

A Comprehensive Review of Solitonic Inequalities in Riemannian Geometry

Bang-Yen Chen*, Majid Ali Choudhary, Mohammed Nisar
and Mohd Danish Siddiqi

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ABSTRACT

In Riemannian geometry, Ricci soliton inequalities are an important field of study that provide profound insights into the geometric and analytic characteristics of Riemannian manifolds. An extensive survey of Ricci soliton inequalities is given in this review article, which also summarizes their historical evolution, core ideas, important findings, and applications. We investigate the complex interactions between curvature conditions and geometric inequalities as well as the several kinds of Ricci solitons, such as expanding, steady, and shrinking solitons. We also go over current developments, unresolved issues, and possible paths for further study in this fascinating area.

Keywords: Ricci soliton, solitonic inequalities, D-homothetic deformation, statistical submersion, hyperbolic Ricci soliton, Hitchin-Thorpe inequality.

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1. Introduction

R. Hamilton [45] introduced in 1982 the concept of Ricci flow. If one has a Riemannian manifold M with metric g_0 , then the Ricci flow is a PDE that evolves the metric tensor:

$$\frac{d}{dt}g(t) = -2\text{Ric}_{g(t)}, \quad g_0 = g(0), \quad (1.1)$$

where $\text{Ric}_{g(t)}$ denotes the Ricci curvature of the metric $g(t)$. Specifically, define the curves $t \mapsto g(t)$ that are solutions of the Ricci flow, given an initial metric $g(0)$ and a maximum interval $[0, T)$ of the variable t , where $0 \leq t < \infty$.

In local coordinates x^j , the system of nonlinear PDE affects the component $g_{jk} = g(e_j, e_k)$ of the metrics $g = g(t)$. The functions depend on variables t in addition to x^j . A relatively complex coordinate variant of condition (1.1) exists as well:

$$\frac{\partial g_{jk}}{\partial t} = -2R_{jk}, \quad \text{where} \quad R_{jk} = \partial_p \Gamma_{jk}^p - \partial_j \Gamma_{pk}^p + \Gamma_{qp}^p \Gamma_{jk}^q - \Gamma_{jq}^p \Gamma_{pk}^q.$$

The Christoffel symbols for the metric $g = g(t)$ are Γ_{jk}^p , as defined by

$$2\Gamma_{jk}^p = g^{pq} (\partial_j g_{kq} + \partial_k g_{jq} - \partial_q g_{jk}),$$

where $[g^{jk}]$ represents the inverse matrix of $[g_{jk}]$ at each point of the coordinate domain, with g^{jk} denoting the contravariant components of the metric.

Hamilton proved in [45] via Ricci flow that if M^3 is a closed 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature, then M^3 also admits a metric of constant positive curvature.

1.1. Ricci flow in the proof of Poincaré conjecture

In [45], Hamilton established the uniqueness and existence of a maximal Ricci flow trajectory on every compact manifold, with any given initial metric $g(0)$. Additionally, he tried to use this finding to support the Poincaré conjecture in three dimensions [67].

"The Hamilton program" was his suggested outline of such a proof, and it comprised a particular set of actions. In 2002, Grigori Perelman [64, 65, 66] completed the final steps.

Perelman also demonstrated the considerably general Thurston geometrization conjecture for three-dimensional manifolds at the same time.

Surgery plays a key role in Perelman's work, as they are required when the Ricci flow encounters a singularity in limited time ($T < \infty$). After the procedure, in a topologically simpler scenario, the Ricci flow is once again employed.

1.2. Ricci solitons - "fixed points" of the Ricci flow

A Riemannian metric $g_0 = g(0)$ is a Ricci soliton on a manifold M . It develops in an inessential manner under the Ricci flow, so up to multiplications by positive constants (also known as "rescalings") and diffeomorphisms, all of the stages $g(t)$ overlap with $g(0)$. Put another way, under the equivalence relationship that was just mentioned, this kind of metric denotes a fixed point in the quotient of the space of metrics on M of the Ricci flow.

The aforementioned idea is evident in the case of compact manifolds since there exists only one maximal Ricci flow trajectory with a given initial metric. A Ricci soliton, in the absence of the compactness condition, is a Riemannian metric g on a manifold M with a solution of equation (1.1), which reflects an inessential development of the Riemannian metric and meets the initial condition $g(0) = g$.

1.3. Applications

In the study of the Ricci flow and differential geometry, Ricci solitons are significant entities. These self-similar solutions to the Ricci flow equation are important in comprehending the creation of singularities and the long-term behavior of the Ricci flow.

Understanding Singularities in Ricci Flow: Near singularities, the behavior of the Ricci flow is commonly modeled by Ricci solitons. They provide insight into the possible shapes and structures that singularities can take [47].

Example: The Cigar soliton is a two-dimensional steady Ricci soliton with the metric:

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

It models a steady soliton that represents the behavior near certain types of singularities in two-dimensions.

Geometric Analysis and Topology: Ricci solitons are used in the classification of manifolds in terms of their curvature properties. They are involved in the research of metrics with non-negative Ricci curvature and Einstein manifolds [20].

Example: Cylindrical Soliton: The product of a round sphere with Euclidean space, such as $(S^{n-1} \times \mathbb{R})$, is an example of a steady Ricci soliton. It models the behavior of Ricci flow near cylindrical singularities.

Mathematical Physics and String Theory: Ricci solitons are used in string theory and the study of the renormalization group flow. They model fixed points and other critical phenomena in the theory of quantum fields [72].

Analysis of Heat Kernels and Eigenvalues: Ricci solitons are used in the study of heat kernels and eigenvalues of the Laplace-Beltrami operator on manifolds. This has applications in spectral geometry and the study of heat diffusion on manifolds [85].

Applications in General Relativity: In general relativity, the Ricci curvature tensor plays a crucial role in describing the curvature of spacetime in terms of Einstein equation [38]. Energy conditions are theoretical

constraints that relate the energy content of matter and fields to the curvature of spacetime. One such condition involves the Ricci curvature tensor.

The energy condition related to the Ricci curvature tensor is called the "weak energy condition" (WEC) [49]. It states that for any timelike vector V^μ , the contraction $R_{\mu\nu}V^\mu V^\nu$ is non-negative, where $R_{\mu\nu}$ represents the components of the Ricci curvature tensor. Mathematically, the weak energy condition can be expressed as:

$$R_{\mu\nu}V^\mu V^\nu \geq 0$$

for all timelike vector fields V^μ . This condition effectively says that any observer's energy density measurement must be non-negative.

Connections to Perelman's Work: Grigori Perelman's work [64, 66] on the geometrization conjecture and the Ricci flow with surgery is strongly related to the research of Ricci soliton inequalities. Comprehending and improving the inequalities linked to Ricci solitons adds to the larger program of comprehending three-manifold geometry and topology.

2. Preliminaries

2.1. Discovery of solitons

A soliton is a type of nonlinear wave that possesses two distinct properties:

- (1) Without changing its shape or speed, a localized wave continues to propagate.
- (2) Localized waves maintain their identities and are stable against mutual collisions.

The first is a condition of solitary waves that has been known since the 19th century in hydrodynamics. The second indicates that the wave possesses a particle-like quality. In contemporary physics, a suffix-on, like phonon and photon, is employed to denote the particle property. Using the particle attribute of a solitary wave, Zabusky and Kruskal [80] called it a "soliton."

The story of how soliton was discovered is fascinating and astounding. The Scottish engineer and scientist John Scott-Russell (1844) made the first on-record observation of the single wave in 1834.

Scott-Russell is credited with coining the term "solitary wave". Scientists like Stokes, Boussinesq, and Rayleigh were interested in this occurrence. Still, no one had an empirical explanation until 1898, when two Dutch physicists named Korteweg and de Vries published their now-famous equation [76] (referred to as *KdV* equation).

$$u_t + \alpha uu_x + \beta u_{xxx} = 0, \quad \alpha \text{ and } \beta \text{ are constants.} \quad (2.1)$$

In this case, x is the coordinate travelling at the velocity of a linearized wave, and $u(x, t)$ represents the height above mean sea level. Equation (2.1) (1-soliton) has a solution for solitary waves:

$$u(x, t) = \frac{3v}{\alpha} \operatorname{sech}^2 [\vartheta], \quad (2.2)$$

with $\vartheta = \frac{1}{2} \sqrt{\frac{v}{\beta}}(x - vt)$.

Pasta, Fermi and Ulam analyzed the equilibrium state approach in a 1-dimensional non-linear lattice in 1955 [40]. An ergodic system, or an equal distribution of energy among all of the modes, was predicted as a consequence of the linear system's nonlinear interactions among its normal modes. Results from a numerical analysis contradicted this idea. The system eventually reverts to its original configuration, despite the energy being distributed unevenly among all the modes.

In 1965, Zabusky and Kruskal used a numerical solution of the *KdV* equation as a model for a nonlinear lattice to discover the recurrent occurrences. They also discovered an unanticipated *KdV* equation feature. Sharply peaks appear on a waveform that was initially smooth. These pulse waves flow through one another after impacts and travel nearly independently at steady speeds. According to a thorough examination, each pulse is a sech^2 -type solitary wave, as shown by (2.2), and these solitary waves exhibit stable particle behaviour, which leads to the discovery of the soliton.

2.2. Establishment of soliton concept

1. How can one verify the soliton's properties analytically?
2. Why do particles like solitons have stability?
3. Is soliton a phenomenon specific to the equation (2.1)?

To construct the concept of solitons, the answers to these questions were essential.

After the independent and dependent variables are appropriately scaled, equation (2.1) takes on its shape

$$u_{xxx} + u_t - 6uu_x = 0. \quad (2.3)$$

A stationary waveform is produced by nonlinear and dispersion effects, which are represented by the KdV equation. Despite mutual interactions, the equation's preserved parameters guarantee the stability of isolated waves. Given the infinite degrees of freedom of the field variable, the dynamic properties of the system are bounded by infinite conservation laws, permitting an arbitrary number of solitons and infinite conserved values.

The inverse scattering approach is used to study the fundamental properties of solitons. In 1967, C. S. Gardner et al. developed in [44] a method for solving the initial-value problem of the Korteweg-deVries equation which is applicable to initial data that approach a constant sufficiently rapidly as $|x| \rightarrow \infty$. The potential function $u(x, t)$ is the solution to equation (2.3):

$$(x, t)\psi - \psi_{xx} = \lambda\psi. \quad (2.4)$$

It is shown that the eigenvalue λ does not rely on time when u evolves as per (2.3). The method of Gardner et al. can be used to predict exactly the "solitons" or solitary waves, which emerge from arbitrary initial conditions. Solutions that describe any finite number of solitons in interaction can be expressed in closed form.

In quantum physics, equation (2.4) is simply the Schrödinger equation. Gelfand-Levitan and Marchenko solved the latter problem for equation (2.4). Further information suggests that the eigenfunction $\psi(x, t)$'s time development is

$$\psi_t = -4\psi_{xxx} + 3u_x\psi + 6u\psi_x. \quad (2.5)$$

In summary, the KdV equation's solution is provided by

$$u(x, t) = 2 \frac{\partial}{\partial x} K(x, x; t), \quad (2.6)$$

where

$$K(x, y; t) + \int_{-\infty}^x K(x, z; t)F(z + y; t)dz + F(x + y; t) = 0, \quad (2.7)$$

and $F(x; t)$ is of the form:

$$F(x; t) = \sum_{n=1}^N c_n^2(t)e^{\eta_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b(k, t)}{a(k, 0)} e^{-ikx} dk. \quad (2.8)$$

The Gelfand-Levitan equation is the integral equation (2.7). It is important to keep in mind that all of the calculations in the example above are linear issues. This resolves the KdV equation's initial-value problem. A soliton is associated with each limited state that has a discrete eigenvalue. In particular, it is simple to obtain the N -soliton solution corresponding to N bound states by solving the Gelfand-Levitan equation where the coefficient for reflection $r(k, 0) = b(k, 0)/a(k, 0)$ is identically zero (the reflectionless potential). Through the exact formulation of the N -soliton solution [77], we can show that the soliton is stable against mutual collisions and that the collisions are pairwise, meaning that they only result in position changes in the solitons. The above-discussed solution methodology is called the inverse scattering method.

It is possible to solve the KdV equation's initial-value problem. At the time, though, it seemed like a coincidence. Five years later, Zakharov and Shabat [82] addressed the nonlinear Schrödinger (NLS) problem by extending on the inverse scattering technique

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0. \quad (2.9)$$

Later, Wadati resolved the equation for modified KdV ($mKdV$) [75].

$$u_t + 6u^2u_x + u_{xxx} = 0. \tag{2.10}$$

Furthermore, we now have more than 100 soliton equations after solving the Sine-Gordon equation [58].

