Istanbul Journal of Mathematics
ijmath 2 (1), 1–27, 2024
DOI: 10.26650/ijmath.2024.00011

RESEARCH ARTICLE

On the holonomic systems for the Gauss hypergeometric function and its confluent family of a matrix argument

H. Kimura[1*]

[1]Kumamoto University, School of Science and Technology, Department of mathematics, 2-39-1, Kurokami, Chuo-ku, Kumamoto 860-8555, Japan

ABSTRACT

We investigate the several special functions defined by a matrix integral on the Hermitian matrix space of size n. They are the matrix argument analogues of the Gauss hypergeometric, Kummer's confluent hypergeometric, the Bessel, the Hermite-Weber and Airy functions which play important roles in the multivariate statistical analysis and the random matrix theory. We give the integral representations for them as functions of eigenvalues of the matrix argument by using the result of Harish-Chandra and Itzykson-Zuber, and give the systems of differential equations for them. We show that these system are holonomic and have the holonomic rank 2^n using the theory of Gröbner basis.

Mathematics Subject Classification (2020): 33C70, 33C80

Keywords: hypergeometric function, matrix integral, holonomic system, Gröbner basis

1. INTRODUCTION

In this paper, we are concerned with the special functions of a matrix argument defined by an integral on the space of complex Hermitian matrices or normal matrices. One of the most important classes of classical special functions may be the Gauss hypergeometric function (HGF) and its confluent family, namely, Kummer's confluent HGF, Bessel function, Hermite-Weber function and Airy function. For example, Gauss, Kummer and Bessel functions are given by the power series

$$
\begin{aligned}
{}_2F_1(a,b,c;x) &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m, \\
{}_1F_1(a,c;x) &= \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m m!} x^m, \\
{}_0F_1(c+1;-x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(c+1)_m m!} x^m,
\end{aligned}
$$

respectively, where $a, b, c \in \mathbb{C}$, x is the complex variable and $(a)_m = \Gamma(a+m)/\Gamma(a)$ is the so-called Pochhammer's symbol defined by the gamma function $\Gamma(a)$. In this paper we consider and study the matrix argument analogues of these classical HGF family. The matrix argument analogues of Gauss, Kummer and Bessel are studied in connection with the multivariate statistical analysis Muirhead (1982) and with the analysis on symmetric cones Faraut and A. Koranyi (1994). We also want to add in this list the matrix argument analogues of Hermite-Weber and Airy functions, which have been studied in Inamasu and Kimura, (2021).

Let us explain our motivation of our study. The above mentioned classical HGF family is sometimes displayed schematically as

$$
\begin{array}{ccccccc}
 & & & & \text{Bessel} & & \\
 & & & \nearrow & & \searrow & \\
\text{Gauss} & \rightarrow & \text{Kummer} & & & & \text{Airy,} \\
 & & & \searrow & & \nearrow & \\
 & & & & \text{Hermite}-\text{Weber} & &
\end{array}
$$

where each arrow implies some kind of limiting process called confluence. These functions are studied by using various aspects

Corresponding Author: Hironobu Kimura **E-mail:** hiro.kimu@gmail.com
Submitted: 08.04.2024 • **Last Revision Received:** 07.06.2024 • **Accepted:** 07.06.2024

of the functions: the power series expressions, the integral representations, the differential equations, the contiguity relations. Here we focus on the aspects of differential equations and integral representations. The differential equations and the integral representations for them are given as follows.

Differential equations:

$$\begin{aligned}
\text{Gauss}: &\quad x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0,\\
\text{Kummer}: &\quad xy'' + (c-x)y' - ay = 0,\\
\text{Bessel}: &\quad xy'' + (c+1)y' + y = 0,\\
\text{Hermite-Weber}: &\quad y'' - xy' + cy = 0,\\
\text{Airy}: &\quad y'' - xy = 0.
\end{aligned}$$

Integral representations:

$$\begin{aligned}
{}_2F_1(a,b,c;x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt\\
{}_1F_1(a,c;x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{xt}dt,\\
{}_0F_1(c+1;-x) &= \int_C t^{c-1}e^{xt-\frac{1}{t}}dt,\\
H(c;x) &= \int_C t^{-c-1}e^{xt-\frac{1}{2}t^2}dt,\\
Ai(x) &= \int_C e^{xt-\frac{1}{3}t^3}dt,
\end{aligned}$$

where C is an appropriate path in the complex t-plane. Note that we took the path $\overrightarrow{01}$ as the path of integration for the Gauss' case and the Kummer's case so that the integrals give the power series expressions. If one takes another appropriate paths of integration, we get various solutions to the differential equations (see Iwasaki et al. (1991)). We should comment on the Bessel equation. In many literatures, it has the form $z^2w'' + zw' + (z^2 - c^2)w = 0$. If one perform, for this equation, the change of unknown $w \mapsto y$ by $w = z^c y$ and then the change of independent variable $z \mapsto x$ by $x = z^2/4$, we get the differential equation we gave in the list.

The Gauss HGF and its confluent family appear in many research fields of mathematics and mathematical physics and play important roles. For example, it is known that the Gauss, Kummer, Hermite-Weber, Bessel and Airy functions appear as particular solutions of the Painlevé equations P_6, P_5, P_4, P_3 and P_2, respectively Iwasaki et al. (1991).

It is also known that they are understood as simple cases of Gelfand's HGF on the complex Grassmannian manifold $\mathrm{Gr}(r,N)$, the set of r-dimensional subspaces in $\mathbb{C}^N$. Roughly speaking, Gelfand's HGF on $\mathrm{Gr}(r,N)$ is defined as follows. First we consider the maximal abelian subgroup H_λ of $\mathrm{GL}(N)$ obtained as the centralizer of a regular element a of $\mathrm{GL}(N)$, where a is in the Jordan normal form and its cell structure is described by the partition λ of N. Then Gelfand's HGF of type λ on $\mathrm{Gr}(r,N)$ is defined as the Radon transform of a character of the universal covering group $\tilde{H}_\lambda$. In this context, the Gauss, Kummer, Bessel, Hermite-Weber and Airy functions are identified with Gelfand's HGFs on $\mathrm{Gr}(2,4)$ corresponding to the partitions $(1,1,1,1), (2,1,1), (2,2), (3,1)$ and (4), respectively.

Taking into account of these facts, we think it is natural to study the extension of classical HGF family to the functions of a matrix argument including those of the Hermite-Weber and Airy functions. It should be commented that the Airy function of a matrix argument, defined by a Hermitian matrix integral in Section 2.2, already played an important role in the resolution of Witten's conjecture on the 2-dimensional quantum gravity by M. Kontsevich Kontsevich (1992).

In Inamasu and Kimura, (2021), we discussed the relation of the HGFs of a matrix argument, defined by the integrals on the space $\mathscr{H}(n)$ of Hermitian matrices, to some semi-classical orthogonal polynomials and to the polynomial solutions to the quantum Painlevé systems (see also Nagoya (2011)). We stated in Inamasu and Kimura, (2021) a conjecture on the explict form of the systems of partial differential equations characterizing the Hermite-Weber and Airy functions of a matrix argument. We give the answer (Theorem 3.1) to this conjecture deriving the systems of differential equations for a matrix argument analogue of the Gauss and its confluent family defined by the matrix integrals (Definition 2.1). It should be mentioned that the differential equations for the matrix argument analogues of Gauss, Kummer and Bessel were obtained in Muirhead (1970) by J. Muirhead. He handled the functions given by the series expansion in terms of zonal polynomials and derived the differential equations characterizing them. Our approach is different from his. We treat the functions defined by the integrals with various possible choices of domain of integration in deriving the differential equations. On the other hand, the functions treated by Muirhead correspond to the integrals with a particular choice of domain of integration, see Proposition 2.8. Since we use the matrix integrals on $\mathscr{H}(n)$ or on the space of normal matrices to define the HGFs of a matrix argument, we call them the HGFs of matrix integral type.

Another main result of this paper is Theorem 5.1 on the holonomicity of the systems and on their holonomic ranks which give the dimension of the solution space for the systems at a generic point. This theorem is proved by computing a Gröbner basis for the ideal in the ring of differential operators generated by the differential operators characterizing the HGFs.

This paper is organized as follows. In Section 2, we introduce the HGFs defined by an integral on the Hermitian matrix or normal matrix space. We give the expressions of HGFs as the functions of eigenvalues of the variable matrix. The main tools are the Harish-Chandra and Itsykson-Zuber integral formulas. In Section 3, we give the systems of differential equations for the HGFs of matrix integral type as the functions of eigenvalues of the matrix argument (Theorem 3.1). Section 4 is devoted to the proof of this theorem. In Section 5, we discuss the holonomicity and the holonomic rank of the systems (Theorem 5.1).

2. HGF OF MATRIX INTEGRAL TYPE

2.1. Integrals on Hermitian matrix space

Let $\mathscr{H}(n)$ be the set of $n \times n$ complex Hermitian matrices. It is a real vector space of dimension n^2. For $Y = (Y_{ij}) \in \mathscr{H}(n)$, let dY denote the volume element on $\mathscr{H}(n)$, which is the usual Euclidean volume element

$$dY = \bigwedge_{i=1}^{n} dY_{ii} \bigwedge_{i<j} \left(d\mathrm{Re}(Y_{ij}) \wedge d\mathrm{Im}(Y_{ij})\right),$$

where we fix some order of indices in the right hand side.

The matrix integral version of the gamma function and the beta function are defined by

$$\Gamma_n(a) = \int_{Y>0} |Y|^{a-n}\mathrm{etr}(-Y)\,dY,$$
$$B_n(a,b) := \int_{0<Y<I} |Y|^{a-n}|I-Y|^{b-n}\,dY,$$

respectively, where $Y \in \mathscr{H}(n)$, $|Y|$ is the determinant of Y, $\mathrm{tr}\,Y$ is the trace of Y, $\mathrm{etr}(Y) := \exp(\,\mathrm{tr}\,(Y))$ and the integral is taken on the set of positive definite Hermitian matrices $Y > 0$ for the gamma function and on the subset of $\mathscr{H}(n)$ satisfying $Y > 0$ and $I - Y > 0$ for the beta function. The gamma integral converges for $\mathrm{Re}(a) > n-1$ and the beta integral for $\mathrm{Re}(a) > n-1, \mathrm{Re}(b) > n-1$, and they define holomorphic functions there.

Proposition. *(see Faraut and A. Koranyi (1994)) The following formulas hold.*
(i) $\Gamma_n(a) = \pi^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} \Gamma(a+i-1)$.
(ii) $B_n(a,b) = \frac{\Gamma_n(a)\Gamma_n(b)}{\Gamma_n(a+b)}$.

2.2. HGF of matrix integral type

We introduced the family of HGFs of matrix integral type in Inamasu and Kimura, (2021). We recall them.

Definition 2.1. For $X \in \mathscr{H}(n)$, put

$$I_G(a,b,c;X) = \int_C |Y|^{a-n}|I-Y|^{c-a-n}|I-XY|^{-b}\,dY,$$
$$I_K(a,c;X) = \int_C |Y|^{a-n}|I-Y|^{c-a-n}\mathrm{etr}(XY)\,dY$$
$$I_B(c;X) = \int_C |Y|^{c-n}\mathrm{etr}(XY - Y^{-1})\,dY,$$
$$I_{HW}(c;X) = \int_C |Y|^{-c-n}\mathrm{etr}(XY - \frac{1}{2}Y^2)\,dY,$$
$$I_A(X) = \int_C \mathrm{etr}(XY - \frac{1}{3}Y^3)\,dY,$$

where C is an appropriate domain of integration in $\mathscr{H}(n)$ or in the space of normal matrices of size n for which the differentiation with respect to the entries of X can be interchanged with the integration.

Comparing the above integrals with the integral representations for the classical hypergeometric family in the introduction, one may recognize that they are extensions of the classical HGF family to functions with a matrix argument. In fact, Muirhead treated in Muirhead (1970) the extension of Gauss and Kummer to the functions of a matrix argument expressed by the series in terms of zonal polynomials. They are denoted by ${}_2F_1(a, b, c; X)$ and ${}_1F_1(a, c; X)$ and have the integral representations:

$$ {}_2F_1(a, b, c; X) = \frac{\Gamma_n(c)}{\Gamma_n(a)\Gamma_n(c-a)} \int_{0<Y<I} |Y|^{a-n}|I-Y|^{c-a-n}|I-XY|^{-b}\, dY, $$
$$ {}_1F_1(a, c; X) = \frac{\Gamma_n(c)}{\Gamma_n(a)\Gamma_n(c-a)} \int_{0<Y<I} |Y|^{a-n}|I-Y|^{c-a-n}\mathrm{etr}(XY)\, dY. $$

It should be mentioned on the choice of domains of integration C for the integrals in Definition 2.1. We required that C is chosen so that the differentiation with respect to the entries of X can be interchanged with the integration, and that we can apply the Stokes theorem. For example, to define the Airy function of matrix integral type, we consider the integral in the space of normal matrices. In this case, taking into account that a normal matrix is a matrix which is transformed to a diagonal matrix with complex eigenvalues by conjugating with a unitary matrix, we see in Proposition 2.6 that the matrix integral can be reduced to the integral on the space of eigenvalues. Then we may take the domain of integration C in the normal matrix space which, after a reduction of the integral, becomes an n-cycle of a locally finite homology group of the space of eigenvalues $y = (y_1, \dots, y_n) \in \mathbb{C}^n$ on which the integrand decreases to 0 exponentially when $|y| \to \infty$. See Hien (2007) for this kind of homology groups.

Remark 2.2. The matrix integrals in Definition 2.1 define functions of the eigenvalues $x_1, \dots, x_n$ of X, see the next subsection.

2.3. Integrals on the eigenvalues

For the HGFs of matrix integral type, we want to rewrite them to the integrals on the space of eigenvalues $y = (y_1, \dots, y_n)$ of $Y \in \mathscr{H}(n)$. To this end we need the following integral formulas . Let $\mathscr{U}(n)$ denote the group of unitary matrices of size n.

