ with $n_t=2(p_0+q_0)m_t$, $(m_t\in\mathbb{N},\lim_{t\to\infty}m_t<+\infty)$. In this case, we get $sin2k_{n_t}^0\xi=sin2\eta_{n_t}\xi, cos2k_{n_t}^0\xi=cos2\eta_{n_t}\xi, sink_{n_t}^0\pi=sin\eta_{n_t}\pi, cosk_{n_t}^0\pi=cos\eta_{n_t}\pi$.

The set $X(L):=\{x_n^j:n=1,2,\dots,j=\overline{1,n}\}$ is called the set of nodal points of the boundary value problem L.

Theorem 3. Suppose that $\{x_n^j\}\in X$, be chosen such that $\lim_{|n|\to+\infty}x_n^j=x$. Then there exists finit limits

$$f_1(x):=\lim_{t\to+\infty}\left(k_{n_t}^0x_{n_t}^{j(n_t)}-\left(j_{(n_t)}-\tfrac{1}{2}\right)\pi\right), x_{n_t}^{j(n_t)}\in[0,\xi), \quad (12)$$

$$g_1(x):=\lim_{t\to+\infty}k_{n_t}^0\left(k_{n_t}^0x_{n_t}^{j(n_t)}-\left(j_{(n_t)}-\tfrac{1}{2}\right)\pi-\varepsilon_{n_t}x_{n_t}^{j(n_t)}\right), x_{n_t}^{j(n_t)}\in[0,\xi), \quad (13)$$

$$f_2(x):=\lim_{t\to+\infty}\left(k_{n_t}^0x_{n_t}^{j(n_t)}-\left(j_{(n_t)}-\tfrac{1}{2}\right)\pi\right), x_{n_t}^{j(n_t)}\in(\xi,\pi], \quad (14)$$

$$g_2(x)\coloneqq\lim_{t\to+\infty}k_{n_t}^0\left(k_{n_t}^0x_{n_t}^{j(n_t)}-\left(j_{(n_t)}-\tfrac{1}{2}\right)\pi-arctan\left(\tfrac{\beta}{2}\left(1+cos2k_{n_t}^0\xi\right)\right)\right), x_{n_t}^{j(n_t)}\in(\xi,\pi] \quad (15)$$

where

$$\lim_{|t|\to+\infty}d_{n_t}=d_0, \lim_{|t|\to+\infty}A_{n_t}\left(x_{n_t}^{j(n_t)}\right)=A_0(x).$$

Proof. If we consider equations (10), (11), we obtain

$$k_{n_t}^0 x_{n_t}^{j(n_t)} - \left(j_{(n_t)} - \frac{1}{2}\right)\pi = \varepsilon_{n_t} x_{n_t}^{j(n_t)} - \frac{1}{k_{n_t}^0}\left(h + \frac{1}{2}\int_0^{x_{n_t}^{j(n_t)}} q(t)dt\right) + o\left(\frac{1}{k_{n_t}^0}\right), x_{n_t}^{j(n_t)} \in [0,\xi), \quad (16)$$

$$k_{n_t}^0\left(k_{n_t}^0 x_{n_t}^{j(n_t)} - \left(j_{(n_t)} - \frac{1}{2}\right)\pi - \varepsilon_{n_t} x_{n_t}^{j(n_t)}\right) = -\left(h + \frac{1}{2}\int_0^{x_{n_t}^{j(n_t)}} q(t)dt\right) + o(1), x_{n_t}^{j(n_t)} \in [0,\xi), \quad (17)$$

$$k_{n_t}^0 x_{n_t}^{j(n_t)} - \left(j_{(n_t)} - \frac{1}{2}\right)\pi = arctan\left(\frac{\beta}{2}\left(1 + cos2k_{n_t}^0\xi\right)\right) + \frac{4A_{n_t}\left(x_{n_t}^{j(n_t)}\right)}{k_{n_t}^0\left(4 + \beta^2\left(1 + cos2k_{n_t}^0\xi\right)^2\right)} + o\left(\frac{1}{k_{n_t}^0}\right), x_{n_t}^{j(n_t)} \in (\xi,\pi], \quad (18)$$