We may precisely define the soliton using the inverse scattering method. Transferring the field variables to the scattering data is known as a canonical transformation. The scattering data space contains definitions for the action-angle variables. Because of this, the soliton equation is a fully integrable system, and the soliton is a system fundamental mode [81].

The greatest advance in the theory of solitons is represented by the discovery of the inverse scattering method. The inverse scattering method can be thought of as the Fourier transformation expanded into nonlinear problems if the scattering data space is viewed as an extension of the momentum space. The Fourier transform's development allowed for the solution of the diffusion problem in 1811. It was refined into a cohesive approach for solving nonlinear evolution problems after more than 150 years.

2.3. Ricci soliton

A Ricci soliton is a complete Riemannian manifold (M, g) if and only if there exists a smooth vector field V such that

$$\text{Ric}(g) - \lambda g + \frac{1}{2}\mathcal{L}_V g = 0.$$

Here, $\lambda \in \mathbb{R}$ is the constant. In this case, the Lie derivative is represented by \mathcal{L} , and the Ricci curvature tensor by Ric . (M, g) is a gradient Ricci soliton if $f : M \rightarrow \mathbb{R}$ exists and $V = \nabla f$. The soliton equation is then expressed as:

$$\text{Ric}(g) + \nabla^2 f = \lambda g.$$

Note that the Einstein equation is the solution of the above equations when either $V = 0$ or $f = 0$.

Role of V in Ricci solitons [20, 34]

1. Characterization of soliton Type:

- Gradient Solitons: For any smooth function f , $V = \nabla f$ is a gradient vector field. In this case, the Ricci soliton equation reduces to $\text{Ric} + \nabla \nabla f - \lambda g = 0$. The potential function in this case is denoted by f .
- Rotational Solitons: For rotational solitons, V generates a 1-parameter family of isometries, implying certain symmetries in the manifold's structure.
- Homogeneous Solitons: In homogeneous solitons, V exhibits a high degree of symmetry, often reflecting the underlying symmetry group of the manifold.

2. Geometric Interpretation: V governs the self-similar evolution of the soliton under the Ricci flow. It determines how the metric g changes along the flow direction induced by V .

3. Physical Interpretation: V can be interpreted as a flow vector field that describes how the metric evolves to maintain the soliton condition. Depending on the sign of λ , V either contracts, expands, or keeps the metric unchanged (in the case of steady solitons).

2.4. Self-similar Ricci flow solutions

The Ricci flow equation has a self-similar solution that is produced by a Ricci soliton (M, g_0) .

$$\partial_t g_t = -2 \text{Ric}(g_t).$$

In particular, putting

$$\sigma(t) = 1 - 2\lambda t,$$

and integrating the vector field $X(t) := \frac{1}{\sigma(t)}V$ to obtain a family of diffeomorphisms Ψ_t , with Ψ_0 the identity map, provides a solution of the Ricci flow by taking

$$g_t = \sigma(t)\Psi_t^*(g_0).$$

Hence, up to diffeomorphism and depending on the sign of λ , a Ricci soliton homothetically shrinks, remains steady or expands under Ricci flow.

2.5. Types of Ricci soliton with example

(i) Shrinking ($\lambda > 0$)

- Einstein manifolds with positive scalar curvature [4].
- Gradient compact Kahler-Ricci shrinkers [15, 53, 78].
- The 4-dimensional shrinker given by Bamler et al. [2].
- The 4-dimensional shrinker given by Feldman Feldman et al. [39].
- Round shrinking cylinder $S^{n-1} \times \mathbb{R}, n \geq 3$ [48].
- Round shrinking sphere $S^n, n \geq 2$ [34].
- Shrinking Gaussian soliton $(\mathbb{R}^n, g_{\text{eucl}}, f(x) = \frac{\lambda}{2}|x|^2)$ [48].

In a shrinking Ricci soliton, the metric contracts under the Ricci flow, resembling a shrinking of the manifold. A classic example is the Gaussian soliton on Euclidean space. Consider Euclidean space with the metric:

$$g = dx^2 + dy^2 + dz^2.$$

The Gaussian soliton is given by the metric:

$$g = (1 + 4t)^{-\frac{1}{2}}(dx^2 + dy^2 + dz^2),$$

where t is time. This soliton shrinks isotropically as time progresses.

(ii) Steady ($\lambda = 0$)

- The 2-dimensional cigar soliton $(\mathbb{R}^2, g = \frac{dx^2 + dy^2}{1 + x^2 + y^2}, V = -2(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}))$.
- The Bryant soliton with rotational symmetry of $3d$ and its extension to higher dimensions [12].
- Ricci flat manifold.

The sphere with its standard metric is among the most basic examples. We will use the standard round metric derived from the ambient Euclidean space to represent the n -dimensional sphere immersed in \mathbb{R}^{n+1} as S^n .

The standard metric on S^n is given by

$$g = d\beta_1^2 + \cos^2 \beta_1 d\beta_2^2 + \dots + \cos^2 \beta_1 \dots \cos^2 \beta_{n-1} d\beta_n^2,$$

where $\{\beta_1, \dots, \beta_n\}$ is a standard spherical coordinate system.

The metric g evolves under the following equation under the Ricci flow:

$$\frac{\partial g}{\partial t} + 2 \text{Ric} = 0,$$

where Ric is the Ricci curvature tensor. The metric itself determines the Ricci curvature of a round sphere, which is $\text{Ric} = \frac{2(n-1)}{n}g$. As a result, the equation for Ricci flow becomes:

$$\frac{\partial g}{\partial t} = -\frac{4(n-1)}{n}g.$$

Now, consider a time-dependent metric $g(t)$ on S^n given by:

$$g(t) = e^{4(n-1)t/n}g(0).$$

The Ricci flow equation is satisfied by this metric, which may be directly confirmed. Furthermore, we may retrieve the initial metric $g(0)$ if we set $t = 0$. As a result, $g(0)$ is implied to be a stable Ricci soliton and a fixed point of the Ricci flow.

(iii) Expanding ($\lambda < 0$)

- Negative scalar curvature Einstein manifolds: Kähler Ricci solitons expansion on the complex line bundles $O(-k), k > n$ over $\mathbb{C}P^n, n \geq 1$.

In an expanding Ricci soliton, the metric expands under the Ricci flow. An example is the Bryant soliton [59]. Consider the affine 3-space with the warped product metric:

$$g = dx^2 + e^{2x}(dy^2 + dz^2).$$

This metric represents a soliton with positive curvature and expansion in the x -direction.

2.6. Different inequalities in Riemannian geometry

Inequalities play a crucial role in the study of Riemannian manifolds, which are geometric spaces that generalize the concept of curved surfaces. Here are several key reasons why inequalities are important in this context:

1. Geometric Inequalities: Bonnet-Myers theorem [52] states that if M is a complete Riemannian manifold such that the Ricci curvature of M satisfies

$$\text{Ric}_p(v) \geq \frac{1}{r^2} > 0$$

for all $p \in N$ and for all $v \in T_pM$, then M is compact and the diameter $\text{diam}(M) \leq \pi r$.

2. Topological Inequalities: The Bishop-Gromov inequality provides a bound on a Riemannian manifold's diameter in terms of its sectional curvature [23]. It states explicitly that for any two points p and q in a Riemannian manifold M , we have

$$\text{dist}(p, q) \leq \frac{\pi}{\sqrt{\sec(M)}},$$

where the distance between points p and q is indicated by $\text{dist}(p, q)$.

3. Rigidity Theorems: The sphere theorem, also known as the quarter-pinched sphere theorem, strongly restricts the topology of manifolds admitting metrics with a particular curvature bound. The sphere theorem states that if M is a complete, simply-connected, Riemannian n -manifold with sectional curvature taking values in the interval $(1, 4]$, then M is homeomorphic to the n -sphere.

4. Gradient Inequalities: The Laplacian of a function f and the gradient's norm on a Riemannian manifold are related by the Bochner inequality [69]:

$$\|\nabla^2 f\| \geq \frac{1}{n} \|\text{Hess}(f)\|^2,$$

where $\nabla^2 f$ denotes the Hessian of f , and $\text{Hess}(f)$ denotes the Hessian operator.

5. Isoperimetric Inequalities: A basic finding in geometry that establishes a relationship between the area and the perimeter of a closed surface or curve in Euclidean space is known as the isoperimetric inequality.

Let A be the area enclosed by a closed curve or surface in Euclidean space and let P be the perimeter or circumference of the same curve or surface. Then, the isoperimetric inequality states that for any such curve or surface,

$$4\pi A \leq P^2.$$

In two dimension, equality is maintained only when the curve or surface is a circle, and in three dimensions, it is a sphere. In other words, among all closed curves or surfaces with a given area, the one with the smallest perimeter or circumference is the circle (or sphere), and the ratio of its perimeter (or circumference) to its area is $\frac{P}{A} = \frac{1}{r}$, where r is the radius of the circle (or sphere).

According to Cheeger's inequality, the first eigenvalue of the Laplacian operator has a lower constraint in terms of the isoperimetric constant of a Riemannian manifold.

6. Functional Inequalities: A function f on a Riemannian manifold M has a constant C such that, according to the Poincaré inequality,

$$\|f - \bar{f}\|_2 \leq C \|\nabla f\|_2,$$

where \bar{f} represents the average of f , the gradient of f is represented by ∇f , and $\|\cdot\|_2$ represents the L^2 norm.

2.7. Definitions

Conformal Ricci flow equations : A. E. Fishcher [41] introduced a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The resulting equations are named the conformal Ricci flow equations because of the role that conformal geometry plays in constraining the scalar curvature and because these equations are the vector field sum of a conformal flow

equation and a Ricci flow equation. These new equations are given by

$$2 \left(\frac{1}{n}g + \text{Ric } g \right) + \frac{\partial g}{\partial t} = -pg, \quad R(g) = -1,$$

for a metric g that is dynamically changing and a static scalar field p . The conformal Ricci flow equations and the Navier-Stokes equations are the same in fluid mechanics:

$$\nu \Delta v + \nabla_v v + \text{grad } p + \frac{\partial v}{\partial t} = 0, \quad \text{div } v = 0.$$

This comparison results in the time-dependent scalar field p and the term "conformal pressure". Like the real physical pressure in fluid mechanics, which maintains the fluid's incompressibility, the conformal pressure acts as a Lagrange multiplier to conformally adjust the metric flow in order to preserve the scalar curvature constraint.

Conformal Ricci flow equations with the Einstein constant $-\frac{1}{n}$ have Einstein metrics at their locations of equilibrium. The formula $-2 \left(\frac{1}{n}g + \text{Ric } g \right)$ thus functions as a nonlinear restorative force and quantifies the flow's divergence from an equilibrium point. Unless it is in an equilibrium location, the conformal pressure $p \geq 0$ is positive. The constraint force $-pg$ conformally deforms g to maintain the scalar curvature by acting pointwise orthogonally to the nonlinear restoring force $-2 \left(\frac{1}{n}g + \text{Ric } g \right)$.

Conformal Ricci soliton: [34, 16] A conformal Ricci soliton generalizes Ricci soliton concept by allowing a conformal change in the metric. Formally, a conformal Ricci soliton is defined by

$$\text{Ric}(g) + \mathcal{L}_X g = \lambda g + \mu g$$

where μ is a differentiable function on M .

This equation indicates that the Ricci curvature, modified by the Lie derivative of the metric, is equal to a scalar multiple of the metric plus a term that varies smoothly over the manifold.

Conformal η Ricci soliton: A. Bhattacharyya and N. Basu [3] introduced in 2015 the conformal type Ricci soliton, which is a generalization of the classic type Ricci soliton and is represented by the equation

$$\mathcal{L}_V g + 2 \text{Ric} = \left[2\lambda - p - \frac{2}{n} \right] g, \tag{2.11}$$

where λ represents a constant and p the conformal pressure. It is crucial to keep in mind that the conformal type Ricci soliton contains a self-similar solution to Fisher's conformal type Ricci flow equation. Following this, a multitude of investigations into conformal type Ricci solitons have been carried out on a range of geometric structures, including $(LCS)_n$ -manifolds [42] and generalized Sasakian space forms [43].