Proposition 2.3. *(Weyl integration formula) We have*

$$ \int f(Y)dY = \pi^{\frac{n(n-1)}{2}} \left(\prod_{p=1}^{n} p!\right)^{-1} \int f(gyg^*)\Delta(y)^2 dy dg, $$

where $Y \sim y = \mathrm{diag}(y_1, \dots, y_n)$ by $Y = gyg^*$ with $g \in \mathscr{U}(n)$, $\Delta(y) = \prod_{i<j}(y_i - y_j)$, $dy = dy_1 \cdots dy_n$, and dg is the normalized Haar measure on the unitary group $\mathscr{U}(n)$.

We also need the following results due to Harish-Chandra and Itzykson-Zuber. We refer to Balantekin (2000); Bleher and Kuijlaars (2004); Deift, (2000); Harnad and Orlov (2007); Mehta (1991) for these formulas.

Proposition 2.4. *Let A, B be normal matrices of size n diagonalized as*

$$ A \sim \mathrm{diag}(a_1, \dots, a_n), \quad B \sim \mathrm{diag}(b_1, \dots, b_n), $$

and assume that $a_i \neq a_j,\ b_i \neq b_j$ for $i \neq j$. For $t \in \mathbb{C}$, we have

$$ \int_{\mathscr{U}(n)} (\det(1 - tAgBg^*))^{-\alpha} dg = \prod_{p=1}^{n-1} \frac{p!}{(\alpha - n + 1)_p} \frac{\det[(1 - ta_ib_j)^{-\alpha+n-1}]}{\Delta(a)\Delta(b)}. $$

Proposition 2.5. *Let A, B be as in Proposition 2.4. For $t \in \mathbb{C}$, we have*

$$ \int_{\mathscr{U}(n)} \exp[t\,\mathrm{tr}\,(AgBg^*)]dg = \left(\prod_{p=1}^{n-1} p!\right) \frac{\det(e^{ta_ib_j})}{\Delta(a)\Delta(b)}. $$

By applying Propositions 2.3, 2.4 to the integrals in Definition 2.1, we obtain the following result.

Proposition 2.6. *Assume that $X \in \mathscr{H}(n)$ has distinct eigenvalues $x_1, \dots, x_n$. Then we have*

$$I_G(a,b,c;X) = C_0 \int_D \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det\left((1-x_j y_k)^{-b+n-1}\right) \frac{\Delta(y)}{\Delta(x)} dy,$$

$$I_K(a,c;X) = C_1 \int_D \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det(e^{x_j y_k}) \frac{\Delta(y)}{\Delta(x)} dy,$$

$$I_B(c;X) = C_1 \int_D \prod_{i=1}^n y_i^{c-n} e^{-1/y_i} \det\left(e^{x_j y_k}\right) \frac{\Delta(y)}{\Delta(x)} dy,$$

$$I_{HW}(c;X) = C_1 \int_D \prod_{i=1}^n y_i^{-c-n} e^{-\frac{1}{2}y_i^2} \det\left(e^{x_j y_k}\right) \frac{\Delta(y)}{\Delta(x)} dy,$$

$$I_A(X) = C_1 \int_D \prod_{i=1}^n e^{-\frac{1}{3}y_i^3} \det\left(e^{x_j y_k}\right) \frac{\Delta(y)}{\Delta(x)} dy,$$

where $C_0 = \pi^{\frac{n(n-1)}{2}}(n!\prod_{p=1}^{n-1}(b-n+1)_p)^{-1}$, $C_1 = \pi^{\frac{n(n-1)}{2}}(n!)^{-1}$, and D is a twisted n-cycle of the homology group defined by the integrand.

Proof. We show the assertion for $I_G(a,b,c;X)$ for the sake of completeness of presentation. We apply the Weyl integration formula to $f(Y) = |Y|^{c_1}|I-Y|^{c_2}|I-XY|^{-b}$ with $c_1 = a-n, c_2 = c-a-n$. Note that

$$\begin{aligned} f(gyg^*) &= |gyg^*|^{c_1}|I-gyg^*|^{c_2}|I-Xgyg^*|^{-b} \\ &= |y|^{c_1}|I-y|^{c_2}|I-Xgyg^*|^{-b} \\ &= \prod_{i=1}^n y_i^{c_1}(1-y_i)^{c_2} \cdot |I-Xgyg^*|^{-b}. \end{aligned}$$

Putting this in the Weyl formula and using Proposition 2.4 for $t=1$, we have

$$\begin{aligned} I_G(a,b,c;X) &= \pi^{\frac{n(n-1)}{2}}(\prod_{p=1}^n p!)^{-1} \int_D \left(\int_{\mathscr{U}(n)} |I-Xgyg^*|^{-b} dg\right) \prod_{i=1}^n y_i^{c_1}(1-y_i)^{c_2}\Delta(y)^2 dy \\ &= C_0 \int_D \frac{\det\left((1-x_j y_k)^{-b+n-1}\right)}{\Delta(x)\Delta(y)} \prod_{i=1}^n y_i^{c_1}(1-y_i)^{c_2}\Delta(y)^2 dy. \\ &= C_0 \int_D \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det\left((1-x_j y_k)^{-b+n-1}\right) \frac{\Delta(y)}{\Delta(x)} dy. \end{aligned}$$

The expressions for the other HGFs can be obtained in a similar way by using Proposition 2.5.

Remark 2.7. For the Airy integral $I_A(X)$, we can take an n-cycle D in the rapidly decay homology group Hien (2007). Let γ_1, γ_2 be the paths in $\mathbb{C}$ as in Figure 1. Then $D_{i_1,\dots,i_n} = \gamma_{i_1} \times \cdots \times \gamma_{i_n}$ for $i_1,\dots,i_n \in \{1,2\}$ gives an n-cycle and there are 2^n choices.

Now the following statement is easily deduced from Proposition 2.6.

Proposition 2.8. *(1) For ${}_2F_1(a,b,c;X)$, we assume that $X \in \mathscr{H}(n)$ has distinct eigenvalues $x_1,\dots,x_n$. Then we have*

$$\begin{aligned} {}_2F_1(a,b,c;X) &= C_2 \int_{(0,1)^n} \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det\left((1-x_j y_k)^{-b+n-1}\right) \frac{\Delta(y)}{\Delta(x)} dy \\ &= n!C_2 \int_{(0,1)^n} \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n}(1-x_i y_i)^{-b+n-1} \frac{\Delta(y)}{\Delta(x)} dy, \end{aligned}$$

where $C_2 = \frac{\Gamma_n(c)}{\Gamma_n(a)\Gamma_n(c-a)} C_0$.

(2) For ${}_1F_1(a,b,c;X)$, we assume that $X \in \mathscr{H}(n)$ has distinct eigenvalues $x_1,\dots,x_n$. Then we have

$$\begin{aligned} {}_1F_1(a,c;X) &= C_3 \int_{(0,1)^n} \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det(e^{x_j y_k}) \frac{\Delta(y)}{\Delta(x)} dy \\ &= n!C_3 \int_{(0,1)^n} \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} e^{x_i y_i} \frac{\Delta(y)}{\Delta(x)} dy, \end{aligned}$$

where $C_3 = \frac{\Gamma_n(c)}{\Gamma_n(a)\Gamma_n(c-a)} C_1$.

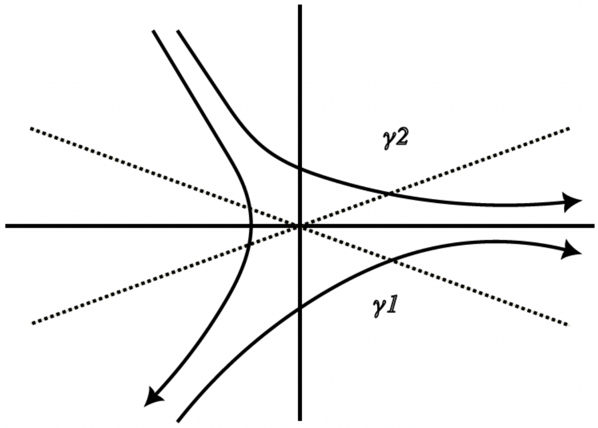

Figure 1. Figure 1

Proof. We show (2). The first representation for ${}_1F_1(a, c; X)$ is obvious by Propostion 2.6. We show the second one for ${}_1F_1$. Put $U(y) := \prod_{i=1}^{n} y_i^{a-n}(1-y_i)^{c-a-n}$ and consider the expansion

$$\det(e^{x_j y_k}) = \sum_{\sigma\in\mathfrak{S}_n} (\operatorname{sgn}\sigma) e^{x_1 y_{\sigma(1)}} \cdots e^{x_n y_{\sigma(n)}},$$

where $\mathfrak{S}_n$ is the symmetric group of degree n. Then we have

$$\frac{\Delta(x)}{C_3}\, {}_1F_1(a, c; X) = \sum_{\sigma\in\mathfrak{S}_n} \int_{(0,1)^n} (\operatorname{sgn}\sigma) U(y) \prod_{i=1}^{n} e^{x_i y_{\sigma(i)}} \Delta(y) dy.$$

Consider the integral in the right hand side for any fixed $\sigma \in \mathfrak{S}_n$ and make a change of variables $y \to y'$ defined by $y'_i = y_{\sigma(i)}$ $(i = 1, \dots, n)$. Note that, by the change $y \to y'$, the function $U(y)$ and the domain of integration $(0, 1)^n$ are invariant, and $\Delta(y') = (\operatorname{sgn}\sigma)\Delta(y)$. Hence the integrals in the right hand side are all equal to

$$\int_{(0,1)^n} \prod_{i=1}^{n} y_i^{a-n}(1-y_i)^{c-a-n} e^{x_i y_i} \Delta(y) dy.$$

This establishes the second representation for ${}_1F_1$.

3. SYSTEM OF DIFFERENTIAL EQUATIONS FOR HGF

We give the systems of differential equations satisfied by the family of HGFs of matrix integral type given in Definition 2.1. We assume that the domain of integration C for these integrals is chosen so that the interchange of derivation with respect to X and the integration with respect to Y is allowed and the Stokes theorem can be applied. Let ∂_i denote the partial derivation $\frac{\partial}{\partial x_i}$.

Theorem 3.1. *The HGF $I_*(X)$ $(* = G, K, B, HW, A)$ satisfies, as a function of eigenvalues of X, the following system of differential equations $\mathcal{S}_*$.*

Gauss $\mathcal{S}_G$:

$$x_i(1-x_i)\partial_i^2 F + \{c-(n-1)-(a+b+1-(n-1))x_i\}\partial_i F + \sum_{j(\neq i)} \frac{x_i(1-x_i)\partial_i F - x_j(1-x_j)\partial_j F}{x_i - x_j} - abF = 0, \quad 1 \le i \le n.$$

Kummer $\mathcal{S}_K$:

$$x_i\partial_i^2 F + \{c-(n-1)-x_i\}\partial_i F + \sum_{j(\neq i)} \frac{x_i\partial_i F - x_j\partial_j F}{x_i - x_j} - aF = 0, \quad 1 \le i \le n.$$

Bessel $\mathcal{S}_B$:

$$x_i\partial_i^2 F + \{c+1\}\partial_i F + \sum_{j(\neq i)} \frac{x_i\partial_i F - x_j\partial_j F}{x_i - x_j} + F = 0, \quad 1 \le i \le n.$$

Hermite-Weber $\mathcal{S}_{HW}$:

$$\partial_i^2 F - x_i\partial_i F + \sum_{j(\neq i)} \frac{\partial_i F - \partial_j F}{x_i - x_j} + cF = 0, \quad 1 \le i \le n. \tag{1}$$

Airy $\mathcal{S}_A$:

$$\partial_i^2 F + \sum_{j(\neq i)} \frac{\partial_i F - \partial_j F}{x_i - x_j} - x_i F = 0, \quad 1 \le i \le n. \tag{2}$$

The proof of the theorem is given in the next section.

As a particular case of Theorem 3.1, we have the following result, which was given by Muirhead in Muirhead (1970).

Proposition 3.2. *(1) ${}_2F_1(a,b,c;X)$, as a function of eigenvalues of X, is characterized as the holomorphic solution F to the system $\mathcal{S}_G$ which is symmetric in the variables and satisfies $F(0)=1$.*

(2) ${}_1F_1(a,c;X)$, as a function of eigenvalues of X, is characterized as the holomorphic solution F to the system $\mathcal{S}_K$, which is symmetric in the variables and satisfies $F(0)=1$.

Once we get the system of differential equations $\mathcal{S}_*$ $(*=G,K,B,HW,A)$, we can consider it as defined on $\mathbb{C}^n$. In Theorem 5.1 of the last section, we show that these systems are holonomic on the Zariski open set $\Omega_* \subset \mathbb{C}^n$ and their holonomic rank is 2^n, namely the systems are equivalent to the completely integrable Pfaffian systems of rank 2^n.

4. PROOF OF THEOREM 3.1

In this section, we use Y_{ij} $(1 \le i,j \le n)$, the entries of matrix integration variable Y, as the independent variables of the real space $\mathscr{H}(n)$ instead of $Y_{ii}, \mathrm{Re}(Y_{ij}), \mathrm{Im}(Y_{ij})$ $(1 \le i < j \le n)$. Note that, since

$$\mathrm{Re}(Y_{ij}) = \frac{Y_{ij}+Y_{ji}}{2}, \quad \mathrm{Im}(Y_{ij}) = \frac{Y_{ij}-Y_{ji}}{2\sqrt{-1}}, \quad (1 \le i \le j \le n),$$

we have

$$dY = \bigwedge_{i=1}^{n} dY_{ii} \bigwedge_{i<j} \left(\frac{\sqrt{-1}}{2} Y_{ij} \wedge Y_{ji} \right).$$

4.1. Lemmas

Let $X=(X_{ij}) \in \mathscr{H}(n)$ be diagonalized as $x = UXU^\dagger, x = \mathrm{diag}(x_1,\dots,x_n)$ by a unitary matrix U, where $U^\dagger$ is the hermitian conjugate of U, namely $U^\dagger = {}^t\bar{U}$. Assume that $x_1,\dots,x_n$ are distinct. Note that x and U depends on X. The following lemmata are known (Adler and Moerbeke (1992), p50). For the sake of completeness of presentation, we give their proof.