$$k_{n_t}^0\left(k_{n_t}^0 x_{n_t}^{j(n_t)} - \left(j_{(n_t)} - \frac{1}{2}\right)\pi - arctan\left(\frac{\beta}{2}\left(1 + cos2k_{n_t}^0\xi\right)\right)\right) = \frac{4A_{n_t}\left(x_{n_t}^{j(n_t)}\right)}{4 + \beta^2\left(1 + cos2k_{n_t}^0\xi\right)^2} + o(1), x_{n_t}^{j(n_t)} \in (\xi,\pi]. \quad (19)$$

Since,

$\lim_{|n|\to+\infty} x_n^j = x$ from this (16)- (19) we determine that as $|t| \to +\infty$ the limits of left side of (16)-(19) exist. Theorem 2 is completed.

Let's give the uniqueness theorem, give constractive prosedure for solving the inverse nodal problem.

Theorem 4. Let $X_{(0)} \subset X$ be a subset of nodes which is dense $(0,\pi)$. Then, the specification of uniquely determines the potential q a.e. on $(0,\pi)$. The potential q can be established via the following algorithm:

1. For each $x \in [0,\pi]$, we select a sequence $\{x_n^k\} \subset X_{(0)}$ such that $\lim_{|n|\to+\infty} x_n^k = x$.
2. The function q is determined as

$$q(x) = -2g_1'^{(x)}, x \in [0,\xi), \quad (20)$$

$$q(x) = \frac{2g_2'(x)}{1+\beta^2 cos^4\eta_0\xi}, x \in (\xi,\pi], \quad (21)$$

$$h = -g_1(0), \quad (22)$$

$$\beta = \frac{tanf_2(\pi)}{cos^2\eta_0\xi}. \quad (23)$$

Proof. The functions $f_i(x), g_i(x), (i = 1,2)$ of given by equations (12)-(15) can be easily found with the help equations (10) and (11). Then, if $x = 0$ is taken in the expression of the $g_1(x)$ function, (22) is obtained. It is easily seen that if the $g_1(x)$ function is differentiated once with respect to x, the equation (20) is obtained. Similarly, if the $g_2(x)$ function is differentiated once with respect to x, (21) is obtained. If x=π is taken in the expression of the $f_2(x)$ function, (23) is obtained.

CONCLUSION

Solvable models of quantum mechanics have an important place in the literature. As can be seen, these models are generally expressed with Hamilton operators or Schrödinger operators with singular coefficients. Many of the problems expressed by these models are related to the solution of spectral inverse problems for differential operators with singular coefficients. However, many problems in mathematical physics are reduced to the study of differential operators whose coefficients are generalized functions.

For example, the stationary vibrations of a spring-tied homogeneous wire fixed at both ends, density $R'(x) = a\delta(x - x_0)$ ($\delta(x)$ −Dirac function) and stiffness $R(x)$ at point x_0, whose domain set is

$$D(L_0) = \{y(x) \in W_2^2[0,1] : y'(x_0 + 0) - y'(x_0 - 0) = ay(x_0), x_0 \in (0,1); y(0) = 0 = y(1)\}$$

and is expressed by the differential operator given as $L_0 = -\frac{d^2}{dx^2}$ in Hilbert space $L_2[0,1]$. The correct (regular) definition of such operators and the study of their spectral properties have increased the interest in inverse problems and remain up-to-date.

In the problem considered in this study, the inverse nodal problem for the Sturm- Liouville operator with Dirac δ- type potential is investigated. With the help of nodal points, the coefficients $q(x), h, \beta$ are determined uniquely for this problem.

Conflict of Interest

The article author declare that there is no conflict of interest**.**

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