Gradient Ricci soliton refers, once more, to the situation in which the potential vector field V in (2.11) is the gradient of a smooth function f on M . The Ricci soliton potential function is the name given to this function, f . J. T. Cho and M. Kimura [29] introduced the η -Ricci soliton in complex space forms for the first time. Later, C. Calin and M. Crasmareanu [13] studied it on Hopf hypersurfaces. A η -Ricci soliton is said to be accepted by the Riemannian manifold (M, g) when, for a smooth vector field V , the metric g satisfies the following equation:

$$2\mu\eta \otimes \eta + \mathcal{L}_V g + \mathcal{L}_V g + 2\lambda g + 2S = 0, \tag{2.12}$$

where \mathcal{L}_V is the Lie derivative in the direction of V , S , the Ricci tensor, and λ, μ , are real constants. Observe that when $\mu = 0$, the η -Ricci soliton becomes a Ricci soliton.

The combination of equations (2.11) and (2.12): The following equation represents the conformal η -Ricci soliton, which was established by M. D. Siddiqi [71].

$$2\mu\eta \otimes \eta + \mathcal{L}_V g + \left[2\lambda - p - \frac{2}{n} \right] g + 2S = 0,$$

where n is the dimension of the manifold, p is scalar field which is non-dynamical, \mathcal{L}_V is the Lie derivative along V and S the Ricci tensor's direction.

Hitchin–Thorpe inequality: [50, 74, 37] This is a significant finding in differential geometry, particularly for the investigation of four-dimensional manifolds. It provides a link between the Euler characteristic and the signature of an oriented compact 4-manifold accepting a Riemannian metric with non-negative scalar curvature.

For a differentiable compact orientated 4-manifold, the signature $\tau(M)$ and the Euler characteristic $\chi(M)$ of M are related by the Hitchin–Thorpe inequality as follows:

$$3|\tau(M)| \leq 2\chi(M).$$

Hyperbolic Ricci soliton: [1] If a vector field X on M has real scalars λ and μ , such that a Riemannian manifold (M^n, g) , then the structure is known as hyperbolic Ricci soliton structure:

$$\text{Ric} + \frac{1}{2}\mathcal{L}_X(\mathcal{L}_X g) + \lambda\mathcal{L}_X g = \mu g.$$

As previously stated, depending on whether λ is positive, zero, or negative, a hyperbolic Ricci soliton is said to be expanding, stable, or shrinking. Furthermore, the parameter μ represents the rate of change of our solutions and has geometric meaning. The equation above reduces to Einstein equation when $X = 0$ or X is a Killing vector field.

The gradient hyperbolic structure (M, g, X, λ, μ) is a hyperbolic Ricci soliton structure if there exists a function f , which is often referred to as the potential function, such that $X = \nabla f$. Hence, (2.7) becomes

$$\text{Ric} + 2\lambda \text{Hess } f + \mathcal{L}_{\nabla f} \text{Hess } f = \mu g.$$

Example: Let $(M^n, g(t)) = (\mathbb{R}^n, g_{\text{can}})$ be a static Euclidean space, then it is a stationary solution to the hyperbolic Ricci flow as given below

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}, \quad g(0) = g_0, \quad \frac{\partial g_{ij}}{\partial t} = 0,$$

and is therefore a constant hyperbolic soliton. A hyperbolic Ricci soliton that expands or shrinks modulo diffeomorphisms could also be applied to this solution.

3. Solitonic inequalities on Sasakian manifolds

3.1. Contact metric structure and D-Homothetic deformation

An odd-dimensional Riemannian manifold (M, g) is called an *almost contact metric manifold* [8, 73, 79] if there exist a $(1, 1)$ -tensor field ϕ , a vector field ξ and a one-form η , such that

$$\begin{aligned} \eta(\xi) &= 1, \quad \phi^2(X) = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \\ \eta(X) &= g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \end{aligned}$$

for $X, Y \in \mathfrak{X}(M)$. An almost contact metric manifold is called a *contact metric manifold* if it satisfies

$$g(X, \phi Y) = d\eta(X, Y).$$

An almost contact metric manifold M is called *normal* if the tensor field

$$N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$$

vanishes identically, where $[\phi, \phi]$ is called the *Nijenhuis tensor* of ϕ . A normal contact metric manifold is said to be a *Sasakian manifold*. An almost contact metric manifold is called *almost cosymplectic* if it satisfies $d\eta = 0$ and $d\phi = 0$ applied to a $(1, 1)$ -tensor. An almost cosymplectic manifold is called *cosymplectic* if it satisfies $\nabla\phi = 0$ and $\nabla\xi = 0$. An almost contact metric manifold \bar{M} is called a *Kenmotsu manifold* if it satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

A *Sasakian space form* is a Sasakian manifold with a constant ϕ -sectional curvature.

Scaling a manifold's metric tensor by a positive constant is commonly referred to as a homothetic deformation. A homothetic deformation by a constant $c > 0$ yields a new metric \tilde{g} defined as follows if g is the metric tensor on a manifold M such that $\tilde{g} = c^2g$. This idea is extended by D-homothetic deformation, which applies a differential scale that can change over the manifold. It is frequently examined in relation to specific geometric structures, including paracontact or virtually contact metric structures.

They explored the behavior of virtually contact metric structures which are common in contact geometry under scaling transformations using D-homothetic deformations. Assume that a manifold M has a metric structure (ϕ, ξ, η, g) which is almost contact. *This case has the following: a vector field (the Reeb vector field), a 1-form η , a metric tensor g , and a (1,1)-tensor field ϕ .*

A D-homothetic deformation involves transforming these components :

$$\tilde{\xi} = \frac{1}{c}\xi, \tilde{\phi} = \phi, \quad \tilde{\eta} = c\eta, \quad \tilde{g} = gc^2 + (c^2 - 1)\eta \otimes \eta,$$

where c is a positive constant.

3.2. Paracontact structure and para-Sasakian metric manifold

A $(2n + 1)$ -dimensional smooth manifold M is said to have an almost paracontact structure if it admits a vector field ξ , (1,1)-tensor field ϕ and a 1-form η satisfying the following three conditions:

(i) $\phi^2 = I - \eta \otimes \xi$,

(ii) $\eta(\xi) = 1$,

(iii) ϕ induces on the $2n$ -dimensional distribution $\mathcal{D} \equiv \ker(\eta)$, an almost paracomplex structure \mathcal{P} i.e., $\mathcal{P}^2 \equiv I$, I the identity map, and the eigensubbundles \mathcal{D}^+ and \mathcal{D}^- , corresponding to the eigenvalues $1, -1$ of \mathcal{P} respectively, have equal dimension n ; and hence $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$.

The vector field ξ is called characteristic vector field or Reeb vector field. An immediate consequence of those relations are $\phi\xi = 0$ and $\eta \circ \phi = 0$. The tensor field ϕ induces an almost paracomplex structure on each fibre of $\ker(\eta)$ i.e., the eigendistributions corresponding to eigenvalues 1 and -1 have same dimension n . A pseudo-Riemannian metric g is said to be compatible with the almost paracontact structure if it satisfies

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{3.1}$$

holds for arbitrary vector fields X and Y and (M, ϕ, ξ, η, g) is called an *almost paracontact metric manifold*.

The normality of a paracontact metric manifold (M, ϕ, ξ, η, g) is equivalent to vanishing of the (1, 2)-torsion tensor defined by $N_\phi(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi$, where

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[X, \phi Y] - \phi[\phi X, Y]$$

for any vector fields $X, Y \in \mathfrak{X}(M)$. A normal paracontact metric manifold is called a *para-Sasakian metric manifold*.

3.3. Inequalities involving Ricci soliton on para Sasakian manifolds

In order to investigate the connection between extrinsic and intrinsic invariants, M. A. Choudhary et al. [33] examined in 2022 the scalar curvature for submanifolds of Ricci solitons. They obtained the following.

Lemma 3.1. [33] *Let $(\tilde{M}, g, V, \Lambda)$ be a Ricci soliton and M a contact pseudo-slant submanifolds of an (ϵ) -para Sasakian form $\tilde{M}(k)$. We obtain*

$$\begin{aligned} \operatorname{div}(V) - \mathcal{A}_1 + \|\sigma\|^2 \{ \epsilon(k + 1) (-\epsilon \cos^2 \theta - 2) + (n - 1)(k - 3) \} (n) \\ + \mathcal{K}_2 (\epsilon (2\epsilon - \cos^2 \theta - 1) + 5 - n) + n\Lambda = 0, \end{aligned} \tag{3.2}$$

where $\mathcal{A}_1 = n^2\|H\|^2$ and $\mathcal{K}_2 = \sum_{\alpha=n+1}^m \sum_{i,j \neq 1}^n (\sigma_{ij}^\alpha)^2$.

Theorem 3.1. [33] *Let $(\tilde{M}, g, V, \lambda)$ be a Ricci soliton and M be a contact pseudo-slant submanifold of an (ϵ) -para Sasakian form $\tilde{M}(k)$. Then we have*

$$\begin{aligned} \operatorname{div}(V) \leq n(n - 1)\|H\|^2 - n\Lambda - \{ \epsilon(k + 1) (-2 - \epsilon \cos^2 \theta) + (n)(n - 1)(k - 3) \} \\ - \mathcal{K}_2 (5 - n + \epsilon (2\epsilon - 1 - \cos^2 \theta)). \end{aligned}$$

A. M. Blaga and M. Crasmareanu [7] proposed an inequality for a lower boundary of the geometry of g in terms of gradient Ricci soliton for a function ψ on ambient space M such that

$$\|Hess\|_g^2 - \frac{1}{n}(\Delta\psi)^2 \leq \|S\|_g^2, \tag{3.3}$$

where $Hess$ denotes the Hessian of the differentiable function ψ on M . Let the soliton vector field V satisfy $V = \nabla\psi$. The authors used (3.3) to obtain the following theorem.

Theorem 3.2. [33] Consider a gradient Ricci soliton $(\tilde{M}, g, \nabla\psi, \Lambda)$ with a gradient-type soliton vector field V . Let M be a contact pseudo-slant submanifold of an (ϵ) -PS space form $\tilde{M}(k)$. Then we have

$$\|S\|_g^2 \geq \|Hess\|_g^2 - n^3\|H\|^4 + \|\sigma\|^4 + n\Lambda^2 + \{\epsilon(k+1)(-2 - \epsilon\cos^2\theta) + (n-1)(k-3)\}^2 + \frac{1}{n}(\mathcal{K}_2)^2(5 - n + \epsilon(2\epsilon - 1 - \cos^2\theta))^2.$$

Theorem 3.3. [33] Let M be a submanifold of a (ϵ) -para Sasakian form $\tilde{M}(k)$, with complete umbilical contact pseudo-slant. Assume that 3.2 holds true in its entirety. Next,

$$\|S\|_g^2 \geq \|Hess\|_g^2 + n\Lambda^2 + \{\epsilon(k+1)(-2 - \epsilon\cos^2\theta)\epsilon + (n-1)(k-3)\}^2 + \frac{1}{n}(\mathcal{K}_2)^2(5 - n + \epsilon(2\epsilon - 1 - \cos^2\theta))^2.$$

Choudhary et al [33] discovered an inequality for pseudo-slant submanifolds of the ϵ -para Sasakian manifold, notably concerning gradient Ricci soliton and gradient type vector field.

3.4. Sasakian manifold using generalized Wintgen inequalities

Let ∇ be the Levi-Civita connection on an almost para-contact metric manifold (M, ϕ, ξ, η, g) . Then M called an (ϵ) -PS manifold if it satisfies

$$(\nabla_X\phi)Y = -g(\phi X, \phi Y)\xi - \epsilon\eta(Y)\phi^2X$$

for any vector fields X, Y on M . Let $\tilde{M}(k)$ denote an (ϵ) -PS manifold with constant ϕ -para holomorphic sectional curvature k .

In [32], Choudhary et al. obtained inequalities for Ricci solitons in order to find relationships between extrinsic and intrinsic invariants. More precisely, they proved the following results involving inequalities of Ricci soliton.

Theorem 3.4. Let M be a submanifold of an (ϵ) -PS space form $\tilde{M}(k)$ adopting a Ricci soliton with a potential vector field $V \in TM$ of Ricci soliton. Assume that $(\tilde{M}, g, V, \Lambda)$ is a Ricci soliton. The Ricci soliton is therefore expanding, steady, and shrinking in accordance with

1. $\frac{n}{2}\|H\|^2 < \frac{1}{n}(\text{div}(V) + \|\sigma\|^2 + \tau)$,
2. $\frac{n}{2}\|H\|^2 = \frac{1}{n}(\text{div}(V) + \|\sigma\|^2 + \tau)$, and
3. $\frac{n}{2}\|H\|^2 > \frac{1}{n}(\text{div}(V) + \|\sigma\|^2 + \tau)$, respectively.