Lemma 4.1. *The following equalities hold.*

$$\frac{\partial x_\alpha}{\partial X_{ij}} = U_{\alpha i} U^\dagger_{j\alpha}, \tag{3}$$

$$(x_\alpha - x_\beta)\left(\frac{\partial U}{\partial X_{ij}} U^\dagger\right)_{\alpha\beta} = U_{\alpha i} U^\dagger_{j\beta}, \quad \text{if } \alpha \neq \beta. \tag{4}$$

Proof. Differentiate the both sides of $x = UXU^\dagger$ with respect to X_{ij}. Using the identity

$$\frac{\partial U}{\partial X_{ij}} U^\dagger + U \frac{\partial U^\dagger}{\partial X_{ij}} = 0, \tag{5}$$

which comes from $UU^\dagger = I$, we have

$$\begin{aligned} \frac{\partial x}{\partial X_{ij}} &= \left(\frac{\partial U}{\partial X_{ij}} U^\dagger\right) x + UE_{ij}U^\dagger + x\left(U \frac{\partial U^\dagger}{\partial X_{ij}}\right) \\ &= \left(\frac{\partial U}{\partial X_{ij}} U^\dagger\right) x + UE_{ij}U^\dagger - x\left(\frac{\partial U}{\partial X_{ij}} U^\dagger\right), \end{aligned} \tag{6}$$

where E_{ij} is the (i,j) matrix unit, namely the $n \times n$ matrix whose only non-zero entry is 1 at the (i,j)-entry. Comparing the (α,α)-entry of both sides of (6), we get (3) and comparing the (α,β)-entry with $\alpha \neq \beta$, we get (4).

Lemma 4.2. *For (α,β) with $1\le\alpha,\beta\le n$, we have the equalities:*

$$\sum_{i,j} U^\dagger_{i\beta}\frac{\partial x_\alpha}{\partial X_{ij}}U_{\beta j}=\delta_{\alpha\beta}. \tag{7}$$

$$\sum_{i,j,k} U^\dagger_{i\beta}\frac{\partial^2 x_\alpha}{\partial X_{ij}\partial X_{jk}}U_{\beta k}=\begin{cases}\frac{1}{x_\alpha-x_\beta}, & \text{if } \alpha\neq\beta,\\ \sum_{\gamma\neq\alpha}\frac{1}{x_\alpha-x_\gamma}, & \text{if } \alpha=\beta.\end{cases} \tag{8}$$

$$\sum_{a,b,p,q} U^\dagger_{al}U^\dagger_{ps}U_{sb}U_{lq}\frac{\partial^2 x_\alpha}{\partial X_{ab}\partial X_{pq}}=\begin{cases}\frac{1}{x_l-x_s}, & \text{if } \alpha=l, l\neq s,\\ -\frac{1}{x_l-x_s}, & \text{if } \alpha=s, l\neq s,\\ 0, & \text{otherwise}.\end{cases} \tag{9}$$

Proof. From the equality (3) of Lemma 4.1, we have

$$\sum_{i,j} U^\dagger_{i\beta}\frac{\partial x_\alpha}{\partial X_{ij}}U_{\beta j}=\sum_{i,j}U^\dagger_{i\beta}U_{\alpha i}U^\dagger_{j\alpha}U_{\beta j}=\delta_{\alpha\beta}\delta_{\beta\alpha}=\delta_{\alpha\beta}.$$

To show the second equality, differentiate the both sides of $\frac{\partial x_\alpha}{\partial X_{ij}}=U_{\alpha i}U^\dagger_{j\alpha}$ with respect to X_{jk} and obtain $\frac{\partial^2 x_\alpha}{\partial X_{ij}\partial X_{jk}}=\frac{\partial U_{\alpha i}}{\partial X_{jk}}U^\dagger_{j\alpha}+U_{\alpha i}\frac{\partial U^\dagger_{j\alpha}}{\partial X_{jk}}$. Denote the left hand side of (8) as $A(\alpha,\beta)$. Then

$$\begin{aligned}A(\alpha,\beta)&=\sum_{i,j,k}U^\dagger_{i\beta}\frac{\partial U_{\alpha i}}{X_{jk}}U^\dagger_{j\alpha}U_{\beta k}+\sum_{i,j,k}U^\dagger_{i\beta}U_{\alpha i}\frac{\partial U^\dagger_{j\alpha}}{\partial X_{jk}}U_{\beta k}\\ &=\sum_{j,k}\left(\frac{\partial U}{\partial X_{jk}}U^\dagger\right)_{\alpha\beta}U^\dagger_{j\alpha}U_{\beta k}+\delta_{\alpha\beta}\sum_{j,k}\frac{\partial U^\dagger_{j\alpha}}{\partial X_{jk}}U_{\beta k}=:A_1(\alpha,\beta)+A_2(\alpha,\beta).\end{aligned}$$

In the case $\alpha\neq\beta$, the contribution to A comes only from A_1. Using the equality (4) of Lemma 4.1, we have

$$A_1(\alpha,\beta)=\sum_{j,k}\frac{1}{x_\alpha-x_\beta}U_{\alpha j}U^\dagger_{k\beta}U^\dagger_{j\alpha}U_{\beta k}=\frac{1}{x_\alpha-x_\beta}.$$

In the case $\alpha=\beta$, using (5) and $U^\dagger_{j\alpha}U_{\alpha i}=\delta_{ij}-\sum_{\gamma\neq\alpha}U^\dagger_{j\gamma}U_{\gamma i}$, we have

$$\begin{aligned}A_1(\alpha,\alpha)&=-\sum_{j,k}\left(U\frac{\partial U^\dagger}{\partial X_{jk}}\right)_{\alpha\alpha}U^\dagger_{j\alpha}U_{\alpha k}=-\sum_{i,j,k}U_{\alpha i}\frac{\partial U^\dagger_{i\alpha}}{\partial X_{jk}}U^\dagger_{j\alpha}U_{\alpha k}\\ &=\sum_{\gamma\neq\alpha}\sum_{i,j,k}\frac{\partial U^\dagger_{i\alpha}}{\partial X_{jk}}U^\dagger_{j\gamma}U_{\gamma i}U_{\alpha k}-A_2(\alpha,\alpha).\end{aligned}$$

Hence using the identity (5) and Lemma 4.1, we have

$$\begin{aligned}A(\alpha,\alpha)&=A_1(\alpha,\alpha)+A_2(\alpha,\alpha)\\ &=\sum_{\gamma\neq\alpha}\sum_{j,k}\left(U\frac{\partial U^\dagger}{\partial X_{jk}}\right)_{\gamma\alpha}U^\dagger_{j\gamma}U_{\alpha k}=-\sum_{\gamma\neq\alpha}\sum_{j,k}\left(\frac{\partial U}{\partial X_{jk}}U^\dagger\right)_{\gamma\alpha}U^\dagger_{j\gamma}U_{\alpha k}\\ &=-\sum_{\gamma\neq\alpha}\frac{1}{x_\gamma-x_\alpha}\sum_{j,k}U_{\gamma j}U^\dagger_{k\alpha}U^\dagger_{j\gamma}U_{\alpha k}=-\sum_{\gamma\neq\alpha}\frac{1}{x_\gamma-x_\alpha}.\end{aligned}$$

To obtain the equality (9), differentiate the both sides of $\frac{\partial x_\alpha}{\partial X_{pq}}=U_{\alpha p}U^\dagger_{q\alpha}$ with respect to X_{ab} to get

$$\frac{\partial^2 x_\alpha}{\partial X_{ab}\partial X_{pq}}=\frac{\partial U_{\alpha p}}{\partial X_{ab}}U^\dagger_{q\alpha}+U_{\alpha p}\frac{\partial U^\dagger_{q\alpha}}{\partial X_{ab}}.$$

Denote the left hand side of (9) as $A(\alpha, l, s)$. Then

$$\begin{aligned}
A(\alpha, l, s) &= \sum_{a,b,p,q} U^\dagger_{al} U^\dagger_{ps} U_{sb} U_{lq} \left(\frac{\partial U_{\alpha p}}{\partial X_{ab}} U^\dagger_{q\alpha} + U_{\alpha p} \frac{\partial U^\dagger_{q\alpha}}{\partial X_{ab}} \right) \\
&= \sum_{a,b,p,q} U^\dagger_{al} U^\dagger_{ps} U_{sb} U_{lq} \frac{\partial U_{\alpha p}}{\partial X_{ab}} U^\dagger_{q\alpha} + \sum_{a,b,p,q} U^\dagger_{al} U^\dagger_{ps} U_{sb} U_{lq} U_{\alpha p} \frac{\partial U^\dagger_{q\alpha}}{\partial X_{ab}} \\
&= \sum_{a,b,q} U^\dagger_{al} U_{sb} U_{lq} \left(\frac{\partial U}{\partial X_{ab}} U^\dagger \right)_{\alpha s} U^\dagger_{q\alpha} + \sum_{a,b,p} U^\dagger_{al} U^\dagger_{ps} U_{sb} U_{\alpha p} \left(U \frac{\partial U^\dagger}{\partial X_{ab}} \right)_{l\alpha} \\
&= \sum_{a,b} U^\dagger_{al} U_{sb} \delta_{l\alpha} \left(\frac{\partial U}{\partial X_{ab}} U^\dagger \right)_{\alpha s} + \sum_{a,b} U^\dagger_{al} U_{sb} \delta_{\alpha s} \left(U \frac{\partial U^\dagger}{\partial X_{ab}} \right)_{l\alpha} \\
&= A_1(\alpha, l, s) + A_2(\alpha, l, s).
\end{aligned}$$

Let us compute the first term A_1. In the case $\alpha \neq l$, this term vanishes. So assume $\alpha = l$. When $l \neq s$, we have

$$A_1(l, l, s) = \sum_{a,b} U^\dagger_{al} U_{sb} \left(\frac{\partial U}{\partial X_{ab}} U^\dagger \right)_{ls} = \sum_{a,b} U^\dagger_{al} U_{sb} \frac{1}{x_l - x_s} U_{la} U^\dagger_{bs} = \frac{1}{x_l - x_s}.$$

Let us compute the term A_2 . In the case $\alpha \neq s$, this term vanishes. So assume $\alpha = s$. When $l \neq s$, we have

$$A_2(s, l, s) = \sum_{a,b} U^\dagger_{al} U_{sb} \delta_{\alpha s} \left(U \frac{\partial U^\dagger}{\partial X_{ab}} \right)_{l\alpha} = - \sum_{a,b} U^\dagger_{al} U_{sb} \left(\frac{\partial U}{\partial X_{ab}} U^\dagger \right)_{ls} = - \frac{1}{x_l - x_s}.$$

When $\alpha = l = s$, we have

$$\begin{aligned}
A(l, l, l) &= \sum_{a,b} U^\dagger_{al} U_{lb} \left(\frac{\partial U}{\partial X_{ab}} U^\dagger \right)_{ll} + \sum_{a,b} U^\dagger_{al} U_{lb} \left(U \frac{\partial U^\dagger}{\partial X_{ab}} \right)_{ll} \\
&= \sum_{a,b} U^\dagger_{al} U_{lb} \left(\frac{\partial U}{\partial X_{ab}} U^\dagger + U \frac{\partial U^\dagger}{\partial X_{ab}} \right)_{ll} = 0.
\end{aligned}$$

Thus we have proved the equality (9).

4.2. Gauss case

In the Gauss case, we put

$$F(X) = \int_C |Y|^{c_1} |I - Y|^{c_2} |I - XY|^{c_3} dY = \int_C \exp f(Y) dY, \quad X, Y \in \mathscr{H}(n), \tag{10}$$

where $c_1 = a - n, c_2 = c - a - n, c_3 = -b$,

$$f(Y) = c_1 \log |Y| + c_2 \log |I - Y| + c_3 \log |I - XY|,$$

and C is the domain of integration explained in the last paragraph of Section 2.2. By virtue of this choice of C, we can interchange the operations of differentiation with respect to X_{ij} and integration with respect to Y. In the following we will not write C in the integrals for the sake of simplicity. For a function $g(Y)$ of Y, we use the notation:

$$\langle g \rangle := \int g(Y) \exp f \, dY.$$

Lemma 4.3. *For any $1 \leq i, j \leq n$, we have*

$$\frac{\partial f}{\partial Y_{ji}} = c_1 (Y^{-1})_{ij} - c_2 \left((I - Y)^{-1} \right)_{ij} - c_3 \left((I - XY)^{-1} X \right)_{ij}, \tag{11}$$

$$\frac{\partial f}{\partial X_{ji}} = -c_3 \left(Y (I - XY)^{-1} \right)_{ij}. \tag{12}$$

Proof. We see that

$$\begin{aligned}\frac{\partial f}{\partial Y_{ji}} &= \frac{\partial}{\partial Y_{ji}}(c_1\log|Y| + c_2\log|I-Y| + c_3\log|I-XY|)\\ &= c_1\frac{1}{|Y|}\frac{\partial|Y|}{\partial Y_{ji}} + c_2\frac{1}{|I-Y|}\frac{\partial|I-Y|}{\partial Y_{ji}} + c_3\frac{1}{|I-XY|}\frac{\partial|I-XY|}{\partial Y_{ji}}\\ &= c_1\frac{1}{|Y|}C_{ji}(Y) - c_2\frac{1}{|I-Y|}C_{ji}(I-Y) + c_3\frac{1}{|I-XY|}\sum_{k=1}^{n}(-X_{kj})C_{ki}(I-XY),\end{aligned}$$

where $C_{ji}(Y)$ is the (j,i)-cofactor of $|Y|$, and we used $\frac{\partial}{\partial Y_{ji}}(XY)_{ki} = X_{kj}$ to compute the last term. Then noting that $\frac{1}{|Y|}C_{ji}(Y) = (Y^{-1})_{ij}$, we get (11). The equality (12) is shown in a similar way.