Corollary 3.1. Let M be a submanifold of an (ϵ) -PS space form $\tilde{M}(k)$, and let $(\tilde{M}, g, V, \Lambda)$ be a gradient Ricci soliton with a potential vector field $V = \nabla(\psi) \in TM$ of gradient type. Gradient Ricci soliton is then admitted by the submanifold M of $\tilde{M}(k)$ if and only if

$$\frac{n}{2}\|H\|^2 < \frac{1}{n}(\Delta(\psi) + \|\sigma\|^2 + \tau).$$

Theorem 3.5. Let M be a submanifold of an (ϵ) -PS space form $\tilde{M}(k)$, and let $(\tilde{M}, g, V = \nabla(\psi), \Lambda)$ be a gradient shrinking Ricci soliton with $V = \nabla(\psi) \in TM$ of gradient type. Then

$$|W| \leq \frac{2}{n\|H\|^2 - 2\|\sigma\|^2 - 2n\Lambda - 2\Delta(\psi)} \sqrt{\frac{2(n-1)}{n-2}},$$

where W denotes the Weyl conformal tensor of M .

Theorem 3.6. Let $(\tilde{M}, g, V = \nabla(\psi), \Lambda)$ be a gradient Ricci soliton of gradient type, and let M be a CPS-submanifold of an (ϵ) -PS space form $\tilde{M}(k)$. Then

$$|W| \leq \left\{ \left(\frac{k-3}{4} \right) (2p+q) + \left(\frac{k+1}{4} \right) (-2\epsilon - \epsilon^2 \cos^2 \theta) \right\} (2p+q+1) + \left(\frac{k+1}{4} \right) (4-2p-q+2\epsilon^2 - \epsilon - \epsilon \cos^2 \theta) + (2p+q+1)^2 \|H\|^2 - \|\sigma\|^2.$$

Theorem 3.7. Given $V = \nabla(\psi) \in TM$ of gradient type and M a completely umbilical CPS-submanifold of an (ϵ) -PS space form $\tilde{M}(c)$, let $(\tilde{M}, g, V = \nabla(\psi), \Lambda)$ be a gradient shrinking Ricci soliton. Then

$$|W| \leq \left\{ (2p+q) \left(\frac{k-3}{4} \right) + (-2\epsilon - \epsilon^2 \cos^2 \theta) \left(\frac{k+1}{4} \right) \right\} (2p+q+1) + (4-2p-q+2\epsilon^2 - \epsilon - \epsilon \cos^2 \theta) \left(\frac{k+1}{4} \right) + \|H\|^2 (2p+q+1)^2 - \|\sigma\|^2. \tag{5.12}$$

4. Solitonic inequalities on Kenmotsu manifolds

4.1. In a D -homothetically deformed Kenmotsu manifold

In 2022, a D -homothetically deformed Kenmotsu manifold was investigated by A. M. Blaga [5] for almost Riemann and almost Ricci solitons. A gradient vector field, a solenoidal vector field, or the Reeb vector field of the deformed structure are possible vector fields for these solitons. In some situations, it is possible to obtain the scalar and Ricci curvatures explicitly.

In [5], Blaga derived the following results involving Ricci soliton inequalities.

Theorem 4.1. [5] In a $(2n+1)$ -dimensional D -homothetically deformed Kenmotsu manifold $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$, if a gradient almost Riemann soliton is defined by $(V = \overline{\text{grad}}(f), \bar{\lambda})$, then we have

$$|\overline{\text{Ric}}|_{\bar{g}}^2 \geq (2n-1)^2 [v],$$

where $v = |\overline{\text{Hess}}(f)|_{\bar{g}}^2 - \frac{1}{2n+1} (\bar{\Delta}(f))^2$.

Corollary 4.1. [5] In a $(2n+1)$ -dimensional Kenmotsu manifold $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ that is D -homothetically deformed, if a solenoidal gradient almost Riemann soliton is defined by $(V = \overline{\text{grad}}(f), \bar{\lambda})$, then

$$|\overline{\text{Ric}}|_{\bar{g}}^2 \geq (2n-1)^2 |\overline{\text{Hess}}(f)|_{\bar{g}}^2.$$

Proposition 4.1. [5] In a $(2n+1)$ -dimensional D -homothetically deformed Kenmotsu manifold $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$, if $(V = \overline{\text{grad}}(f), \bar{\lambda})$ defines a gradient almost Riemann soliton, then

$$\begin{aligned} & (2n-1)^2 |\overline{\text{Hess}}(f)|_{\bar{g}}^2 - 4n \frac{a-1}{a} \text{scal} - 4n^2 (2n+1) \left(\frac{a-1}{a} \right)^2 \\ & - \frac{(2n-1)^2}{2n+1} (\bar{\Delta}(f))^2 - \frac{2(2n-1)^2}{2n+1} \frac{a-1}{a} [\xi(f) - \xi(\xi(f))] \bar{\Delta}(f) \\ & + \frac{2n(2n-1)^2}{2n+1} \left(\frac{a-1}{a} \right)^2 (\xi(f))^2 - \frac{2(2n-1)^2 (n+na+a)(a-1)}{(2n+1)a^2} (\xi(\xi(f)))^2 \\ & + \frac{2(2n-1)^2 (2n+a)(a-1)}{(2n+1)a^2} \xi(f) \cdot \xi(\xi(f)) \leq |\overline{\text{Ric}}|_{\bar{g}}^2. \end{aligned}$$

Corollary 4.2. [5] As per Proposition 4.1's hypothesis, if V is \bar{g} -orthogonal to ξ , then

$$\begin{aligned} & (2n-1)^2 |\overline{\text{Hess}}(f)|_{\bar{g}}^2 - \frac{(2n-1)^2}{2n+1} (\bar{\Delta}(f))^2 + \frac{4n(2n-1)(a-1)}{a} \bar{\Delta}(f) \\ & + \frac{4n^2(2n+1)(a^2-1)}{a^2} \leq |\overline{\text{Ric}}|_{\bar{g}}^2. \end{aligned}$$

Furthermore, if f is a function that is Δ -harmonic, then

$$|\text{Ric}|_g^2 \geq (2n - 1)^2 |\text{Hess}(f)|_g^2 + \frac{4n^2(2n + 1)(a^2 - 1)}{a^2}.$$

Proposition 4.2. [5] $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is a $(2n + 1)$ -dimensional D -homothetically deformed Kenmotsu manifold if a solenoidal gradient almost Riemann soliton is defined by $(V = \overline{\text{grad}}(f), \bar{\lambda})$, then

$$|\text{Ric}|_g^2 \geq (2n - 1)^2 |\text{Hess}(f)|_g^2 + \frac{a^2 - 1}{a^2} [4n^2 - (2n - 1)^2 (\xi(\xi(f)))^2].$$

Theorem 4.2. [5] In a $(2n + 1)$ -dimensional Kenmotsu manifold $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ that is D -homothetically deformed, if the gradient defined by $(V = \overline{\text{grad}}(f), \bar{\lambda})$ is almost Ricci soliton, then

$$|\overline{\text{Ric}}|_g^2 \geq |\overline{\text{Hess}}(f)|_g^2 - \frac{1}{2n + 1} (\overline{\Delta}(f))^2.$$

Corollary 4.3. [5] If $(V = \overline{\text{grad}}(f), \bar{\lambda})$ defines a solenoidal gradient almost Ricci soliton in a $(2n+1)$ -dimensional D -homothetically deformed Kenmotsu manifold $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$, then

$$|\overline{\text{Ric}}|_g^2 \geq |\overline{\text{Hess}}(f)|_g^2.$$

Proposition 4.3. [5] In a $(2n + 1)$ -dimensional D -homothetically deformed Kenmotsu manifold $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$, if $(V = \overline{\text{grad}}(f), \bar{\lambda})$ defines a gradient almost Ricci soliton, then

$$\begin{aligned} |\text{Ric}|_g^2 \geq & |\text{Hess}(f)|_g^2 - 4n \frac{a-1}{a} \text{scal} - 4n^2(2n+1) \left(\frac{a-1}{a}\right)^2 - \frac{1}{2n+1} (\Delta(f))^2 \\ & - \frac{2}{2n+1} \frac{a-1}{a} [\xi(f) - \xi(\xi(f))] \Delta(f) + \frac{(a-1)2(2n+a)}{(2n+1)a^2} \xi(f) \cdot \xi(\xi(f)) \\ & + \frac{2n}{2n+1} \left(\frac{a-1}{a}\right)^2 (\xi(f))^2 - \frac{(a-1)2(n+na+a)}{(2n+1)a^2} (\xi(\xi(f)))^2. \end{aligned}$$

Corollary 4.4. [5] As per Proposition 4.2's hypotheses, if V is \bar{g} -orthogonal to ξ , then we obtain

$$|\text{Ric}|_g^2 \geq |\text{Hess}(f)|_g^2 - \frac{1}{2n+1} (\Delta(f))^2 + \frac{(a-1)4n}{a} \Delta(f) + \frac{(2n+1)(a^2-1)4n^2}{a^2}.$$

Furthermore, if f is a function that is Δ -harmonic, then

$$|\text{Ric}|_g^2 \geq |\text{Hess}(f)|_g^2 + \frac{4n^2(2n+1)(a^2-1)}{a^2}.$$

Proposition 4.4. [5] In a $(2n+1)$ -dimensional D -homothetically deformed Kenmotsu manifold $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$, if $(V = \overline{\text{grad}}(f), \bar{\lambda})$ defines a solenoidal gradient almost Ricci soliton, then

$$|\text{Ric}|_g^2 \geq |\text{Hess}(f)|_g^2 + \frac{a^2 - 1}{a^2} [4n^2 - (\xi(\xi(f)))^2].$$

4.2. Indefinite Kenmotsu manifolds as a framework for conformal η -Ricci solitons

Y. L. Li et al. [57] investigated certain unique varieties of Ricci tensor in relation to ϵ -Kenmotsu manifolds' conformal η -Ricci soliton in 2022. Subsequently, the writer examined certain curvature circumstances that permit conformal η -Ricci solitons on ϵ -Kenmotsu manifolds.

Corollary 4.5. [57] An n -dimensional ϵ -Kenmotsu manifold (M, g) admits a conformal Ricci soliton (g, ξ, λ) . Consequently, (M, g) becomes a η -Einstein manifold, and λ satisfies $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - 1)$. Moreover, we have:

1. In the event that ξ is spacelike, the soliton is either expanding, stable, or contracting as, $(\frac{p}{2} + \frac{1}{n}) > (1 - n)$, $(\frac{p}{2} + \frac{1}{n}) = (1 - n)$ or $(\frac{p}{2} + \frac{1}{n}) < (1 - n)$; and

2. In the event that ξ is timelike, the soliton is expanding, steady, or shrinking as, $(\frac{p}{2} + \frac{1}{n}) > (n - 1)$, $(\frac{p}{2} + \frac{1}{n}) = (n - 1)$ or $(\frac{p}{2} + \frac{1}{n}) < (n - 1)$.

Corollary 4.6. [57] $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - 1)$ describes the relationship between the scalars λ and μ on the manifold (M, g) , which is a η -Einstein manifold. An n -dimensional ϵ -Kenmotsu manifold (M, g) is constant multiple of ξ if it admits a conformal Ricci soliton (g, V, λ, μ) such that V is pointwise collinear with ξ . Moreover, we have:

1. In the event that ξ is spacelike, the soliton is either expanding, steady, or shrinking as, $(\frac{p}{2} + \frac{1}{n}) + n > 1$, $(\frac{p}{2} + \frac{1}{n}) + n = 1$ or $(\frac{p}{2} + \frac{1}{n}) + n < 1$; and

2. In the event that ξ is timelike, the soliton is expanding, steady, or shrinking as, $(\frac{p}{2} + \frac{1}{n}) + 1 > n$, $(\frac{p}{2} + \frac{1}{n}) + 1 = n$ or $(\frac{p}{2} + \frac{1}{n}) + 1 < n$.

Corollary 4.7. [57] If an n -dimensional ϵ -Kenmotsu manifold has a Codazzi type Ricci tensor and admits a conformal η -Ricci soliton (g, ξ, λ, μ) , then

1. if ξ is spacelike then the soliton is expanding, steady or shrinking according to, $(\frac{p}{2} + \frac{1}{n}) + n > 2$, $(\frac{p}{2} + \frac{1}{n}) + n = 2$ or $(\frac{p}{2} + \frac{1}{n}) + n < 2$; and

2. if ξ is timelike then the soliton is expanding, steady or shrinking according to $(\frac{p}{2} + \frac{1}{n}) + 2 > n$, $(\frac{p}{2} + \frac{1}{n}) + 2 = n$ or $(\frac{p}{2} + \frac{1}{n}) + 2 < n$.

Theorem 4.3. [57] Given an n -dimensional ϵ -Kenmotsu manifold (M, g) , let (g, ξ, λ, μ) be a conformal η -Ricci soliton. If the manifold meets the curvature criterion $R(X, Y) \cdot S = 0$, it can be said to be Ricci semi-symmetric. In this case, $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - 2)$ and $\mu = 1$. Additionally,

1. In the event that ξ is spacelike, the soliton is either expanding, steady, or shrinking as, $(\frac{p}{2} + \frac{1}{n}) > (2 - n)$, $(\frac{p}{2} + \frac{1}{n}) = (2 - n)$ or $(\frac{p}{2} + \frac{1}{n}) < (2 - n)$; and

2. In the event that ξ is timelike, the soliton is either expanding, steady, or shrinking as, $(\frac{p}{2} + \frac{1}{n}) + (2 - n) > 0$, $(\frac{p}{2} + \frac{1}{n}) + (2 - n) = 0$ or $(\frac{p}{2} + \frac{1}{n}) + (2 - n) < 0$.

5. Solitonic inequalities on Kähler manifolds

5.1. A finding on compactness for Kähler Ricci soliton

Cao and Sesum [19] showed results of compactness for Kähler Ricci gradient compact shrinking solitons. In case that (M_i, g_i) represents a series of Kähler Ricci solitons with real-dimension with $n \geq 4$, curvatures with uniformly bounded $L^{n/2}$ norms, Ricci curvatures with uniform boundedness from below, and $\mu(g_i, 1/2) \geq A$ (where μ is the Perelman functional), A subsequence (M_i, g_i) leads to a compact orbifold (M_∞, g_∞) with finitely many isolated singularities, where g_∞ is a Kähler Ricci soliton metric in the sense of an orbifold (fulfils the Kähler Ricci soliton equation in a lifting about singular points and smoothly extends in some gauge to a metric that satisfies the soliton equation away from singular points.).

Cao and Sesum proved the following theorem

Theorem 5.1. [19] Let (M_i, g_i) be a sequence of Kähler Ricci solitons of real-dimension $n \geq 6$, with $c_1(M_i) > 0$, satisfying

$$g_i(t) - \text{Ric}(g_i(t)) = \partial\bar{\partial}u_i(t) = \frac{d}{dt}g_i(t),$$

with $\nabla_j \nabla_{\bar{k}} u_i = \bar{\nabla}_{\bar{j}} \nabla_k u_i = 0$, such that

(i) $\int_{M_i} |\text{Rm}|^{n/2} dV_{g_i} \leq C_1,$

(ii) $\text{Ric}(g_i) \geq -C_2$ and

(iii) $A \leq \mu(g_i, 1/2),$

for some uniform constants C_1, C_2, A independent of i . Then there there exists a subsequence (M_i, g_i) converging to (Y, \bar{g}) , where Y is an orbifold with finitely many isolated singularities and \bar{g} is a Kähler Ricci soliton in an orbifold sense.

5.2. On the Kähler Ricci flow and Harnack's inequality

In 1992, H.-D. Cao [14] studied Richard Hamilton's Ricci flow (see [45, 46]):

$$\frac{\partial}{\partial t} g_{ij} = -R_{ij} + g_{ij}. \tag{5.1}$$

On a compact n -dimensional Kähler manifold X with positive holomorphic bisectional curvature. The solution of (5.1) yields the following Harnack estimate for the Ricci tensor.

Theorem 5.2. [14] *Given a compact Kähler manifold X with positive bisectional curvature and $0 \leq t < \infty$, let g_{ij} be a solution of (5.1). Given any pair $x \in X$ and $v \in T_x X$, let*

$$Q_{ij} = \frac{\partial}{\partial t} R_{ij} + R_{ik} R_{kj} - R_{ij} + R_{ij,k} v^k + R_{ij,\bar{k}} v^{\bar{k}} + R_{ijk\bar{i}} v^k v^{\bar{i}} + \frac{1}{1 - e^{-t}} R_{ij}.$$

Then, for any $t > 0$, and $w \in T_x X, w \neq 0$, we have $Q_{ij} w^i w^j > 0$.

Cao obtained gradient estimates for the scalar curvature and the determinant of the Ricci tensor of the metric g_{ij} by taking different traces of Q_{ij} .

Corollary 5.1. [14] *The estimate is satisfied by the scalar curvature R the following:*

$$\frac{\partial R}{\partial t} + R_{,k} v^k + R_{,\bar{k}} v^{\bar{k}} + R_{ij} v^i v^{\bar{j}} + \frac{R}{1 - e^{-t}} > 0,$$

for all $t > 0, x \in X$ and $v \in T_x X$. In particular, taking $v_i = -R_{,i}/R$, we obtain

$$\frac{\partial R}{\partial t} - \frac{|DR|^2}{R} + \frac{R}{1 - e^{-t}} > 0,$$

for all $t > 0, x \in X$.

Corollary 5.2. [14] *Put $\phi = \det(R_{ij}) / \det(g_{ij})$. Then we have*

$$\frac{\partial \phi}{\partial t} - \frac{|D\phi|^2}{n\phi} + \frac{n\phi}{1 - e^{-t}} > 0,$$

for all $t > 0, x \in X$.

The author got the following Harnack inequalities for the scalar curvature and the determinant of the Ricci tensor by integrating the above estimates as in [55].

Theorem 5.3. [14] *Let g_{ij} be the solution of (5.1) as in 5.2. Then for any $x, y \in X$ and $0 < t_1 < t_2 < \infty$, we have:*

(i) *The scalar curvature R satisfies the inequality*

$$R(x, t_1) \leq \frac{e^{t_2} - 1}{e^{t_1} - 1} e^{\Delta/4} R(y, t_2).$$

(ii) *The determinant of the Ricci tensor $\phi = \det(R_{ij}) / \det(g_{ij})$ satisfies the inequality*

$$\phi(x, t_1) \leq \left(\frac{e^{t_2} - 1}{e^{t_1} - 1} \right)^n e^{n\Delta/4} \phi(y, t_2),$$

where Δ is defined as

$$\Delta = \Delta(x, y; t_1, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} \left(|\gamma'(s)|_s^2 \right) ds.$$

Taking the infimum over all curves from x to y , where $|\gamma'(s)|_s$ is the velocity of γ at time s .

6. Solitonic inequalities for compactness of a manifold

6.1. Ricci soliton structures and non-compact manifolds as extremes of log-Sobolev inequality

In 2019, M. Rimoldi and G. Veronelli [68] showed that exponential decline occurs at the extremals' infinity when Ricci curvature is likewise constrained from above. These analytical results led them to prove that a gradient Ricci soliton structure is supported by non-trivial shrinking Ricci solitons, under the same conditions.

Theorem 6.1. [68] Given a non-compact complete Riemannian manifold (connected) (M^m, g) , let us assume that

$$-(m - 1)K \leq \text{Ric} \text{ and } \text{inj}_{(M,g)} \geq i_0 > 0, \tag{6.1}$$

where $\text{inj}_{(M,g)}$ denotes the injectivity radius of (M, g) .

In the case $K \in [0, +\infty]$, $i_0 \in \mathbb{R}^+$, there exists a smooth extremal v for the log-Sobolev functional \mathcal{L} if $\lambda < \lambda_\infty$. Furthermore, we suppose that in place of the bound $\text{Ric} \geq -(m - 1)K$, assume that $(m - 1)K \geq |\text{Ric}|$. After fixing a point $o \in M$, the extremal v fulfills the following positive constants: $C, c > 0$ such that

$$Ce^{-cd^2(x,o)} \geq v(x), \quad x \in M. \tag{6.2}$$

Remark 6.1. [68]

(a) The scalar curvature R is contained in the log-Sobolev functional \mathcal{L} , however deleting the term holding the scalar curvature does not change the outcome. There isn't much to change in the proof.

(b) See [83] for remarks on the condition's universality at infinity $\lambda < \lambda_\infty$, as well as for several examples of Riemannian manifolds that meet this condition.

Rimoldi and Veronelli obtained the following geometric consequence, which was also demonstrated in [68], by utilizing Theorem 6.1.

Theorem 6.2. [68] Consider a connected full non-compact Riemannian manifold (M^m, g) that supports a Ricci soliton structure that is shrinking. Assume that K and i_0 are positive constants such that

$$(m - 1)K \geq |\text{Ric}|, \text{ and } \text{inj}_{(M,g)} \geq i_0 > 0.$$

Moreover, $\lambda < \lambda_\infty$. A shrinking gradient Ricci soliton structure is then also supported by (M^m, g) .

6.2. Theorems of compactness for gradient Ricci solitons

X. Zhang's compactness theorem [84] from 2006 requires a subsequence with uniformly bounded curvatures when a sequence of compact gradient Ricci solitons of dimension n converges to a compact orbifold with finitely many isolated singularities.

The author of [84] discussed the finding of compactness for Ricci solitons in this work. When the underlying manifolds are closed (compact, boundaryless), it is simple to show that the steady and expanding Ricci solitons are Einstein metrics. As a result, the author focused on the shrinking situation and developed the Theorem that followed.

Theorem 6.3. [84] Assume that (M_α, g_α) is a sequence of shrinking gradient Ricci solitons with dimension $n \geq 4$, meaning that it fulfills the subsequent equation:

$$g_\alpha - \text{Ric}(g_\alpha) = \nabla du_\alpha$$

such that

- (1) $\text{Ric}(g_\alpha) \geq -C_1 g_\alpha$,
- (2) $\text{diam}(M_\alpha, g_\alpha) \leq C_2$,
- (3) $\text{Vol}(M_\alpha, g_\alpha) \geq C_3$ and
- (4) $\int_{M_\alpha} |\text{Rm}|^{\frac{n}{2}} dV_{g_\alpha} \leq C_4$,

for some uniform positive constants C_1, C_2, C_3, C_4 . Then there exists a subsequence of (M_α, g_α) converging to (M_∞, g_∞) in the Cheeger–Gromov sense, where M_∞ is an orbifold with finitely many isolated singularities and g_∞ is a Ricci soliton in an orbifold sense.

Moreover, if n is odd, then there are no singular points and (M_∞, g_∞) is a smooth gradient Ricci soliton which is diffeomorphic to M_α for α sufficiently large. In this case, (M_α, g_α) (sub)converges smoothly to (M_∞, g_∞) .

Bounds on $\int_M |\text{Rm}|^2$ are implied for Ricci solitons by lower bounds on Ricci curvature and volume, an upper bound on diameter, and a bound on $b_2(M)$. We possess the subsequent corollary.

Corollary 6.1. [84] Given a sequence of shrinking gradient Ricci solitons of dimension 4, let (M_α, g_α) be such that (1) $\text{Ric}(g_\alpha) \geq -C_1 g_\alpha$, (2) $\text{diam}(M_\alpha, g_\alpha) \leq C_2$, (3) $\text{Vol}(M_\alpha, g_\alpha) \geq C_3$ and (4) $b_2(M_\alpha) \leq C_4$ for a few uniformly positive constants C_1, C_2, C_3, C_4 . Then, in the Cheeger–Gromov sense, there is a subsequence (M_α, g_α) that converges to (M_∞, g_∞) , where g_∞ is a Ricci soliton in the sense of an orbifold and M_∞ is an orbifold with finitely many isolated singularities.

X. Zhang then made some deductions and derived certain estimates; specifically, C_1 estimates for functions u_α and a uniform bound for Perelman’s function and the Ricci curvature. The author then calculated the Ricci soliton’s ϵ -regularity estimate.

Proposition 6.1. [84] *If g is a gradient soliton that is either steady or expanding over a compact manifold M , then g has to be an Einstein metric.*

When λ is a positive constant, from the estimate $\sup_{x \in M} u \leq C_8$, we have

$$|\nabla u|^2 \leq n\lambda + 2\lambda \left(\sup_{x \in M} u - \inf_{x \in M} u \right) - R \leq C_{10},$$

where C_{10} is a constant depending only on C_1, C_2, C_3 and λ .

Using the following lemma, the author was able to determine a uniform upper bound for scalar curvature.

Lemma 6.1. [84] *Consider (M_α, g_α) . be a sequence of shrinking Ricci solitons ($\lambda = 1$) satisfying conditions (1), (2), and (3) of Theorem 6.3, and u_α satisfying the constraint*

$$(2\pi)^{-\frac{n}{2}} \int_M e^{-u} dV_g = 1. \tag{6.3}$$

There are positive constants C_4, C_5 that rely exclusively on C_1, C_2 , and C_3 , such that

$$|u_\alpha|_{C^1} \leq C_{11} \text{ and } R(g_\alpha) \leq C_{12}.$$

In the next portion, X. Zhang gave a uniform bound of Perelman’s functional (see [64]), $\mu(g, \frac{1}{2})$, for a sequence of shrinking Ricci solitons (M_α, g_α) meeting conditions (1), (2) and (3) in Theorem 6.3.