Lemma 4.4. *For any* $1 \le i, j \le n$*, we have*

$$-c_3\left\langle\left(Y(I-XY)^{-1}\right)_{ij}\right\rangle = \frac{\partial F}{\partial X_{ji}}, \tag{13}$$

$$-c_3\left\langle\left((I-XY)^{-1}\right)_{ij}\right\rangle = \sum_a X_{ia}\frac{\partial F}{\partial X_{ja}} - \delta_{ij}c_3F. \tag{14}$$

Proof. Differentiate the both sides of (10) with respect to X_{ji} and use (12) to obtain

$$\frac{\partial F}{\partial X_{ji}} = \int\frac{\partial f}{\partial X_{ji}}\exp f(Y)dY = \left\langle\frac{\partial f}{\partial X_{ji}}\right\rangle = -c_3\left\langle\left(Y(I-XY)^{-1}\right)_{ij}\right\rangle.$$

The second equality follows from

$$\begin{aligned}\sum_a X_{ia}\frac{\partial F}{\partial X_{ja}} &= \sum_a X_{ia}(-c_3)\left\langle\left(Y(I-XY)^{-1}\right)_{aj}\right\rangle = -c_3\sum_a\left\langle X_{ia}\left(Y(I-XY)^{-1}\right)_{aj}\right\rangle\\ &= -c_3\left\langle\left(XY(I-XY)^{-1}\right)_{ij}\right\rangle = -c_3\left\langle\left((I-(I-XY))\,(I-XY)^{-1}\right)_{ij}\right\rangle\\ &= -c_3\left\langle(I-XY)^{-1}_{ij}\right\rangle + c_3\delta_{ij}F.\end{aligned}$$

Put

$$\omega = \exp f(Y)\,dY,\quad \omega_{ij} = i_{\partial/\partial Y_{ij}}dY,\quad 1 \le i, j \le n, \tag{15}$$

where $i_{\partial/\partial Y_{ij}}$ is the inner derivation with the vector field $\partial/\partial Y_{ij}$.

Lemma 4.5. *For any* $1 \le i, j \le n$*, we have*

$$\left\langle c_2\left((I-Y)^{-1}\right)_{ij}\right\rangle = \sum_a X_{aj}\frac{\partial F}{\partial X_{ai}} + \delta_{ij}(c_1+c_2+n)F \tag{16}$$

and

$$\begin{aligned}\Big\langle\left((I-XY)^{-1}X\right)_{ij}&\operatorname{tr}\left((I-XY)^{-1}(I-X)Y\right)\\ &+ (c_1+n)\left((I-XY)^{-1}(I-X)\right)_{ij} - c_2\left((I-XY)^{-1}(I-X)Y(I-Y)^{-1}\right)_{ij}\\ &-c_3\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}\Big\rangle = 0.\end{aligned} \tag{17}$$

Proof. To obtain the equality (16), consider $\eta_{ij} = \sum_{k=1}^{n} Y_{ik}\exp f(Y)\,\omega_{jk}$ for $1 \le i, j \le n$. Then using Lemma 4.3,

$$\begin{aligned}d\eta_{ij} &= \left(\sum_{k=1}^{n}\frac{\partial Y_{ik}}{\partial Y_{jk}} + \sum_{k=1}^{n}Y_{ik}\frac{\partial f}{\partial Y_{jk}}\right)\omega\\ &= \left\{n\delta_{ij} + \sum_{k=1}^{n}Y_{ik}\left(c_1(Y^{-1})_{kj} - c_2((I-Y)^{-1})_{kj} - c_3\left((I-XY)^{-1}X\right)_{kj}\right)\right\}\omega\\ &= \left\{(c_1+c_2+n)\delta_{ij} - c_2((I-Y)^{-1})_{ij} - c_3\left(Y(I-XY)^{-1}X\right)_{ij}\right\}\omega.\end{aligned}$$

Since $\int d\eta_{ij} = 0$ by virtue of the Stokes theorem, using (13) we have

$$\left\langle c_2\left((I-Y)^{-1}\right)_{ij}\right\rangle = -c_3\left\langle\left(Y(I-XY)^{-1}X\right)_{ij}\right\rangle + \delta_{ij}(c_1+c_2+n)F$$
$$= \sum_a X_{aj}\frac{\partial F}{\partial X_{ai}} + \delta_{ij}(c_1+c_2+n)F.$$

To obtain the equality (17), put $\eta_{ij} = \sum_{k=1}^n \left((I-XY)^{-1}(I-X)Y\right)_{ik} \exp f(Y)\,\omega_{jk}$ and compute its exterior derivative. We have

$$d\eta_{ij} = \left(\sum_{k=1}^n \frac{\partial}{\partial Y_{jk}}\left((I-XY)^{-1}(I-X)Y\right)_{ik} + \sum_{k=1}^n\left((I-XY)^{-1}(I-X)Y\right)_{ik}\frac{\partial f}{\partial Y_{jk}}\right)\omega. \tag{18}$$

Noting that

$$\frac{\partial (I-XY)^{-1}}{\partial Y_{jk}} = (I-XY)^{-1}\left(\sum_{a=1}^n X_{aj}E_{ak}\right)(I-XY)^{-1},$$

the first terms of (18) are computed as

$$\sum_{k=1}^n \frac{\partial}{\partial Y_{jk}}\left((I-XY)^{-1}(I-X)Y\right)_{ik}$$
$$= \left((I-XY)^{-1}X\right)_{ij}\,\mathrm{tr}\,\left((I-XY)^{-1}(I-X)Y\right) + n\left((I-XY)^{-1}(I-X)\right)_{ij}. \tag{19}$$

Using (11), the second terms of (18) are computed as

$$\sum_{k=1}^n\left((I-XY)^{-1}(I-X)Y\right)_{ik}\frac{\partial f}{\partial Y_{jk}}$$
$$= c_1\left((I-XY)^{-1}(I-X)\right)_{ij} - c_2\left((I-XY)^{-1}(I-X)Y(I-Y)^{-1}\right)_{ij}$$
$$- c_3\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}. \tag{20}$$

Then the equality (17) follows from (18),(19),(20) and $\int d\eta_{ij} = 0$.

We shall derive the system of differential equations for F from (17). So it is necessary to compute

$$A_{ij} := c_2\left\langle\left((I-XY)^{-1}(I-X)Y(I-Y)^{-1}\right)_{ij}\right\rangle,$$
$$B_{ij} := c_3\left\langle\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}\right\rangle.$$

To compute A_{ij}, note that

$$(I-XY)^{-1}(I-X)Y(I-Y)^{-1} = -(I-XY)^{-1} + (I-Y)^{-1}.$$

Then from (14) and (16) we have

$$c_3A_{ij} = \left\langle -c_2c_3\left((I-XY)^{-1}\right)_{ij}\right\rangle + \left\langle c_2c_3\left((I-Y)^{-1}\right)_{ij}\right\rangle$$
$$= c_2\left(\sum_a X_{ia}\frac{\partial F}{\partial X_{ja}} - \delta_{ij}c_3F\right) + c_3\left(\sum_a X_{aj}\frac{\partial F}{\partial X_{ai}} + \delta_{ij}(c_1+c_2+n)F\right)$$
$$= c_2\sum_a X_{ia}\frac{\partial F}{\partial X_{ja}} + c_3\sum_a X_{aj}\frac{\partial F}{\partial X_{ai}} + \delta_{ij}(c_1+n)c_3F. \tag{21}$$

To compute B_{ij}, note that

$$\left\langle\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}\right\rangle$$
$$= \sum_a X_{aj}\sum_b\left\langle\left((I-XY)^{-1}(I-X)\right)_{ib}\left(Y(I-XY)^{-1}\right)_{ba}\right\rangle.$$

Taking this into account, we differentiate $\left\langle \left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle$ with respect to X_{ab} and get

$$\begin{aligned}&\frac{\partial}{\partial X_{ab}}\left((I-XY)^{-1}(I-X)\right)_{ib}\\&=\left((I-XY)^{-1}(\sum_k Y_{bk}E_{ak})(I-XY)^{-1}(I-X)\right)_{ib}-\left((I-XY)^{-1}E_{ab}\right)_{ib}\\&=(I-XY)^{-1}_{ia}\left(Y(I-XY)^{-1}(I-X)\right)_{bb}-(I-XY)^{-1}_{ia}.\end{aligned}$$

Using (12) we have

$$\begin{aligned}\frac{\partial}{\partial X_{ab}}\left\langle\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle&\\&=\left\langle(I-XY)^{-1}_{ia}\left(Y(I-XY)^{-1}(I-X)\right)_{bb}-(I-XY)^{-1}_{ia}\right.\\&\qquad\left.-c_3\left((I-XY)^{-1}(I-X)\right)_{ib}\left(Y(I-XY)^{-1}\right)_{ba}\right\rangle.\end{aligned}$$

Then

$$\begin{aligned}&\sum_a X_{aj}\sum_b\frac{\partial}{\partial X_{ab}}\left\langle\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle\\&=\sum_a X_{aj}\left\langle(I-XY)^{-1}_{ia}\operatorname{tr}\left(Y(I-XY)^{-1}(I-X)\right)-n(I-XY)^{-1}_{ia}\right.\\&\qquad\left.-c_3\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}\right)_{ia}\right\rangle\\&=\left\langle\left((I-XY)^{-1}X\right)_{ij}\operatorname{tr}\left(Y(I-XY)^{-1}(I-X)\right)-n\left((I-XY)^{-1}X\right)_{ij}\right.\\&\qquad\left.-c_3\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}\right\rangle.\end{aligned}$$

Thus we have

$$\begin{aligned}B_{ij}=-\sum_a X_{aj}\sum_b\frac{\partial}{\partial X_{ab}}\left\langle\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle-n\left\langle\left((I-XY)^{-1}X\right)_{ij}\right\rangle\\+\left\langle\left((I-XY)^{-1}X\right)_{ij}\operatorname{tr}\left(Y(I-XY)^{-1}(I-X)\right)\right\rangle.\end{aligned}$$

Hence the relation (17) becomes

$$\begin{aligned}\sum_a X_{aj}\sum_b\frac{\partial}{\partial X_{ab}}\left\langle c_3\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle&\\+\left\langle(c_1+n)c_3\left((I-XY)^{-1}\right)_{ij}\right\rangle-\left\langle c_1c_3\left((I-XY)^{-1}X\right)_{ij}\right\rangle-c_3A_{ij}&=0.\end{aligned}\tag{22}$$

We assert that this relation gives the differential equations for F.

Lemma 4.6. *The function F, defined by (10), satisfies the differential equations*

$$\begin{aligned}\sum_{a,b,p,q}X_{aj}(I-X)_{pb}X_{iq}\frac{\partial^2F}{\partial X_{ab}\partial X_{pq}}+\sum_{b,p}X_{ij}(I-X)_{pb}\frac{\partial F}{\partial X_{pb}}&\\-c_3\sum_{a,b}X_{aj}(I-X)_{ib}\frac{\partial F}{\partial X_{ab}}-(c_1+n)\sum_{p,q}X_{pj}X_{iq}\frac{\partial F}{\partial X_{pq}}&\\+(c_1+c_2+n)\sum_a X_{ia}\frac{\partial F}{\partial X_{ja}}+c_3\sum_a X_{aj}\frac{\partial F}{\partial X_{ai}}+(c_1+n)c_3X_{ij}F=0,\ 1\le i,j\le n.&\end{aligned}\tag{23}$$

Proof. We express all the terms in (22) in terms of F and its derivatives. The first term in (22) is computed as follow. From (14), we have

$$\left\langle c_3\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle=-\sum_p(I-X)_{pb}\left(\sum_q X_{iq}\frac{\partial F}{\partial X_{pq}}-\delta_{ip}c_3F\right).$$

Differentiate the both sides with respect to X_{ab}. Then, from the right hand side, we have

$$-\sum_p (I-X)_{pb}\left(\sum_q X_{iq}\frac{\partial^2 F}{\partial X_{ab}\partial X_{pq}} + \delta_{ia}\frac{\partial F}{\partial X_{pb}} - \delta_{ip}c_3\frac{\partial F}{\partial X_{ab}}\right) + \left(\sum_q X_{iq}\frac{\partial F}{\partial X_{aq}} - \delta_{ia}c_3 F\right).$$

Hence we have

$$\begin{aligned}\sum_a X_{aj}\sum_b \frac{\partial}{\partial X_{ab}}\left\langle c_3\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle \\ = -\sum_{a,b,p,q} X_{aj}(I-X)_{pb}X_{iq}\frac{\partial^2 F}{\partial X_{ab}\partial X_{pq}} - \sum_{b,p} X_{ij}(I-X)_{pb}\frac{\partial F}{\partial X_{pb}} \\ + c_3\sum_{a,b} X_{aj}(I-X)_{ib}\frac{\partial F}{\partial X_{ab}} + \sum_{a,b,q} X_{aj}X_{iq}\frac{\partial F}{\partial X_{aq}} - nX_{ij}c_3 F.\end{aligned}$$

Then using (14), (21) and

$$\begin{aligned}\left\langle c_3\left((I-XY)^{-1}X\right)_{ij}\right\rangle &= -\sum_p X_{pj}\left\langle -c_3\left((I-XY)^{-1}\right)_{ip}\right\rangle \\ &= -\sum_{p,q} X_{pj}X_{iq}\frac{\partial F}{\partial X_{pq}} + X_{ij}c_3 F,\end{aligned}$$

we obtain the differential equations (23) from (22).

Theorem 3.1 for the Gauss HGF of matrix integral type is the following.

Proposition 4.7. *As a function of* eigenvalues $x_1,\dots,x_n$ *of* X, $I_G(a,b,c;X)$ *satisfies the system*

$$\begin{aligned}x_l(1-x_l)\frac{\partial^2 F}{\partial x_l^2} + \sum_{\alpha\neq l}\frac{x_l(1-x_l)\frac{\partial F}{\partial x_l} - x_\alpha(1-x_\alpha)\frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} \\ + \{(c-(n-1)) - (a+b+1-(n-1))x_l\}\frac{\partial F}{\partial x_l} - abF = 0, \quad 1\le l\le n. \quad (24)\end{aligned}$$

We give the proof of this proposition. Take any $1 \le l \le n$ and fix it. Multiply the both sides of (23) by $U^\dagger_{jl}U_{li}$ and take a sum for $i,j = 1,\dots,n$. We compute the term which comes from the first term of the left hand side of (23):

$$I := \sum_{i,j} U^\dagger_{jl}\cdot\left(\sum_{a,b,p,q} X_{aj}(I-X)_{pb}X_{iq}\frac{\partial^2 F}{\partial X_{ab}\partial X_{pq}}\right)\cdot U_{li}.$$

Noting that

$$\begin{aligned}\frac{\partial F}{\partial X_{pq}} &= \sum_\alpha \frac{\partial x_\alpha}{\partial X_{pq}}\frac{\partial F}{\partial x_\alpha}, \\ \frac{\partial^2 F}{\partial X_{ab}\partial X_{pq}} &= \sum_\alpha \frac{\partial^2 x_\alpha}{\partial X_{ab}\partial X_{pq}}\frac{\partial F}{\partial x_\alpha} + \sum_{\alpha,\beta}\frac{\partial x_\alpha}{\partial X_{pq}}\frac{\partial x_\beta}{\partial X_{ab}}\frac{\partial^2 F}{\partial x_\alpha\partial x_\beta},\end{aligned}$$

we write I as $I = I_1 + I_2$ with

$$\begin{aligned}I_1 &= \sum_{\alpha,\beta}\frac{\partial^2 F}{\partial x_\alpha\partial x_\beta}\sum_{i,j,a,b,p,q} U^\dagger_{jl}X_{aj}(I-X)_{pb}X_{iq}\frac{\partial x_\alpha}{\partial X_{pq}}\frac{\partial x_\beta}{\partial X_{ab}}U_{li}, \\ I_2 &= \sum_\alpha \frac{\partial F}{\partial x_\alpha}\sum_{a,b,p,q} U^\dagger_{jl}X_{aj}(I-X)_{pb}X_{iq}\frac{\partial^2 x_\alpha}{\partial X_{ab}\partial X_{pq}}U_{li}.\end{aligned}$$

For I_1, using the equality (3) of Lemma 4.1 and $x = UXU^\dagger$, we have

$$\begin{aligned}
I_1 &= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \sum_{i,j,a,b,p,q} U^\dagger_{jl} X_{aj} (I-X)_{pb} X_{iq} U_{\alpha p} U^\dagger_{q\alpha} U_{\beta a} U^\dagger_{b\beta} U_{li} \\
&= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \left(\sum_{j,a} U_{\beta a} X_{aj} U^\dagger_{jl} \right) \left(\sum_{p,b} U_{\alpha p} (I-X)_{pb} U^\dagger_{b\beta} \right) \left(\sum_{i,q} U_{li} X_{iq} U^\dagger_{q\alpha} \right) \\
&= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} (\delta_{\beta l} x_l)(\delta_{\alpha\beta}(1-x_\alpha))(\delta_{l\alpha} x_l) \\
&= x_l^2 (1-x_l) \frac{\partial^2 F}{\partial x_l^2}.
\end{aligned}$$

Next we compute I_2. Note that, from $X = U^\dagger x U$, we have $X_{aj} = \sum_r U^\dagger_{ar} x_r U_{rj}$, etc. By virtue of (9) of Lemma 4.2 we have

$$\begin{aligned}
I_2 &= \sum_\alpha \frac{\partial F}{\partial x_\alpha} \sum_{i,j,a,b,p,q} U^\dagger_{jl} X_{aj} (I-X)_{pb} X_{iq} \frac{\partial^2 x_\alpha}{\partial X_{ab} \partial X_{pq}} U_{li} \\
&= \sum_\alpha \frac{\partial F}{\partial x_\alpha} \sum_{i,j,a,b,p,q} \sum_{r,s,u} U^\dagger_{jl} U^\dagger_{ar} x_r U_{rj} U^\dagger_{ps} (1-x_s) U_{sb} U^\dagger_{iu} x_u U_{uq} \frac{\partial^2 x_\alpha}{\partial X_{ab} \partial X_{pq}} U_{li} \\
&= x_l^2 \sum_{\alpha,s} \frac{\partial F}{\partial x_\alpha} (1-x_s) \sum_{a,b,p,q} U^\dagger_{al} U^\dagger_{ps} U_{sb} U_{lq} \frac{\partial^2 x_\alpha}{\partial X_{ab} \partial X_{pq}} \\
&= x_l^2 \left\{ \sum_{\alpha \neq l} \frac{\partial F}{\partial x_s} (1-x_\alpha) \frac{-1}{x_l - x_\alpha} + \frac{\partial F}{\partial x_l} \sum_{s \neq l} (1-x_s) \frac{1}{x_l - x_s} \right\} \\
&= x_l^2 \sum_{\alpha \neq l} \frac{(1-x_\alpha)\frac{\partial F}{\partial x_l} - (1-x_\alpha)\frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha}. \\
&= x_l^2 \sum_{\alpha \neq l} \frac{(1-x_l)\frac{\partial F}{\partial x_l} - (1-x_\alpha)\frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} + (n-1) x_l^2 \frac{\partial F}{\partial x_l}
\end{aligned}$$

Thus we have

$$I = x_l \left\{ x_l (1-x_l) \frac{\partial^2 F}{\partial x_l^2} + x_l \sum_{\alpha \neq l} \frac{(1-x_l)\frac{\partial F}{\partial x_l} - (1-x_\alpha)\frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} + (n-1) x_l \frac{\partial F}{\partial x_l} \right\}. \tag{25}$$

To compute the contribution, which comes from the other terms of the left hand side of (23), we need the following lemma, which can be shown in a similar way as above using Lemma 4.2.

Lemma 4.8. *We have*

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{b,p} X_{ij} (I-X)_{pb} \frac{\partial F}{\partial X_{pb}} \right) \cdot U_{li} = x_l \cdot \sum_\alpha (I - x_\alpha) \frac{\partial F}{\partial x_\alpha}, \tag{26}$$

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{a,b} X_{aj} (I-X)_{ib} \frac{\partial F}{\partial X_{ab}} \right) \cdot U_{li} = x_l (1-x_l) \frac{\partial F}{\partial x_l}. \tag{27}$$

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{a,b} X_{aj} X_{ib} \frac{\partial F}{\partial X_{ab}} \right) \cdot U_{li} = x_l^2 \frac{\partial F}{\partial x_l} \tag{28}$$

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{a} X_{ia} \frac{\partial F}{\partial X_{ja}} \right) \cdot U_{li} = x_l \frac{\partial F}{\partial x_l}, \tag{29}$$

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{a} X_{aj} \frac{\partial F}{\partial X_{ai}} \right) \cdot U_{li} = x_l \frac{\partial F}{\partial x_l}, \tag{30}$$

By the help of (25) and Lemma 4.8, we can derive from (23) the equation

$$x_l(1-x_l)\frac{\partial^2 F}{\partial x_l^2} + x_l \sum_{\alpha \neq l} \frac{(1-x_l)\frac{\partial F}{\partial x_l} - (1-x_\alpha)\frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} + (n-1)x_l \frac{\partial F}{\partial x_l}$$
$$+ \sum_\alpha (1-x_\alpha)\frac{\partial F}{\partial x_\alpha} - c_3(1-x_l)\frac{\partial F}{\partial x_l} - (c_1+n)x_l\frac{\partial F}{\partial x_l}$$
$$+ (c_1+c_2+c_3+n)\frac{\partial F}{\partial x_l} + (c_1+n)c_3 F = 0.$$

Using

$$\begin{aligned}\sum_\alpha (1-x_\alpha)\frac{\partial F}{\partial x_\alpha} &= \sum_{\alpha\neq l} \frac{x_l - x_\alpha}{x_l - x_\alpha}(1-x_\alpha)\frac{\partial F}{\partial x_\alpha} + (1-x_l)\frac{\partial F}{\partial x_l}, \\ &= \sum_{\alpha\neq l} \frac{x_l(1-x_\alpha)}{x_l - x_\alpha}\frac{\partial F}{\partial x_\alpha} - \sum_{\alpha\neq l} \frac{x_\alpha(1-x_\alpha)}{x_l - x_\alpha}\frac{\partial F}{\partial x_\alpha} + (1-x_l)\frac{\partial F}{\partial x_l},\end{aligned}$$

we obtain the differential equation

$$x_l(1-x_l)\frac{\partial^2 F}{\partial x_l^2} + \sum_{\alpha\neq l} \frac{x_l(1-x_l)\frac{\partial F}{\partial x_l} - x_\alpha(1-x_\alpha)\frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha}$$
$$+ \{(c_1+c_2+n-1) - (c_1-c_3-2)x_l\}\frac{\partial F}{\partial x_l} + (c_1+n)c_3 F = 0.$$

Recovering the original parameters $c_1 = a-n, c_2 = c-a-n, c_3 = -b$, we obtain the desired differential equations (24) and finish the proof of Proposition 4.7.

4.3. Kummer case

We prove Theorem 3.1 for Kummer's HGF of matrix integral type following the same line of thought as in the Gauss case. Put

$$F(X) = \int_C \mathrm{etr}\,(XY)\,|Y|^{c_1}|I-Y|^{c_2}dY = \int_C \exp f(Y)dY, \quad X, Y \in \mathscr{H}(n), \tag{31}$$

where $c_1 = a-n, c_2 = c-a-n$, C is a domain of integration which allows us to apply the Stokes theorem, and

$$f(Y) = \mathrm{tr}\,(XY) + c_1 \log|Y| + c_2 \log|I-Y|.$$

The usage of the symbol $\langle g\rangle$ for a function $g(Y)$ is the same as in the Gauss case. A simple computation shows the following.

Lemma 4.9. *For any $1 \le i, j \le n$, we have*

$$\frac{\partial f}{\partial X_{ij}} = Y_{ji}, \quad \frac{\partial f}{\partial Y_{ij}} = X_{ji} + c_1(Y^{-1})_{ji} + c_2((I-Y)^{-1})_{ji}.$$

Lemma 4.10. *The function F, defined by (31), satisfies the differential equations*

$$\sum_{k,m} X_{kj}\frac{\partial^2 F}{\partial X_{km}\partial X_{mi}} - \sum_k X_{kj}\frac{\partial F}{\partial X_{ki}} + (c_1+c_2+n)\frac{\partial F}{\partial X_{ji}}$$
$$+ \delta_{ij}\left\{\sum_k \frac{\partial F}{\partial X_{kk}} - (c_1+n)F\right\} = 0. \quad 1 \le i, j \le n. \tag{32}$$

Proof. Let ω_{jk}, ω be those by (15) with $f(Y)$ in (31). Consider (n^2-1)-form

$$\eta_{ij} = \sum_{k=1}^n (Y(I-Y))_{ik} \exp f(Y)\,\omega_{jk}, \quad 1 \le i, j \le n$$

and compute $d\eta_{ij}$. Using Lemma 4.9, we have

$$\begin{aligned}
d\eta_{ij} &= \sum_{k=1}^{n} \frac{\partial}{\partial Y_{jk}} (Y(I-Y))_{ik}\,\omega + \sum_{k=1}^{n} (Y(I-Y))_{ik} \frac{\partial f}{\partial Y_{jk}}\omega \\
&= \sum_{k=1}^{n} (\delta_{ij} - \delta_{ij} Y_{kk} - Y_{ij}) \cdot \omega \\
&\quad + \sum_{k=1}^{n} (Y(I-Y))_{ik} \left(c_1 (Y^{-1})_{kj} - c_2((I-Y)^{-1})_{kj} + X_{kj} \right) \omega \\
&= \left\{ n\delta_{ij} - \delta_{ij} \operatorname{tr} Y - nY_{ij} + c_1 (I-Y)_{ij} - c_2 Y_{ij} + (Y(I-Y)X)_{ij} \right\} \omega \\
&= \left\{ (c_1+n)\delta_{ij} - \delta_{ij} \operatorname{tr} Y - (c_1+c_2+n) Y_{ij} + \sum_k Y_{ik} X_{kj} - \sum_{k,m} Y_{im} Y_{mk} X_{kj} \right\} \omega.
\end{aligned}$$

Then the Stokes theorem implies

$$(c_1+n)\delta_{ij}\langle 1 \rangle - \delta_{ij} \sum_k \langle Y_{kk} \rangle - (c_1+c_2+n)\left\langle Y_{ij} \right\rangle + \sum_k X_{kj} \langle Y_{ik} \rangle - \sum_{k,m} X_{kj} \langle Y_{im} Y_{mk} \rangle = 0. \tag{33}$$

Since $\langle 1 \rangle = F$ by definition and $\langle Y_{ab} \rangle = \partial F / \partial X_{ba}$ by virtue of Lemma 4.9, the equality (33) implies the differential equation (32).

Theorem 3.1 for the Kummer's case is the following.