In [64], Perelman proposed a functional fulfilling

$$W(g, \varphi, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-\varphi} [\tau (R + |\nabla\varphi|^2) + f - n] dV_g$$

under the constraint

$$(4\pi\tau)^{-\frac{n}{2}} \int_M e^{-\varphi} dV_g = 1.$$

Then they defined the functional $\mu(g, \tau) = \inf W(g, \cdot, \tau)$, where all functions meeting the restriction (6.2) are taken over by \inf , and $\tau > 0$.

Lemma 6.2. [84] *If (M, g) is a shrinking gradient Ricci soliton, that is, $g - \text{Ric}(g) = \nabla du$, where u is a minimizer of Perelman’s functional W with regard to metric g and $\tau = \frac{1}{2}$, then u satisfies the constraint (6.3).*

Proposition 6.2. [84] *Let u_α satisfy the constraint (6.3), and let (M_α, g_α) be a sequence of shrinking Ricci solitons ($\lambda = 1$) satisfying conditions (1), (2), and (3) in Theorem 6.3. Then, there is a constant C_6 that depends only on C_1, C_2 , and C_3 , such that $|\mu(g_\alpha, \frac{1}{2})| \leq C_6$.*

The following compactness theorem for Ricci solitons can be obtained with ease using the Gromov-Cheeger compactness theorem.

Proposition 6.3. [84] *Let (M_α, g_α) be a sequence of shrinking gradient Ricci solitons of dimension $n = 3$, such that (1) $\text{Ric}(g_\alpha) \geq -C_1 g_\alpha$, (2) $\text{diam}(M_\alpha, g_\alpha) \leq C_2$ and (3) $\text{Vol}(M_\alpha, g_\alpha) \geq C_3 > 0$. Consider the following uniform constants: C_1, C_2, C_3 . A smooth gradient Ricci soliton (M_∞, g_∞) is then a subsequence in C^∞ topology, and it converges to (M_∞, g_∞) .*

The following mean value inequality was determined by the author using Moser’s iteration argument.

Lemma 6.3. [84] *Given a compact Riemannian manifold (M, g) and a Lipschitz function f that satisfies*

$$f \Delta f \geq -\theta_1 |\nabla f|^2 - \theta_2 f^2 - \theta_3 f^3$$

in a feeble manner. In the event where $\theta_1 \leq \frac{1}{4}$, a constant ϵ will exist, and it will only depend on the dimensions of M, θ_3 , and the lower bound of the Sobolev constant C_s . This means that if $\int_{B_{\mathbb{R}^n}(2r)} f^{\frac{n}{2}} dv_g \leq \epsilon$, then

$$\sup_{B_{\mathbb{R}^n}(\frac{r}{2})} f \leq C_* \left(1 + \frac{1}{r^2} \right) \left(\int_{B_{\mathbb{R}^n}(r)} f^{\frac{n}{2}} dv_g \right)^{\frac{2}{n}},$$

where the Sobolev constant C_s , the lower bound of $\text{Vol}(M)$, and the dimensions of M, θ_2, θ_3 are the only factors that affect C_* .

6.3. Geometry of compact shrinking Ricci solitons

In 2014, B.-Y. Chen and S. Deshmukh [27] proved two characterizations of compact shrinking trivial Ricci solitons. They provided the following results

Theorem 6.4. *Let (M, g, f, λ) be an n -dimensional compact connected shrinking gradient Ricci soliton of positive Ricci curvature. If the Ricci curvature Ric and the scalar curvature S of (M, g) satisfy*

$$Ric(\nabla S, \nabla S) \leq \lambda \left(\|\nabla S\|^2 + \frac{\lambda}{2} (n^2 \lambda^2 - S^2) \right),$$

then S is a solution of the Poisson equation $\Delta \varphi = \sigma$ with $\sigma = \lambda(n\lambda - S)$.

Corollary 6.2. *A compact connected shrinking gradient Ricci soliton (M, g, f, λ) of positive Ricci curvature with $\lambda < \lambda_1$ is trivial if and only if the scalar curvature S satisfies*

$$Ric(\nabla S, \nabla S) \leq \lambda \left(\|\nabla S\|^2 + \frac{\lambda}{2} (n^2 \lambda^2 - S^2) \right), \quad n = \dim M.$$

On a compact Riemannian manifold (M, g) and a function $\varphi : M \rightarrow R$, the average value of φ is a real number defined by

$$\varphi_{av} = \frac{1}{\text{Vol}(M)} \int_M \varphi.$$

Chen and Deshmukh also proved the following characterization of trivial Ricci solitons.

Theorem 6.5. [27] *A compact connected shrinking gradient Ricci soliton (M, g, f, λ) with normalized potential is trivial if and only if $(fS)_{av} \leq \frac{1}{2}n^2\lambda$, where $n = \dim M$ and S is the scalar curvature of (M, g) .*

7. Solitonic inequalities on curvature

7.1. About the Ricci curvature of Lagrangian and isotropic submanifolds in complex space forms

B.-Y. Chen proved in [24] that any isotropic submanifold M^n in a complex space form $\tilde{M}^m(4c)$ satisfies

$$(n-1)c + \frac{n^2}{4}H^2 \geq \overline{Ric}. \tag{7.1}$$

In [24] he also proved that a Lagrangian submanifold of a complex space form satisfying the equality case of (7.1) is a minimal submanifold. Finally, he gave an explanation of the geometry of Lagrangian submanifolds satisfying (7.1) for equality, as long as the second basic form kernel's dimension doesn't change.

Theorem 7.1. [24] *Given a complex space form $\tilde{M}^m(4c)$ and an isotropic submanifold M^n , the Ricci tensor S of M^n satisfies*

$$\left(\frac{n^2}{4}H^2 + (n-1)c \right) g \geq S. \tag{7.2}$$

The equality sign of (7.2) holds identically if and only if M^n is a fully geodesic submanifold or if $n = 2$ and M^n is entirely umbilical.

Then he proved the main result on the minimality of Lagrangian in the following Theorem

Theorem 7.2. [24] *Let M^n be a Lagrangian submanifold of $\tilde{M}^m(4c)$, which is a complex space form. Then*

$$\overline{Ric} \leq (n-1)c + \frac{n^2}{4}H^2. \tag{7.3}$$

Clearly, Theorem 7.2 follows from the following.

Lemma 7.1. [24] *We have (7.3) for every n -dimensional isotropic submanifold of a complex space form $\tilde{M}^m(4c)$. Furthermore, if an isotropic submanifold M^n of $\tilde{M}^m(4c)$ satisfies the equality case of (7.3) at a point p , then $J(T_p M^n)$ is perpendicular to the mean curvature vector \vec{H} at p .*

7.2. Ricci solitons with finite scalar curvature ratio and complete gradient expansion asymptotically

Given (M^n, g, f) , $n \geq 5$ is a complete expanding gradient Ricci soliton with nonnegative Ricci curvature. In 2023, Cao et al., [18], showed that if the scalar asymptotic curvature ratio of (M^n, g, f) is finite (i.e., $\limsup_{r \rightarrow \infty} Rr^2 < \infty$), then the tensor of Riemann curvature must have at least sub-quadratic decay, i.e., $\limsup_{r \rightarrow \infty} |Rm|r^\alpha < \infty$ for any $0 < \alpha < 2$. The following are the main results given by them.

Theorem 7.3. [18] *Let (M^n, g, f) , $n \geq 5$, be an n -dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature $Ric \geq 0$ and finite asymptotic scalar curvature ratio*

$$\limsup_{r \rightarrow \infty} Rr^2 < \infty,$$

where $r = r(x)$ is the distance function to a given base point $x_0 \in M$. Then (M^n, g, f) has finite α -asymptotic curvature ratio for any $0 < \alpha < 2$,

$$A_\alpha := \limsup_{r \rightarrow \infty} |Rm|r^\alpha < \infty.$$

Additionally, there exist constant $C > 0$ depending on n and the geometry of (M^n, g, f) , sequences $\{r_j\} \rightarrow \infty$ and $\{\alpha_j\} \rightarrow 2$ such that

$$|Rm|(x) \leq C(r(x) + 1)^{-\alpha_j}$$

for each $x \in M \setminus B(x_0, r_j + 1)$, where $B(x_0, r_j + 1)$ is the geodesic ball centered at x_0 with radius $r_j + 1$.

7.3. Estimates of curvature for steady Ricci solitons

P.-Y. Chan’s work [22] from 2019 improves the estimate given in [17] for an n -dimensional full non-Ricci flat gradient stable Ricci soliton with a potential function constrained by a constant and a curvature tensor Rm fulfilling $Rm < 5$, $|Rm| < Ce^{-r}$ for some constant $C > 0$. For a four-dimensional complete non-Ricci flat gradient steady Ricci soliton, the decay rate of $|Rm|$ is exponential.

In [60], O. Munteanu, C.-J.A. Sung and J. Wang investigated the solvability of the weighted Poisson equation for a certain class of smooth metric measure spaces. They illustrated the following, for example:

Theorem 7.4. [60] *Given $n \geq 2$, let (M^n, g, f) be a n dimensional complete non Ricci flat gradient steady Ricci soliton. In the event of constant limits, the potential function f is bounded above and $\lim_{r \rightarrow \infty} r|Rm| = 0$, then a positive constant C exists such that*

$$|Rm|(x) \leq C(1 + r(x))^{3(n+1)}e^{-r(x)}, \tag{7.4}$$

where $r = r(x)$ is the distance of x from a fixed point $p_0 \in M$.

Instead of using Green’s function estimate in [60], the author of [22] sharpens the upper bound under weaker curvature decay conditions by studying the curvature features of growing gradient Ricci solitons using Deruelle’s maximum principle [35].

Theorem 7.5. [22] *Let (M^n, g, f) be an n dimensional non-Ricci flat complete gradient steady Ricci soliton, where $n \geq 2$. Let us assume that the curvature tensor Rm satisfies $\limsup_{r \rightarrow \infty} |Rm| < \frac{1}{5}$ and that the potential function f is bounded above by a constant. Next, a positive constant C exists such that*

$$|Rm|(x) \leq Ce^{-r(x)} \text{ on } M, \tag{7.5}$$

where $r = r(x)$ is the distance of x from a fixed point $p_0 \in M$.

Remark 7.1. [22] When Σ and \mathbb{T}^{n-2} represent Hamilton’s cigar soliton and the $n - 2$ dimensional flat torus, respectively, the decay rate is sharp on $\Sigma \times \mathbb{T}^{n-2}$.

Remark 7.2. [22] In Theorem (7.5), there exists a technical constant $\frac{1}{5}$. According to [10] and [11], $\lim_{r \rightarrow \infty} r|Rm| = \sqrt{\frac{(n-1)}{2(n-2)}} > \frac{1}{5}$ for the n dimensional Bryant soliton. Whether the constant $\frac{1}{5}$ is optimal is not obvious at this time.

Remark 7.3. [22] For simplicity’s sake, the author states the Theorem as follows: $\overline{\lim}_{r \rightarrow \infty} r^{-1}f < 1$. This condition on f seems to be weaker than the previous one, but it does not affect the validity of the conclusion (7.5). For f to be bounded above by a constant, it is necessary that $ric \geq 0$ and $S \rightarrow 0$ at infinity (see [21]).

O. Munteanu and J. Wang [61] noted, among other unique characteristics, that the Riemann curvature Rm in a four-dimensional gradient Ricci soliton can be bounded by ∇Ric and Ric .

$$|Rm| \leq A_0 \left(|\text{Ric}| + \frac{|\nabla \text{Ric}|}{|\nabla f|} \right), \tag{7.6}$$

for some universal positive constant A_0 .

Here is another main result of this paper:

Theorem 7.6. [22] *Let (M^4, g, f) be a four-dimensional non-Ricci flat gradient steady Ricci soliton with $\lim_{r \rightarrow \infty} S = 0$. There is a positive constant c such that*

$$|Rm| \leq cS \text{ on } M. \tag{7.7}$$

A 4-dimensional full nontrivial gradient stable Ricci soliton's Riemann curvature Rm decays exponentially if the potential f is constrained from above and $\lim_{r \rightarrow \infty} rS$ is small enough, as was shown in [22].