Proposition 4.11. *As a function of* eigenvalues $x_1, \ldots, x_n$ *of* X, $I_K(a, c; X)$ *satisfies the differential equations*

$$x_l \frac{\partial^2 F}{\partial x_l^2} + (c - n + 1 - x_l) \frac{\partial F}{\partial x_l} + \sum_{\alpha \neq l} \frac{x_l \frac{\partial F}{\partial x_l} - x_\alpha \frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} - aF = 0, \quad 1 \leq l \leq n. \tag{34}$$

Proof. For a fixed $1 \leq l \leq n$, multiply the both sides of (32) by $U_{jl}^\dagger U_{li}$ and take a sum over $i, j = 1, \ldots, n$. The terms containing the second order derivatives are computed as follows. Since

$$\frac{\partial^2 F}{\partial X_{km} \partial X_{mi}} = \sum_\alpha \frac{\partial^2 x_\alpha}{\partial X_{km} \partial X_{mi}} \frac{\partial F}{\partial x_\alpha} + \sum_{\alpha,\beta} \frac{\partial x_\alpha}{\partial X_{mi}} \frac{\partial x_\beta}{\partial X_{km}} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta},$$

we have $\sum_{i,j} U_{jl}^\dagger \left(\sum_{k,m} X_{kj} \frac{\partial^2 F}{\partial X_{km} \partial X_{mi}} \right) U_{li} = I_1 + I_2$ with

$$\begin{aligned}
I_1 &= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \sum_{i,j} U_{jl}^\dagger \left(\sum_{k,m} X_{kj} \frac{\partial x_\alpha}{\partial X_{mi}} \frac{\partial x_\beta}{\partial X_{km}} \right) U_{li}, \\
I_2 &= \sum_\alpha \frac{\partial F}{\partial x_\alpha} \sum_{i,j} U_{jl}^\dagger \left(\sum_{k,m} X_{kj} \frac{\partial^2 x_\alpha}{\partial X_{km} \partial X_{mi}} \right) U_{li}.
\end{aligned}$$

Using Lemma 4.1 and $x = UXU^\dagger$, I_1 is computed as

$$\begin{aligned}
I_1 &= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \sum_{i,j,k,m} X_{kj} U_{jl}^\dagger U_{\alpha m} U_{i\alpha}^\dagger U_{\beta k} U_{m\beta}^\dagger U_{li} \\
&= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \delta_{l\alpha} \delta_{\beta\alpha} \sum_{j,k} U_{\beta k} X_{kj} U_{jl}^\dagger = x_l \frac{\partial^2 F}{\partial x_l^2}.
\end{aligned}$$

Noting that $X_{kj} = \sum_p U_{kp}^\dagger x_p U_{pj}$ and using (8) of Lemma 4.2, I_2 is computed as

$$\begin{aligned}
I_2 &= \sum_\alpha \frac{\partial F}{\partial x_\alpha} \sum_{i,j,k,m} \sum_p U_{jl}^\dagger U_{kp}^\dagger x_p U_{pj} \frac{\partial^2 x_\alpha}{\partial X_{km} \partial X_{mi}} U_{li} \\
&= x_l \sum_\alpha \frac{\partial F}{\partial x_\alpha} \sum_{i,k,m} U_{kl}^\dagger \frac{\partial^2 x_\alpha}{\partial X_{km} \partial X_{mi}} U_{li} = x_l \sum_{\alpha \neq l} \frac{1}{x_\alpha - x_l} \left(\frac{\partial F}{\partial x_\alpha} - \frac{\partial F}{\partial x_l} \right).
\end{aligned}$$

Thus we have

$$\sum_{i,j} U^{\dagger}_{jl} \cdot \left(\sum_{k,m} X_{kj} \frac{\partial^2 F}{\partial X_{km} \partial X_{mi}} \right) \cdot U_{li} = x_l \frac{\partial^2 F}{\partial x_l^2} + x_l \sum_{\alpha \neq l} \frac{1}{x_\alpha - x_l} \left(\frac{\partial F}{\partial x_\alpha} - \frac{\partial F}{\partial x_l} \right). \tag{35}$$

For the other terms of the first derivatives, contribution from the second term in (32) is already computed in (30), and that from the rest is computed by using as

$$\sum_{i,j} U^{\dagger}_{jl} \cdot \left(\frac{\partial F}{\partial X_{ji}} \right) \cdot U_{li} = \frac{\partial F}{\partial x_l}, \tag{36}$$

$$\sum_{i,j} \delta_{ij} U^{\dagger}_{jl} \cdot \left(\sum_{k} \frac{\partial F}{\partial X_{kk}} \right) \cdot U_{li} = \sum_{\alpha} \frac{\partial F}{\partial x_\alpha}. \tag{37}$$

Noting that $\sum_{i,j} \delta_{ij} U^{\dagger}_{jl} U_{li} = 1$, from the differential equation (32), we have

$$x_l \frac{\partial^2 F}{\partial x_l^2} + x_l \sum_{\alpha \neq l} \frac{1}{x_l - x_\alpha} \left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha} \right) + \sum_{\alpha} \frac{\partial F}{\partial x_\alpha} + (c_1 + c_2 + n - x_l) \frac{\partial F}{\partial x_l} - (c_1 + n) F = 0.$$

Using $c_1 = a - n, c_2 = c - a - n$ and

$$\sum_{\alpha} \frac{\partial F}{\partial x_\alpha} = \sum_{\alpha \neq l} \frac{x_l - x_\alpha}{x_l - x_\alpha} \frac{\partial F}{\partial x_\alpha} + \frac{\partial F}{\partial x_l},$$

we have the differential equation (34).

4.4. Bessel case

We prove Theorem 3.1 for the Bessel integral $I_B(c; X)$. Put

$$F(X) = \int \operatorname{etr}\left(XY - Y^{-1} \right) |Y|^{c-n} dY = \int \exp f(Y) dY, \quad X, Y \in \mathscr{H}(n), \tag{38}$$

where

$$f(Y) = \operatorname{tr}(XY - Y^{-1}) + (c - n) \log |Y|.$$

The usage of the symbol $\langle g \rangle$ is same as above. The following lemma is now easy to show.

Lemma 4.12. *For any $1 \le i, j \le n$, we have*

$$\frac{\partial f}{\partial X_{ij}} = \langle Y_{ji} \rangle, \qquad \frac{\partial f}{\partial Y_{ij}} = X_{ji} + (Y^{-2})_{ji} + (c - n)(Y^{-1})_{ji}.$$

Lemma 4.13. *The function F, defined by (38), satisfies the differential equations*

$$\sum_{k,m} X_{kj} \frac{\partial^2 F}{\partial X_{km} \partial X_{mi}} + c \frac{\partial F}{\partial X_{ji}} + \delta_{ij} \left\{ \sum_{k} \frac{\partial F}{\partial X_{kk}} + F \right\} = 0, \quad 1 \le i, j \le n. \tag{39}$$

Proof. For $1 \le i, j \le n$, consider the $(n^2 - 1)$-form

$$\eta_{ij} = \sum_{k=1}^{n} \left(Y^2 \right)_{ik} \exp f(Y)\, \omega_{jk},$$

and compute $d\eta_{ij}$:

$$\begin{aligned} d\eta_{ij} &= \left\{ \sum_{k=1}^{n} \frac{\partial}{\partial Y_{jk}} \left(Y^2 \right)_{ik} + \sum_{k=1}^{n} \left(Y^2 \right)_{ik} \frac{\partial f}{\partial Y_{jk}} \right\} \omega \\ &= \left\{ \sum_{k=1}^{n} \frac{\partial}{\partial Y_{jk}} \left(\sum_{m=1}^{n} Y_{im} Y_{mk} \right) + \sum_{k=1}^{n} \left(Y^2 \right)_{ik} \left(X_{kj} + (Y^{-2})_{kj} + (c - n)(Y^{-1})_{kj} \right) \right\} \omega \\ &= \left\{ \delta_{ij} \left(1 + \sum_{k} Y_{kk} \right) + c Y_{ij} + \left(Y^2 X \right)_{ij} \right\} \omega, \end{aligned}$$

where the usage of ω and ω_{jk} are similar as in the Kummer's case. Then the Stokes theorem implies that

$$\sum_{k,m} X_{kj}\langle Y_{im}Y_{mk}\rangle + c\left\langle Y_{ij}\right\rangle + \delta_{ij}\sum_{k}\langle Y_{kk}\rangle + \delta_{ij}\langle 1\rangle = 0. \tag{40}$$

Since $\langle Y_{ab}\rangle = \partial F/\partial X_{ba}$ by virtue of Lemma 4.12, the equality (40) implies the differential equation (39).

Theorem 3.1 for the Bessel function of matrix integral type is the following.

Proposition 4.14. *As a function of* eigenvalues $x_1,\dots,x_n$ *of* X*, the Bessel integral* $I_B(X)$ *satisfies the differential equations*

$$x_l\frac{\partial^2 F}{\partial x_l^2} + (c+1)\frac{\partial F}{\partial x_l} + \sum_{\alpha\neq l}\frac{x_l\frac{\partial F}{\partial x_l} - x_\alpha\frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} + F = 0, \quad 1\le l\le n. \tag{41}$$

Proof. For a fixed $1\le l\le n$, multiply the both sides of (39) by $U^\dagger_{jl}U_{li}$ and take a sum over $i,j=1,\dots,n$. Using the identities (35), (36) and (37), we have from (39) the equation

$$x_l\frac{\partial^2 F}{\partial x_l^2} + c\frac{\partial F}{\partial x_l} + x_l\sum_{\alpha\neq l}\frac{1}{x_l - x_\alpha}\left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha}\right) + \sum_{\alpha}\frac{\partial F}{\partial x_\alpha} + F = 0.$$

Rewriting it using

$$\sum_{\alpha}\frac{\partial F}{\partial x_\alpha} = \sum_{\alpha\neq l}\frac{x_l - x_\alpha}{x_l - x_\alpha}\frac{\partial F}{\partial x_\alpha} + \frac{\partial F}{\partial x_l},$$

we obtain the differential equation (41).

4.5. Hermite-Weber case

We prove Theorem 3.1 for the Hermite-Weber matrix integral $I_{HW}(c;X)$. Put

$$F(X) = \int_C |Y|^{-c-n}\mathrm{etr}\left(XY - \frac{1}{2}Y^2\right)dY = \int_C \exp f(Y)dY, \quad X\in\mathscr{H}(n), \tag{42}$$

where C is a domain of integration as in the previous cases and

$$f(Y) = (-c-n)\log|Y| + \mathrm{tr}\,(XY - \frac{1}{2}Y^2).$$

The usage of the symbol $\langle g\rangle$ is the same as in the previous cases. The following lemma is shown easily.

Lemma 4.15. *For any* $1\le i,j\le n$*, we have*

$$\frac{\partial f}{\partial X_{ij}} = Y_{ji}, \quad \frac{\partial f}{\partial Y_{ij}} = (-c-n)(Y^{-1})_{ji} + X_{ji} - Y_{ji}. \tag{43}$$

Lemma 4.16. *The function* F*, defined by the integral (42), satisfies the differential equations*

$$\sum_k \frac{\partial^2 F}{\partial X_{jk}\partial X_{ki}} - \sum_k X_{kj}\frac{\partial}{\partial X_{ki}}F + c\delta_{ij}F = 0, \quad 1\le i,j\le n. \tag{44}$$

Proof. For any pair (i,j), define $\eta_{ij} = \sum_{k=1}^n Y_{ik}\exp f(Y)\,\omega_{jk}$ as in the previous cases. Then using Lemma 4.15, we have

$$\begin{aligned} d\eta_{ij} &= \left\{\sum_{k=1}^n\frac{\partial Y_{ik}}{\partial Y_{jk}} + \sum_{k=1}^n Y_{ik}\frac{\partial f}{\partial Y_{jk}}\right\}\omega \\ &= \left\{n\delta_{ij} + \sum_{k=1}^n Y_{ik}\left((-c-n)(Y^{-1})_{kj} + X_{kj} - Y_{kj}\right)\right\}\omega \\ &= \left\{-c\delta_{ij} + \sum_{k=1}^n Y_{ik}X_{kj} - \sum_{k=1}^n Y_{ik}Y_{kj}\right\}\omega. \end{aligned}$$

Since $\int d\eta_{ij} = 0$ by the Stokes theorem, we have

$$\sum_k\left\langle Y_{ik}Y_{kj}\right\rangle - \sum_{k=1}^n X_{kj}\langle Y_{ik}\rangle + c\delta_{ij}\langle 1\rangle = 0. \tag{45}$$

Then we see that the identities (45) lead to the differential equations (44) since Lemma 4.15 implies $\partial F/\partial X_{ab} = \langle Y_{ba}\cdot 1\rangle$.

Theorem 3.1 for the Hermite-Weber case is the following.

Proposition 4.17. *As a function of* eigenvalues $x_1, \dots, x_n$, *the Hermite-Weber integral* $I_{HW}(X)$ *satisfies*

$$\frac{\partial^2 F}{\partial x_l^2} - x_l \frac{\partial F}{\partial x_l} + \sum_{\alpha \neq l} \frac{1}{x_l - x_\alpha} \left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha} \right) + cF = 0, \quad 1 \le i, j \le n. \tag{46}$$

Proof. We proceed as in the previous case. For a fixed $1 \le l \le n$, multiply the both sides of (44) by $U_{jl}^\dagger U_{li}$ and take a sum over $i, j = 1, \dots, n$. Then we easily see that

$$\sum_{i,j,k} U_{jl}^\dagger \cdot \frac{\partial^2 F}{\partial X_{jk} \partial X_{ki}} \cdot U_{li} = \frac{\partial^2 F}{\partial x_l^2} + \sum_{\alpha \neq l} \frac{1}{x_l - x_\alpha} \left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha} \right).$$

For the second term in (44), we use (30). Then we obtain the differential equation (46) from (44).

4.6. Airy case

We prove Theorem 3.1 for the Airy integral $I_A(X)$. Put

$$F(X) = \int_C \operatorname{etr} \left(XY - \frac{1}{3} Y^3 \right) dY = \int_C \exp f(Y) dY, \quad X \in \mathscr{H}(n), \tag{47}$$

where C is a domain of integration explained in the last paragraph of Section 2.2 and

$$f(Y) = \operatorname{tr} \left(XY - \frac{1}{3} Y^3 \right).$$

By virtue of this choice of C, we can interchange the operations of differentiation with respect to X_{ij} integration with respect to Y. See also Remark 2.7. The usage of the symbol $\langle g \rangle$ is the same as in the previous cases. The following lemma is easy.

Lemma 4.18. *For any* $1 \le i, j \le n$, *we have*

$$\frac{\partial f}{\partial X_{ij}} = Y_{ji}, \quad \langle (Y^2)_{ji} \rangle - \langle X_{ji} \rangle = 0. \tag{48}$$

Lemma 4.19. *The function* F *satisfies the differential equations*

$$\sum_k \frac{\partial^2 F}{\partial X_{ik} \partial X_{kj}} - X_{ji} F = 0, \quad 1 \le i, j \le n. \tag{49}$$

Proof. The equation (49) follows from Lemma 4.18 and $\left(Y^2\right)_{ji} = \sum_k Y_{jk} Y_{ki}$.

Theorem 3.1 for the Airy integral is the following, whose proof is similar to that for Proposition 4.17 and is omitted.