Theorem 7.7. [22] *Let (M^4, g, f) be a four-dimensional non-Ricci flat gradient steady Ricci soliton with $\lim_{r \rightarrow \infty} S = 0$. Assume the potential function f is bounded from above by a constant and $\overline{\lim}_{r \rightarrow \infty} rS < \frac{1}{5A_0^2}$, where A_0 is the constant in (7.6). Then there exists a constant $C > 0$ such that $|Rm|(x) \leq Ce^{-r(x)}$ on M .*

8. Solitonic inequalities on stability and instability

8.1. Stability and instability of Ricci solitons

Kröncke [54] investigated the volume-normalized Ricci flow near compact shrinking Ricci Solitons. Any normalized Ricci flow that begins near a compact Ricci Soliton (M, g) , which is a local maximum of Perelman's shrinker entropy, is shown to exist for all time and to converge towards a Ricci Soliton. If g is not a local maximum of the shrinker entropy, then a nontrivial normalized Ricci flow emerges from it.

Lemma 8.1. [54] *Consider a gradient shrinking Ricci soliton (M, g_0) . Then, in the space of metrics, there is a constant $C > 0$ and a $C^{2,\alpha}$ -neighbourhood \mathcal{U} of g_0 such that*

$$\left\| \frac{d}{dt} \Big|_{t=0} f_{g+th} \right\|_{C^{2,\alpha}} \leq C \|h\|_{C^{2,\alpha}}, \quad \left\| \frac{d}{dt} \Big|_{t=0} f_{g+th} \right\|_{H^i} \leq C \|h\|_{H^i}, \quad \left\| \frac{d}{dt} \Big|_{t=0} \tau_{g+th} \right\| \leq C \|h\|_{L^2}, \quad i = 0, 1, 2,$$

for all $g \in \mathcal{U}$, where g, g_0 are Riemannian metric tensors, $f \in C^\infty(M)$, and H, L are the maps defined in [54].

Theorem 8.1. [54] *Consider a Ricci soliton that shrinks gradient shrinkage, (M, g_0) . Then, given g_0 and constants $\sigma \in [1/2, 1)$, $C > 0$, there exists a $C^{2,\alpha}$ neighborhood \mathcal{U} such that*

$$C \left\| \tau \left(\text{Ric}_g + \nabla^2 f_g \right) - \frac{1}{2}g \right\|_{L^2} \geq |v(g) - v(g_0)|^\sigma, \tag{8.1}$$

for all $g \in \mathcal{U}$.

Further, they proved the main results on dynamical stability and instability by considering the τ -flow

$$\dot{g}(t) = -2 \text{Ric}_{g(t)} + \frac{1}{\tau_{g(t)}} g(t). \tag{8.2}$$

It is well defined around a gradient-shrinking Ricci soliton. Note that under the τ -flow, v is nondecreasing. The writers built a modified τ -flow in this way: The collection of diffeomorphisms produced by $X(t) = -\text{grad}_{g(t)} f_{g(t)}$ is denoted as $\tilde{g}(t) := \varphi_t^* g(t)$, where $g(t)$ is a solution of (8.2). Let $\varphi_t \in \text{Diff}(M), t \geq 1$ be the collection of diffeomorphisms. Then

$$\frac{d}{dt} \tilde{g}(t) = -2 \left(\text{Ric}_{\tilde{g}(t)} + \nabla^2 f_{\tilde{g}(t)} \right) + \frac{1}{\tau_{\tilde{g}(t)}} \tilde{g}(t). \tag{8.3}$$

This represents the τ gradient flow concerning the weighted L^2 -measure.

Lemma 8.2. [54] (τ -flow Shi estimates) Assuming that $g(t), t \in [0, T]$ is a solution to the τ -flow (8.2),

$$T^{-1} \geq \sup_{p \in M} |R_{g(t)}|_{g(t)} + \frac{1}{\tau_{g(t)}} \quad \forall t \in [0, T].$$

Then, there exists a constant $C(k)$ such that, for each $k \geq 1$, we have

$$\sup_{p \in M} |\nabla^k R_{g(t)}|_{g(t)} \leq C(k) \cdot T^{-1} t^{-k/2} \quad \forall t \in (0, T].$$

Theorem 8.2. [54] (Dynamical stability) Assume that $k \geq 3$ and that the gradient decreasing Ricci soliton is (M, g) . Suppose that a local maximizer of v is g . Then for every C^k -neighborhood \mathcal{U} of g , there is a C^{k+5} -neighborhood \mathcal{V} for which the following is true: A 1-parameter family of diffeomorphisms φ_t exists for every metric $g_0 \in \mathcal{V}$, such that the modified flow $\varphi_t^* g(t)$ always stays in \mathcal{U} and converges to a gradient shrinking Ricci soliton g_∞ in \mathcal{U} as $t \rightarrow \infty$ for the τ -flow (8.2) beginning at g_0 . Polynomial rate convergence suggests that there are constants $C, \alpha > 0$, such that

$$\|\varphi_t^* g(t) - g_\infty\|_{C^k} \leq C(t+1)^{-\alpha}.$$

Lemma 8.3. [54] Consider a gradient-shrinking Ricci soliton (M, g_0) . Then, for any g_0 , there is a $C^{2,\alpha}$ neighborhood \mathcal{U} with a constant $C > 0$, such that

$$C \|k\|_{C^{2,\alpha}} \|h\|_{H^1} \geq \left| \frac{d^2}{dtds} \right|_{t,s=0} f_{g+sk+th} \|h\|_{H^1} \quad \text{and} \quad C \|k\|_{C^{2,\alpha}} \|h\|_{H^1} \geq \left| \frac{d^2}{dtds} \right|_{t,s=0} \tau_{g+sk+th},$$

for all $g \in \mathcal{U}$.

Proposition 8.1. [54] (Estimates of the third variation of v) Consider a gradient shrinking Ricci soliton (M, g_0) . It is possible for g_0 to have a $C^{2,\alpha}$ -neighborhood \mathcal{U} such that

$$\left| \frac{d^3}{dt^3} \right|_{t=0} v(g+th) \leq C \|h\|_{H^1}^2 \|h\|_{C^{2,\alpha}},$$

for all $g \in \mathcal{U}$ and some constant $C > 0$.

9. Solitonic inequalities on warped product manifolds

For general references on warped product manifolds, we refer to the books [26, 63].

9.1. Inequalities of Ricci soliton in CR-warped product manifold

In 2023, Y. Li et al., [56], applied inequalities by using the notion of CR-warped products introduced in [25].

Theorem 9.1. [56] Assume that $M = M_T \times_f M_\perp$ is a CR-warped product in $\tilde{M}(c)$ that admits a Ricci soliton with a shrinking gradient. Thus, the following disparity is true:

$$\|h\|^2 \geq \beta \|\nabla(\ln f)\|^2 + c\alpha\beta + \beta \sum_{q=1}^{2\alpha} \text{Ric}(E_q, E_q). \tag{9.1}$$

Furthermore, the equality holds if and only if M_\perp is a fully umbilical submanifold of $M(c)$ and M_T is a totally geodesic submanifold.

9.2. Ricci solitons on noncompact warped products

Ricci solitons with complete noncompact warped product gradient were studied by V. Borges and K. Tenenblat [9]. It is proved that the warping function's gradient and nonexistence results are true. When the soliton is expanding or steady, some PDE estimates and rigidity discovered when investigating warped product Einstein manifolds are applied to a wider context. A nonexistence theorem is presented for shrinking soliton; this theorem has no counterpart in the Einstein case and is demonstrated by utilizing the properties of a weighted Laplacian's first eigenvalue. The noncompact warped product Ricci soliton yielded the following results.

Theorem 9.2. [9] Let $M^n \times_h F^m$ be a complete gradient shrinking Ricci soliton and R_F the scalar curvature of F^m . Suppose that M is noncompact, fix $q_0 \in F$ and let μ be defined by $m\mu = R_F(q_0)$. The following is true:

1. If h is not constant, then $\mu > 0$.
2. If $h \leq \sqrt{\frac{\mu}{\lambda}}$, then h is constant.

Theorem 9.3. [9] Consider a complete gradient steady Ricci soliton $M^n \times_h F^m$, where the scalar curvature of F^m is denoted by R_F . Assume that M is noncompact. Fix $q_0 \in F$. Let $m\mu = R_F(q_0)$ define μ . The following is true:

1. If h is not constant, then $\mu > 0$.
2. If $\sup h < +\infty$, then h is constant.

Theorem 9.4. [9] Assume that $M^n \times_h F^m$ is a complete gradient expanding Ricci soliton and that R_F is the scalar curvature of F^m . Let's say M is not compact. Adjust $q_0 \in F$. To define μ , let $m\mu = R_F(q_0)$. The following is true:

1. If $\mu < 0$, then $h \geq \sqrt{\frac{\mu}{\lambda}}$,
2. If $\mu \leq 0$, then $|\nabla \ln h|^2 \leq -\frac{\lambda}{m}$,
3. If $\mu < 0$ and $\sup_M h < +\infty$, then $|\nabla \ln h|^2 \leq -\frac{\lambda}{m} + \frac{2\mu}{m(\sup_M h)^2}$,
4. If $\mu \geq 0$ and $\sup h < +\infty$, then h is constant.

10. Solitonic inequalities on statistical submersion

10.1. Solitons and sharp inequalities for statistical submersion

In 2024, B.-Y. Chen et al. studied in [28] Ricci and scalar curvatures for a given statistical submersion and constructed some inequalities involving the Ricci and scalar curvatures:

$$\text{Ric}(E, F) = \overline{\text{Ric}}(E, F) - g(\mathcal{T}_E F, N^*) + (\hat{\delta}\mathcal{T})(E, F) + g(\mathcal{A}E, \mathcal{A}^*F) - g(\nabla_E^* \sigma, F) \quad (10.1)$$

and

$$\begin{aligned} \text{Ric}(X, Y) = & \hat{\text{Ric}}(X, Y) + g(\nabla_X^* N^*, Y) - g(\mathcal{T}X, \mathcal{T}Y) + (\hat{\delta}\mathcal{A})(X, Y) \\ & + g(\sigma, \mathcal{A}_X Y) - g(\mathcal{A}_X, \mathcal{A}_Y^*) - g(\mathcal{A}_X^*, \mathcal{A}_Y^*), \end{aligned} \quad (10.2)$$

where

$$\begin{aligned} (\hat{\delta}\mathcal{T})(E, F) &= \sum_{i=1}^n g((\nabla_{X_i} \mathcal{T})(E, F), X_i), \quad (\hat{\delta}\mathcal{A})(X, Y) = \sum_{j=1}^m g((\nabla_{E_j} \mathcal{A})(X, Y), E_j), \\ \sum_{i=1}^n g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) &= g(\mathcal{A}_X, \mathcal{A}_Y) = \sum_{j=1}^m g(\mathcal{A}_X^* E_j, \mathcal{A}_Y^* E_j), \quad \sigma = \sum_{i=1}^n \mathcal{A}_{X_i} X_i, \\ \sum_{i=1}^n g(\mathcal{A}_{X_i} E, \mathcal{A}_{X_i} F), \quad g(\mathcal{T}X, \mathcal{T}Y) &= g(\mathcal{A}E, \mathcal{A}F) = \sum_{j=1}^m g(\mathcal{T}_{E_j} X, \mathcal{T}_{E_j} Y). \end{aligned}$$

In which $\{E_j\}$ is an orthonormal basis, \mathcal{T} and \mathcal{A} are O'Neill tensors, and $\delta(X)$ is horizontal divergence.

Siddiqui et al. proved in [28] the following results using (10.1) and (10.2).

Theorem 10.1. [28] Let $\psi : (M, \nabla, g) \rightarrow (N, \hat{\nabla}, \hat{g})$ be a statistical submersion. Then, we have

$$\text{Ric}(E, E) \geq \overline{\text{Ric}}(E, E) - m^2 g(\mathcal{T}_E E, H^*) + (\hat{\delta}\mathcal{T})(E, E) - (\bar{\delta}\sigma)(E, E). \quad (10.3)$$

The equality case holds in (10.3) if and only if $\mathcal{H}(M)$ is integrable.

Since $2\mathcal{A}^0 = \mathcal{A} + \mathcal{A}^*$, we have

Theorem 10.2. [28] Let $\psi : (M, \nabla, g) \rightarrow (N, \hat{\nabla}, \hat{g})$ be a statistical submersion. Then, we have

$$\begin{aligned} \text{Ric}(X, X) \leq & g(\nabla_X^* N^*, X) + \hat{\text{Ric}}(X, X) + (\hat{\delta}\mathcal{A})(X, X) \\ & + g(\sigma, \mathcal{A}_X X) - 2g(\mathcal{A}_X^0, \mathcal{A}_X^*). \end{aligned} \quad (10.4)$$

The equality case holds in (10.4) if and only if each fiber is totally geodesic for $\nabla(\mathcal{T} = 0)$.