Proposition 4.20. *As a function of* eigenvalues $x_1, \dots, x_n$ *of* X, *the Airy integral* $I_A(X)$ *satisfies the differential equation*

$$\frac{\partial^2 F}{\partial x_l^2} + \sum_{\alpha \neq l} \frac{1}{x_l - x_\alpha} \left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha} \right) - x_l F = 0, \quad 1 \le l \le n.$$

5. HOLONOMICITY OF THE SYSTEM FOR HGF

Theorem 5.1. *The system* $\mathcal{S}_*$ $(* = G, K, B, HW, A)$ *is holonomic in* $\Omega_* \subset \mathbb{C}^n$ *and is equivalent to the completely integrable Pfaffian system of rank* 2^n, *where*

$$\Omega_G = \{x \in \mathbb{C}^n \mid \prod_{i=1}^n x_i(x_i - 1) \cdot \Delta(x) \neq 0\},$$

$$\Omega_K = \Omega_B = \{x \in \Omega \mid \prod_{i=1}^n x_i \cdot \Delta(x) \neq 0\},$$

$$\Omega_{HW} = \Omega_A = \{x \in \Omega \mid \Delta(x) \neq 0\}.$$

We prove the theorem in detail for the systems $\mathcal{S}_{HW}, \mathcal{S}_A$ by using the theory of Gröbner basis.

Let $\mathbb{C}[x]$ be the ring of polynomials in $x = (x_1, \ldots, x_n)$ and let R be the localization of $\mathbb{C}[x]$ by $\Delta = \prod_{i<j}(x_i - x_j)$, namely $R = \{f/\Delta^m \mid f \in \mathbb{C}[x], m \in \mathbb{Z}_{\geq 0}\}$ which is also denoted as $\mathbb{C}[x]_\Delta$. We denote by D the ring of differential operators in x with coefficients in R. Any $P \in D$ can be expressed uniquely in the so-called normal form

$$P = \sum_\alpha a_\alpha(x)\partial^\alpha = \sum_\alpha a_\alpha(x)\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}, \quad a_\alpha(x) \in R,$$

where $\sum_\alpha$ is a finite sum with respect to multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. To this $P \in D$ we associate its symbol:

$$\sigma(P) = \sum_\alpha a_\alpha(x)\xi^\alpha = \sum_\alpha a_\alpha(x)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} \in R[\xi].$$

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, let $|\alpha|$ denote the sum $\alpha_1 + \cdots + \alpha_n$.

Let us fix an order in the set of monomials $\{a_\alpha(x)\partial^\alpha\}$ in D as follows. Firstly, we use the lexicographic order as a monomial order $\prec_{lex}$ in $\mathbb{C}[\xi]$, namely, $\xi^\alpha \prec_{lex} \xi^\beta$ means that either $|\alpha| < |\beta|$ holds or $|\alpha| = |\beta|$ and the most left nonzero member of $(\beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n)$ is > 0. Using $\prec_{lex}$, define the order in D as

$$a_\alpha(x)\partial^\alpha \prec b_\beta(x)\partial^\beta \Leftrightarrow \quad \xi^\alpha \prec_{lex} \xi^\beta.$$

For $P \in D$, the initial term $\mathrm{in}_\prec(P)$ is the symbol of the greatest monomial in P with respect to the order $\prec$. For $P, Q \in R$ with $\mathrm{in}_\prec(P) = a(x)\xi^\alpha, \mathrm{in}_\prec(Q) = b(x)\xi^\beta$, let $\gamma = (\max(\alpha_1, \beta_1), \ldots, \max(\alpha_n, \beta_n)) \in \mathbb{Z}^n_{\geq 0}$. Then S-pair $\mathrm{sp}(P, Q)$ for P, Q is defined by

$$\mathrm{sp}(P, Q) = b(x)\partial^{\gamma-\alpha}P - a(x)\partial^{\gamma-\beta}Q.$$

Let $\mathcal{J}$ be a left ideal of D. By $\mathrm{in}_\prec(\mathcal{J})$ we denote the ideal of $R[\xi]$ generated by $\{\mathrm{in}_\prec(P) \mid P \in \mathcal{J}\}$. Let $\{f_1, \ldots, f_d\}$ be a generator of the ideal $\mathcal{J}$. It should be noted that $\{\mathrm{in}_\prec(f_1), \ldots, \mathrm{in}_\prec(f_d)\}$ does not necessarily generate $\mathrm{in}_\prec(\mathcal{J})$, in general. A generator $G = \{g_1, \ldots, g_m\}$ of $\mathcal{J}$ is said to be a Gröbner basis for $\mathcal{J}$ if $(\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_m))$ generates $\mathrm{in}_\prec(\mathcal{J})$, namely $\mathrm{in}_\prec(\mathcal{J}) = \langle \mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_m)\rangle$. We can apply the Buchberger's algorithm to find a Gröbner basis for a given left ideal $\mathcal{J}$ of D.

5.1. Hermite-Weber

Consider the system of differential equations $\mathcal{S}_{HW}$ for the Hermite-Weber function $I_{HW}(c, X)$ and put

$$L_i = \partial_i^2 - x_i\partial_i + \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) + c, \quad 1 \leq i \leq n.$$

Let $\mathcal{J}_{HW} \subset D$ be the left ideal with the generator $G_{HW} = \{L_1, \ldots, L_n\}$.

Proposition 5.2. *G_{WH} is a Gröbner basis of the left ideal $\mathcal{J}_{HW}$.*

Proof. It is enough to show that, for any pair L_i, L_j $(i \neq j)$, the S-pair $\mathrm{sp}(L_i, L_j) \equiv 0$ after applying the division algorithm of Buchberger using G_{WH}. Since the largest term of L_i is ∂_i^2, we have

$$\begin{aligned}
\mathrm{sp}(L_i, L_j) &= \partial_j^2 L_i - \partial_i^2 L_j \\
&= \partial_j^2\left(\partial_i^2 - x_i\partial_i + \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) + c\right) \\
&\quad - \partial_i^2\left(\partial_j^2 - x_j\partial_j + \sum_{k(\neq j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) + c\right) \\
&= A + B + C + D,
\end{aligned}$$

where

$$\begin{aligned}
A &= -x_i\partial_i\partial_j^2 + x_j\partial_j\partial_i^2, \\
B &= \partial_j^2 \cdot \frac{1}{x_i - x_j}(\partial_i - \partial_j) - \partial_i^2 \cdot \frac{1}{x_j - x_i}(\partial_j - \partial_i), \\
C &= \sum_{k(\neq i,j)} \left\{\frac{1}{x_i - x_k}(\partial_i - \partial_k)\partial_j^2 - \frac{1}{x_j - x_k}(\partial_j - \partial_k)\partial_i^2\right\}, \\
D &= c(\partial_j^2 - \partial_i^2).
\end{aligned}$$

We carry out a reduction of A, B, C, D by the division algorithm using the generator G_{WH}. Noting that

$$\partial_i^2 = L_i + x_i\partial_i - \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) - c, \tag{50}$$

we have

$$\begin{aligned}
A &\equiv -x_i\partial_i \left\{ x_j\partial_j - \sum_{k(\neq j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) - c \right\} + x_j\partial_j \left\{ x_i\partial_i - \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) - c \right\} \\
&= \sum_{k(\neq i,j)} \left\{ \frac{x_i}{x_j - x_k}\partial_i(\partial_j - \partial_k) - \frac{x_j}{x_i - x_k}\partial_j(\partial_i - \partial_k) \right\} \\
&+ x_i \left\{ \frac{1}{x_j - x_i}\partial_i(\partial_j - \partial_i) + \frac{1}{(x_j - x_i)^2}(\partial_j - \partial_i) \right\} - x_j \left\{ \frac{1}{x_i - x_j}\partial_j(\partial_i - \partial_j) + \frac{1}{(x_i - x_j)^2}(\partial_i - \partial_j) \right\}, \\
&+ c(x_i\partial_i - x_j\partial_j),
\end{aligned}$$

and

$$\begin{aligned}
B &\equiv \frac{1}{x_i - x_j}(\partial_i - \partial_j)\partial_j^2 + \frac{2}{(x_i - x_j)^2}(\partial_i - \partial_j)\partial_j + \frac{2}{(x_i - x_j)^3}(\partial_i - \partial_j) \\
&\quad - \frac{1}{x_j - x_i}(\partial_j - \partial_i)\partial_i^2 - \frac{2}{(x_j - x_i)^2}(\partial_j - \partial_i)\partial_i - \frac{2}{(x_j - x_i)^3}(\partial_j - \partial_i) \\
&= \frac{1}{x_i - x_j}(\partial_i - \partial_j)(\partial_j^2 - \partial_i^2) - \frac{2}{(x_i - x_j)^2}(\partial_j^2 - \partial_i^2) \\
&=: \frac{1}{x_i - x_j}B_1 - \frac{2}{(x_i - x_j)^2}(\partial_j^2 - \partial_i^2).
\end{aligned}$$

To compute B_1, we use

$$\partial_j^2 - \partial_i^2 = L_j - L_i + x_j\partial_j - x_i\partial_i + \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\}$$

and we have

$$\begin{aligned}
B_1 &\equiv (\partial_i - \partial_j) \left\{ x_j\partial_j - x_i\partial_i + \sum_{k(\neq i,j)} \left(\frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right) \right\} \\
&= x_j(\partial_i - \partial_j)\partial_j - x_i(\partial_j - \partial_i)\partial_i - (\partial_j + \partial_i) \\
&+ \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_j)(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)^2}(\partial_i - \partial_k) \right\} \\
&- \sum_{k(\neq i,j)} \left\{ \frac{1}{x_j - x_k}(\partial_i - \partial_j)(\partial_j - \partial_k) + \frac{1}{(x_j - x_k)^2}(\partial_j - \partial_k) \right\}.
\end{aligned}$$

Similarly, we compute C. Using (50) we have

$$\begin{aligned}
C &\equiv \sum_{k(\neq i,j)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) \left\{ x_j\partial_j - \sum_{\ell(\neq j)} \frac{1}{x_j - x_\ell}(\partial_j - \partial_\ell) - c \right\} \\
&- \sum_{k(\neq i,j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) \left\{ x_i\partial_i - \sum_{\ell(\neq i)} \frac{1}{x_i - x_\ell}(\partial_i - \partial_\ell) - c \right\} \\
&= \sum_{k(\neq i,j)} \left\{ \frac{x_j}{x_i - x_k}(\partial_i - \partial_k)\partial_j - \frac{x_i}{x_j - x_k}(\partial_j - \partial_k)\partial_i \right\} \\
&- c \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\} + C_1,
\end{aligned}$$

where

$$
\begin{aligned}
C_1 = &- \sum_{k(\neq i,j)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) \sum_{\ell(\neq j)} \frac{1}{x_j - x_\ell}(\partial_j - \partial_\ell) \\
&+ \sum_{k(\neq i,j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) \sum_{\ell(\neq i)} \frac{1}{x_i - x_\ell}(\partial_i - \partial_\ell).
\end{aligned}
$$

In C_1, we consider separately the cases $\ell = i, k$ in the first part and the cases $\ell = j, k$ in the second part. Thus

$$
\begin{aligned}
C_1 = &- \sum_{k(\neq i,j)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) \frac{1}{x_j - x_i}(\partial_j - \partial_i) \\
&- \sum_{k(\neq i,j)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) \frac{1}{x_j - x_k}(\partial_j - \partial_k) \\
&- \sum_{k(\neq i,j)} \sum_{\ell(\neq i,j,k)} \frac{1}{(x_i - x_k)(x_j - x_\ell)}(\partial_i - \partial_k)(\partial_j - \partial_\ell) \\
&+ \sum_{k(\neq i,j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) \frac{1}{x_i - x_j}(\partial_i - \partial_j) \\
&+ \sum_{k(\neq i,j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) \frac{1}{x_i - x_k}(\partial_i - \partial_k) \\
&+ \sum_{k(\neq i,j)} \sum_{\ell(\neq i,j,k)} \frac{1}{(x_j - x_k)(x_i - x_\ell)}(\partial_j - \partial_k)(\partial_i - \partial_\ell).
\end{aligned}
$$

Reducing C_1 to the normal form we have

$$
\begin{aligned}
C_1 = &- \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\} (\partial_i - \partial_j) \\
&+ \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right\} \frac{1}{(x_i - x_j)^2}(\partial_i - \partial_j) \\
&- \sum_{k(\neq i,j)} \left\{ \frac{1}{(x_i - x_k)^2 (x_j - x_k)}(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)(x_j - x_k)^2}(\partial_j - \partial_k) \right\},
\end{aligned} \tag{51}
$$

and we get the normal form of C. Collecting the terms in A, C, D containing c as a coefficient, we see that they are equal to

$c(L_j - L_i)$ and is 0 after applying the division algorithm. Also summing up all the other terms in A, B, C, we have

$$
\begin{aligned}
\mathrm{sp}(L_i, L_j) \equiv & \sum_{k(\neq i,j)} \left\{ \frac{x_i}{x_j - x_k} \partial_i (\partial_j - \partial_k) - \frac{x_j}{x_i - x_k} \partial_j (\partial_i - \partial_k) \right\} \\
& + x_i \left\{ \frac{1}{x_j - x_i} \partial_i (\partial_j - \partial_i) + \frac{1}{(x_j - x_i)^2} (\partial_j - \partial_i) \right\} \\
& - x_j \left\{ \frac{1}{x_i - x_j} \partial_j (\partial_i - \partial_j) + \frac{1}{(x_i - x_j)^2} (\partial_i - \partial_j) \right\} \\
& + \frac{1}{x_i - x_j} \left\{ x_j (\partial_i - \partial_j) \partial_j - x_i (\partial_j - \partial_i) \partial_i - (\partial_j + \partial_i) \right\} \\
& + \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k} (\partial_i - \partial_j)(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)^2} (\partial_i - \partial_k) \right\} \\
& - \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_j - x_k} (\partial_i - \partial_j)(\partial_j - \partial_k) + \frac{1}{(x_j - x_k)^2} (\partial_j - \partial_k) \right\} \\
& + \frac{2}{(x_i - x_j)^2} (\partial_i^2 - \partial_j^2) \\
& + \sum_{k(\neq i,j)} \left\{ \frac{x_j}{x_i - x_k} (\partial_i - \partial_k) \partial_j - \frac{x_i}{x_j - x_k} (\partial_j - \partial_k) \partial_i \right\} \\
& - \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k} (\partial_i - \partial_k) - \frac{1}{x_j - x_k} (\partial_j - \partial_k) \right\} (\partial_i - \partial_j) \\
& + \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right\} \frac{1}{(x_i - x_j)^2} (\partial_i - \partial_j) \\
& - \sum_{k(\neq i,j)} \left\{ \frac{1}{(x_i - x_k)^2 (x_j - x_k)} (\partial_i - \partial_k) - \frac{1}{(x_i - x_k)(x_j - x_k)^2} (\partial_j - \partial_k) \right\}.
\end{aligned}
$$

This reduces to

$$
\begin{aligned}
\mathrm{sp}(L_i, L_j) \equiv & \frac{1}{(x_i - x_j)^2} \left\{ 2(\partial_i^2 - \partial_j^2) - 2(x_i \partial_i - x_j \partial_j) + \sum_{k(\neq i,j)} \left(\frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right) (\partial_i - \partial_j) \right\} \\
& + \sum_{k(\neq i,j)} \frac{1}{(x_i - x_j)(x_j - x_k)(x_k - x_i)} \left\{ (\partial_i - \partial_k) + (\partial_j - \partial_k) \right\} \\
& = \frac{2}{(x_i - x_j)^2} (L_i - L_j) \\
& \equiv 0.
\end{aligned}
$$

by applying the division algorithm using G_{HW}. Thus we have shown that G_{HW} is a Gröbner basis for the ideal I_{HW}.

Since G_{HW} is the Gröbner basis of the ideal $\mathscr{I}_{HW}$ and $\mathrm{in}_<(L_i) = \xi_i^2$, we see that $\mathscr{I}_{HW}$ is a zero-dimensional ideal of D, $\mathrm{rank}_R(D/\mathscr{I}_{HW}) = \mathrm{rank}_R(R[\xi]/\langle \xi_1^2, \dots, \xi_n^2 \rangle) = 2^n$ and

$$\{\partial_1^{k_1} \partial_2^{k_2} \cdots \partial_n^{k_n} \mid k_1, \dots, k_n = 0, 1\}$$

gives a basis of R-free module $D/\mathscr{I}_{HW}$, where when $k_1 = \cdots = k_n = 0$, above element is understood to be 1. Thus we have shown the following proposition and completed the proof of Theorem 5.1 for $\mathcal{S}_{HW}$.

Proposition 5.3. *The system $\mathcal{S}_{HW}$ is holonomic on $\mathbb{C}^n \setminus S$, $S = \cup_{i<j} \{x_i - x_j = 0\}$ and the holonomic rank is 2^n.*

5.2. Airy

We show the similar result for the system $\mathcal{S}_A$ for the Airy integral $I_A(X)$ of matrix integral type. Put

$$L_i = \partial_i^2 + \sum_{k(\neq i)} \frac{1}{x_i - x_k} (\partial_i - \partial_k) - x_i, \quad 1 \leq i \leq n,$$

and let $\mathscr{I}_A \subset D$ be the left ideal of D with the generator $G_A = \{L_1, \dots, L_n\}$.

Proposition 5.4. *The generator G_A is a Gröbner basis of the left ideal $\mathcal{I}_A$.*

Proof. For $i \neq j$, let us compute the S-pair $\mathrm{sp}(L_i, L_j)$ and show that $\mathrm{sp}(L_i, L_j) \equiv 0$ after carrying out the division algorithm using G_A.

$$\begin{aligned}
\mathrm{sp}(L_i, L_j) &= \partial_j^2 L_i - \partial_i^2 L_j \\
&= \partial_j^2 \left(\partial_i^2 + \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) - x_i \right) \\
&\quad - \partial_i^2 \left(\partial_j^2 + \sum_{k(\neq j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) - x_j \right) \\
&= B + C + D,
\end{aligned}$$

where

$$\begin{aligned}
B &= \partial_j^2 \cdot \frac{1}{x_i - x_j}(\partial_i - \partial_j) - \partial_i^2 \cdot \frac{1}{x_j - x_i}(\partial_j - \partial_i), \\
C &= \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k)\partial_j^2 - \frac{1}{x_j - x_k}(\partial_j - \partial_k)\partial_i^2 \right\}, \\
D &= -x_i \partial_j^2 + x_j \partial_i^2.
\end{aligned}$$

Note that B, C has the same form as in the proof of Proposition 5.2. We make a reduction of B, C in a similar way. B can be written as

$$B = \frac{1}{x_i - x_j} B_1 + \frac{2}{(x_i - x_j)^2}(\partial_i^2 - \partial_j^2),$$

where $B_1 = (\partial_i - \partial_j)(\partial_j^2 - \partial_i^2)$. Note that

$$\partial_i^2 = L_i - \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) + x_i, \tag{52}$$

we have

$$\partial_j^2 - \partial_i^2 = L_j - L_i + \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\} - (x_i - x_j),$$

and see that

$$\begin{aligned}
B_1 &\equiv (\partial_i - \partial_j) \left\{ \sum_{k(\neq i,j)} \left(\frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right) - (x_i - x_j) \right\} \\
&= \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_j)(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)^2}(\partial_i - \partial_k) \right\} \\
&\quad - \sum_{k(\neq i,j)} \left\{ \frac{1}{x_j - x_k}(\partial_i - \partial_j)(\partial_j - \partial_k) + \frac{1}{(x_j - x_k)^2}(\partial_j - \partial_k) \right\} \\
&\quad - (x_i - x_j)(\partial_i - \partial_j) - 2.
\end{aligned}$$

Similarly we compute C using (52) and get

$$C \equiv C_1 + \sum_{k(\neq i,j)} \left(\frac{x_j}{x_i - x_k}(\partial_i - \partial_k) - \frac{x_i}{x_j - x_k}(\partial_j - \partial_k) \right),$$

where C_1 is the same as in (51). For D we have

$$D \equiv (\partial_i - \partial_j) - \sum_{k(\neq i,j)} \left(\frac{x_j}{x_i - x_k}(\partial_i - \partial_k) - \frac{x_i}{x_j - x_k}(\partial_j - \partial_k) \right).$$

Summing up B, C, D, we have

$$\begin{aligned}
\mathrm{sp}(L_i, L_j) \equiv\ & \frac{2}{(x_i - x_j)^2}(\partial_i^2 - \partial_j^2) - (\partial_i - \partial_j) - \frac{2}{x_i - x_j} \\
& + \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_j)(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)^2}(\partial_i - \partial_k) \right\} \\
& - \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_j - x_k}(\partial_i - \partial_j)(\partial_j - \partial_k) + \frac{1}{(x_j - x_k)^2}(\partial_j - \partial_k) \right\} \\
& - \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\} (\partial_i - \partial_j) \\
& + \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right\} \frac{1}{(x_i - x_j)^2}(\partial_i - \partial_j) \\
& - \sum_{k(\neq i,j)} \left\{ \frac{1}{(x_i - x_k)^2 (x_j - x_k)}(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)(x_j - x_k)^2}(\partial_j - \partial_k) \right\}, \\
& + (\partial_i - \partial_j) - \sum_{k(\neq i,j)} \left(\frac{x_j}{x_i - x_k}(\partial_i - \partial_k) - \frac{x_i}{x_j - x_k}(\partial_j - \partial_k) \right).
\end{aligned}$$

This reduces to

$$\begin{aligned}
\mathrm{sp}(L_i, L_j) \equiv\ & \frac{1}{(x_i - x_j)^2} \left\{ 2(\partial_i^2 - \partial_j^2) - 2(x_i - x_j) + \sum_{k(\neq i,j)} \left(\frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right) (\partial_i - \partial_j) \right\} \\
& + \sum_{k(\neq i,j)} \frac{1}{(x_i - x_j)(x_j - x_k)(x_k - x_i)} \{ (\partial_i - \partial_k) + (\partial_j - \partial_k) \} \\
=\ & \frac{2}{(x_i - x_j)^2}(L_i - L_j) \\
\equiv\ & 0
\end{aligned}$$

by the division algorithm using G_A. Thus we have shown that G_A is a Gröbner basis for the ideal $\mathcal{I}_A$.

Since G_A is the Gröbner basis of the ideal $\mathcal{I}_A$ and $\mathrm{in}_{\prec}(L_i) = \xi_i^2$, we see that $\mathcal{I}_A$ is a zero-dimensional ideal of D, $\mathrm{rank}_R(D/\mathcal{I}_A) = \mathrm{rank}_R(R[\xi]/\langle \xi_1^2, \dots, \xi_n^2 \rangle) = 2^n$ and

$$\{ \partial_1^{k_1} \partial_2^{k_2} \cdots \partial_n^{k_n} \mid k_1, \dots, k_n = 0, 1 \}$$

gives a basis of R-free module $D/\mathcal{I}_A$, where when $k_1 = \cdots = k_n = 0$, above element is understood to be 1. Thus we have shown the following proposition and completed the proof of Theorem 5.1 for $\mathcal{S}_A$.

Proposition 5.5. *The Airy system $\mathcal{S}_A$ is holonomic on $\mathbb{C}^n \setminus S$, $S = \cup_{i<j} \{x_i - x_j = 0\}$ and its holonomic rank is 2^n.*

5.3. Gauss, Kummer, Bessel

In this section, we give the reduced form of the S-polynomial $\mathrm{sp}(L_i, L_j)$ for the systems $\mathcal{S}_G, \mathcal{S}_K, \mathcal{S}_B$ for Gauss, Kummer and Bessel without explicit computation. For the proof of these cases, we must modify the ring $R = \mathbb{C}[x]_\Delta$, which is used in the cases $\mathcal{S}_{HW}, \mathcal{S}_A$, as

$$R = \{ f/g^m \mid f \in \mathbb{C}[x], m \in \mathbb{Z}_{\geq 0} \}$$

with $g = \prod_{i=1}^n x_i(x_i - 1) \cdot \Delta(x)$ for the case $\mathcal{S}_G$ and $g = \prod_{i=1}^n x_i \cdot \Delta(x)$ for the cases $\mathcal{S}_K, \mathcal{S}_B$.

5.3.1. System $\mathcal{S}_G$

Put

$$\begin{aligned}
L_i = x_i(1 - x_i)\partial_i^2 + \{ c - (n-1) - (a + b + 1 - (n-1))x_i \} \partial_i \\
+ \sum_{j(\neq i)} \frac{x_i(1 - x_i)\partial_i - x_j(1 - x_j)\partial_j}{x_i - x_j} - ab
\end{aligned}$$

and $G = \{L_1, \ldots, L_n\}$. Then

$$\mathrm{sp}(L_i, L_j) \equiv \frac{2x_i(x_i-1)x_j(x_j-1)}{(x_i-x_j)^2}(L_i - L_j)$$

after applying the division algorithm using G.

5.3.2. System $\mathcal{S}_K$

Put

$$L_i = x_i\partial_i^2 + \{c-(n-1)-x_i\}\partial_i + \sum_{j(\neq i)} \frac{x_i\partial_i - x_j\partial_j}{x_i - x_j} - a$$

and $G_K = \{L_1, \ldots, L_n\}$. Then

$$\mathrm{sp}(L_i, L_j) \equiv \frac{2x_ix_j}{(x_i-x_j)^2}(L_i - L_j)$$

after applying the division algorithm using G_K.

5.3.3. System $\mathcal{S}_B$

Put

$$L_i = x_i\partial_i^2 + \{c-n+1\}\partial_i + \sum_{j(\neq i)} \frac{x_i\partial_i - x_j\partial_j}{x_i - x_j} + 1$$

and $G_B = \{L_1, \ldots, L_n\}$. Then

$$\mathrm{sp}(L_i, L_j) \equiv \frac{2x_ix_j}{(x_i-x_j)^2}(L_i - L_j)$$

after applying the division algorithm using G_B.

Peer Review: Externally peer-reviewed.
Conflict of Interest: Author declared no conflict of interest.
Financial Disclosure: This work was supported by JSPS KAKENHI Grant Number JP19K03521.

LIST OF AUTHOR ORCIDS

H. Kimura https://orcid.org/0009-0000-2284-9820

REFERENCES

M. Adler and P. van Moerbeke, 1992, A matrix integral solution to two-dimensional W_p-gravity. Commun. Math. Phys. **147**, 25-56.
A.B. Balantekin, 2000, Character expansions, Itzykson-Zuber integrals, and the QCD partition function. Phys. Rev. D (3) 62, no. 8, 085017.
P. M. Bleher and A. B. J. Kuijlaars, 2004, Random matrices with external source and multiple orthogonal polynomials. Int. Math. Res. Not. , no. 3, 109–129.
P. Deift, 2000, Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach, Courant Lecture Notes in Mathematics **3**, Amer. Math. Soc., Providence RI.
J. Faraut and A. Koranyi. 1994, Analysis on symmetric cones, Oxford Math. monographs.
J. Harnad and A. Yu. Orlov, 2007, Fermionic construction of tau functions and random processes. Phys. D 235 no. 1-2, 168–206.
M. Hien, 2007, Periods for irregular singular connections on surface, Math. Ann. 337, 631–669.
K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, 1991, From Gauss to Painlevé. Vieweg Verlag.
K. Inamasu and H. Kimura, 2021, Matrix hypergeometric functions, semi-classical orthogonal polynomials and quantum Painleéequations. Integral Transforms Spec. Funct. 32, no. 5-8, 528–544.

H. Kimura and T. Koitabashi, 1996, Normalizer of maximal abelian subgroup of GL$(n)$and general hypergeometric functions. Kumamoto J. Math. **9**, 13-43.
M. Kontsevich, 1992, Intersection theory on the moduli space of curves and the matrix Airy function. Comm. Math. Phys. 147, no. 1, 1–23.
M.L. Mehta, 1991, Random matrices. Second edition, Academic Press, Boston, MA.
R. J. Muirhead, 1970, Systems of partial differential equations for hypergeometric functions of matrix arguments. Ann. Math. Statistics 41, 991-1001.
R. J. Muirhead, 1982, Aspects of Multivariate Statistical Theory, John Wiley & Sons.
H. Nagoya, 2011, Hypergeometric solutions to Schrödinger equations for the quantum Painlevé equations. J. Math. Phys. 52, no. 8, 083509.