Theorem 10.3. [28] Let $\psi : (M, \nabla, g) \rightarrow (N, \hat{\nabla}, \hat{g})$ be a statistical submersion. Then, we have

$$2R \geq 2\bar{R} - m^2g(H, H^*). \tag{10.5}$$

In (10.5), the equality case is satisfied if and only if \mathcal{T} or \mathcal{T}^* is a multiple of the other. Specifically, every fiber is either completely geodesic with regard to $\nabla(\mathcal{T} = 0)$ or completely geodesic concerning $\nabla^*(\mathcal{T}^* = 0)$.

Theorem 10.4. [28] Let $\psi : (M, \nabla, g) \rightarrow (N, \hat{\nabla}, \hat{g})$ be a statistical submersion. Then, we have

$$2R \leq 2\hat{R} + g(\sigma, \sigma). \tag{10.6}$$

The equality case holds in (10.6) if and only if $\mathcal{A}_{\mathcal{H}}\mathcal{H} = 0$, where \mathcal{H} is map defined in [28].

Taking into account relations (10.1) and (10.2), they derived the following equation

$$R - \bar{R} - \hat{R} = -2g(\mathcal{A}, \mathcal{A}) + g(\mathcal{A}, \mathcal{A}^*) - g(\mathcal{T}, \mathcal{T}^*) - g(N, N^*) - \hat{\delta}N - \hat{\delta}^*N^* - \bar{\delta}\sigma + \bar{\delta}^*\sigma + g(\sigma, \sigma). \tag{10.7}$$

The scalar curvatures of the vertical and horizontal spaces of M are denoted by \bar{R} and \hat{R} . Here

$$g(\mathcal{T}, \mathcal{T}^*) = \sum_{i=1}^n g(\mathcal{T}X_i, \mathcal{T}^*X_i), \quad g(\mathcal{A}, \mathcal{A}) = \sum_{i=1}^n g(\mathcal{A}X_i, \mathcal{A}X_i), \quad g(\mathcal{A}, \mathcal{A}^*) = \sum_{i=1}^n g(\mathcal{A}X_i, \mathcal{A}^*X_i).$$

By using the Cauchy-Buniakowski-Schwarz inequality and equation (10.7), we have the following Theorem.

Theorem 10.5. [28] Let $\psi : (M, \nabla, g) \rightarrow (N, \hat{\nabla}, \hat{g})$ be a statistical submersion. Then, we have

$$R \geq \bar{R} + \hat{R} - 2\|\mathcal{A}\|^2 + g(\mathcal{A}, \mathcal{A}^*) - \|\mathcal{T}\| \|\mathcal{T}^*\| - g(N, N^*) - \hat{\delta}N - \hat{\delta}^*N^* - \bar{\delta}\sigma + \bar{\delta}^*\sigma + g(\sigma, \sigma). \tag{10.8}$$

Note that in (10.8), the equality case is satisfied if and only if \mathcal{T} or \mathcal{T}^* is a multiple of the other. Specifically, every fiber is either completely geodesic concerning $\nabla(\mathcal{T} = 0)$ or completely geodesic about $\nabla^*(\mathcal{T}^* = 0)$.

Results given in this section were applied in [28] to investigate Ricci-Bourguignon solitons on statistical submersions with conformal or gradient potential vector field.

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Competing interests

The authors declare that they have no competing interests.

Author’s contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

- [1] Azami, S., Fasihi-Ramandi, G.: Hyperbolic Ricci soliton on warped product manifolds. *Filomat*, 37(20), 6843–6853 (2023).
- [2] Bamler, R. H., Cifarelli, C., Conlon, R. J., Deruelle, A.: A new complete two-dimensional shrinking gradient Kähler Ricci soliton. *Geometric and Functional Analysis*, 1–16, (2024).
- [3] Basu, N., Bhattacharyya, A.: Conformal Ricci soliton in Kenmotsu manifold. *Global Journal of Advanced Research on Classical and Modern Geometries*, 4(1), 15–21 (2015).
- [4] Besse, A. L.: *Einstein manifolds*. Springer (2007).
- [5] Blaga, A. M.: Geometric solitons in a D-homothetically deformed Kenmotsu manifold. *Filomat*. 36(1), 175–186 (2022).
- [6] Blaga, A. M., Chen, B.-Y.: Gradient solitons on statistical manifolds. *J. Geom. Phys.* 164, Paper No. 104195, 10 pp (2021).

- [7] Blaga, A. M., Crasmareanu, M.: Inequalities for gradient Einstein and Ricci solitons. *Facta Universitatis, Series: Mathematics and Informatics*, 351–356 (2020).
- [8] Blair, D. E.: *Riemannian geometry of contact and symplectic manifolds*. Progr. Math., vol. 203, Birkhäuser Boston, Inc., Boston, MA, (2010).
- [9] Borges, V., Tenenblat, K.: Ricci almost solitons on semi-Riemannian warped products. *Mathematische Nachrichten* 295(1), 22–43 (2022).
- [10] Brendle, S.: Rotational symmetry of self-similar solutions to the Ricci flow. *Inventiones Mathematicae*, 194(3), 731–764 (2013).
- [11] Brendle, S.: Rotational symmetry of Ricci solitons in higher dimensions. *Journal of Differential Geometry*, 97(2), 191–214 (2014).
- [12] Bryant, R. L.: Ricci flow solitons in dimension three with $SO(3)$ -symmetries. Department of Mathematics, Duke University (2005). <https://services.math.duke.edu/~bryant/3DRotSymRicciSolitons.pdf>
- [13] Calin, C., Crasmareanu, M.: η -Ricci solitons on Hopf hypersurfaces in complex space forms. *Rev. Roumaine Math. Pures Appl*, 57(1), 55–63 (2012).
- [14] Cao, H.-D.: On Harnack’s inequalities for the Kähler Ricci flow. *Inventiones Mathematicae*, 109(1), 247–263 (1992).
- [15] Cao, H.-D.: Existence of gradient Kähler Ricci solitons. In *Elliptic and parabolic methods in geometry*, pp. 1–16, AK Peters/CRC Press (1996).
- [16] Cao, H.-D.: Recent progress on Ricci solitons. *ArXiv Preprint ArXiv:0908.2006* (2009).
- [17] Cao, H.-D., Cui, X.: Curvature estimates for four-dimensional gradient steady Ricci solitons. *The Journal of Geometric Analysis*, 30, 511–525 (2020).
- [18] Cao, H.-D., Liu, T., Xie, J.: Complete gradient expanding Ricci solitons with finite asymptotic scalar curvature ratio. *Calculus of Variations and Partial Differential Equations*, 62(2), Paper No. 48 (2023).
- [19] Cao, H.-D., Sesum, N.: A compactness result for Kähler Ricci solitons. *Advances in Mathematics*, 211(2), 794–818 (2007).
- [20] Cao, H.-D., Zhou, D.: On complete gradient shrinking Ricci solitons. *Journal of Differential Geometry*, 85(2), 175–186 (2010).
- [21] Carrillo, J., Ni, L.: Sharp logarithmic Sobolev inequalities on gradient solitons and applications. *Communications in Analysis and Geometry*, 17(4), 721–753 (2009).
- [22] Chan, P.-Y.: Curvature estimates for steady Ricci solitons. *Transactions of the American Mathematical Society*, 372(12), 8985–9008 (2019).
- [23] Cheeger, J., Ebin, D. G., Ebin, D. G.: *Comparison theorems in Riemannian geometry*. North-Holland publishing company Amsterdam (1975).
- [24] Chen, B.-Y.: On Ricci curvature of isotropic and Lagrangian submanifolds in complex space forms. *Archiv der Mathematik*, 74(2), 154–160 (2000).
- [25] Chen, B.-Y.: Geometry of warped product CR-submanifolds in Kaehler manifolds. *Monatshefte für Mathematik* 133(3), 177–195 (2001).
- [26] Chen, B.-Y.: *Differential geometry of warped product manifolds and submanifolds*. World Scientific, Hackensack, NJ (2017).
- [27] Chen, B.-Y., Deshmukh, S.: Geometry of compact shrinking Ricci solitons. *Balkan J. Geom. Appl*, 19(1), 13–21 (2014).
- [28] Chen, B.-Y., Siddiqi, M. D., Siddiqui, A. N.: On Ricci-Bourguignon solitons for statistical submersions. *Bull. Korean Math. Soc. (In press)* (2024).
- [29] Cho, J. T., Kimura, M.: Ricci solitons and real hypersurfaces in a complex space form. *Tohoku Mathematical Journal*, 61(2), 205–212 (2009).
- [30] Choudhary, A. M., Siddiqi, M. D., Bahadır, O., Uddin, S.: Hypersurfaces of metallic Riemannian manifolds as k-almost Newton-Ricci solitons. *Filomat*, 37(7), 2187–2197 (2023).
- [31] Choudhary, M. A., Blaga, A. M.: Generalized Wintgen inequality for slant submanifolds in metallic Riemannian space forms. *Journal of Geometry*, 112(2), 26 (2021).
- [32] Choudhary, M. A., Das, L. S., Siddiqi, M. D., Bahadır, O.: Generalized Wintgen Inequalities for (ϵ) -Para Sasakian Manifold. In *Geometry of Submanifolds and Applications* (pp. 133–145). Springer (2024).
- [33] Choudhary, M. A., Khan, M. N. I., Siddiqi, M. D.: Some basic inequalities on para Sasakian manifold. *Symmetry*, 14(12), 2585 (2022).
- [34] Chow, B., Lu, P., Ni, L.: *Hamilton’s Ricci flow* (Vol. 77). American Mathematical Society, Science Press (2023).
- [35] Deruelle, A.: Asymptotic estimates and compactness of expanding gradient Ricci solitons. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V*, 17(2), 485–530 (2017).
- [36] Deshmukh, S., Alsodais, H., Bin Turki, N.: Some results on Ricci almost solitons. *Symmetry*, 13(3), 430 (2021).
- [37] Donaldson, S. K., Kronheimer, P. B.: *The geometry of four-manifolds*. Oxford university press (1997).
- [38] Einstein, A.: Die feldgleichungen der gravitation. *Sitzungsberichte Der Königlich Preußischen Akademie Der Wissenschaften*, 844–847 (1915).
- [39] Feldman, M., Ilmanen, T., Knopf, D.: Rotationally symmetric shrinking and expanding gradient Kähler Ricci solitons. *Journal of Differential Geometry*, 65(2), 169–209 (2003).
- [40] Fermi, E., Pasta, P., Ulam, S., Tsingou, M.: *Studies of the nonlinear problems* [Techreport]. Los Alamos National Lab., Los Alamos, NM, USA (1955).
- [41] Fischer, A. E.: An introduction to conformal Ricci flow. *Classical and Quantum Gravity*, 21(3), S171–S218 (2004).
- [42] Ganguly, D., Bhattacharyya, A.: A study on conformal Ricci solitons in the framework of $(LCS)_n$ -manifolds. *Ganita*, 70(2), 201–216 (2020).
- [43] Ganguly, D., Dey, S., Ali, A., Bhattacharyya, A.: Conformal Ricci soliton and Quasi-Yamabe soliton on generalized Sasakian space form. *Journal of Geometry and Physics*, 169, Paper No. 104339 (2021).
- [44] Gardner, C. S., Greene, J. M., Kruskal, M. D., Miura, R. M.: Method for solving the Korteweg-deVries equation. *Physical Review Letters*, 19(19), 1095 (1967).
- [45] Hamilton, R. S.: Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry*, 17(2), 255–306 (1982).
- [46] Hamilton, R. S.: Four-manifolds with positive curvature operator. *Journal of Differential Geometry*, 24(2), 153–179 (1986).
- [47] Hamilton, R. S.: The Ricci flow on surfaces, *Mathematics and general relativity*. *Contemp. Math.*, 71, 237–261 (1988).
- [48] Hamilton, R.: The formations of singularities in the Ricci Flow. *Surveys in Differential Geometry*, 2(1), 7–136 (1993).
- [49] Hawking, S. W., Ellis, G. F.: *The large scale structure of space-time*. Cambridge University Press (2023).
- [50] Hitchin, N.: Compact four-dimensional Einstein manifolds. *Journal of Differential Geometry*, 9(3), 435–441 (1974).
- [51] Hretcanu, C.-E., Crasmareanu, M.: Metallic structures on Riemannian manifolds. *Revista de la Uni6 Matemática Argentina*, 54(2), 15–27 (2013).
- [52] Jost, J.: *Riemannian geometry and geometric analysis*. 6th edition, Springer (2011).
- [53] Koiso, N.: On rotationally symmetric Hamilton’s equation for Kähler-Einstein metrics. In *Recent topics in differential and analytic geometry* (pp. 327–337). Elsevier (1990).

- [54] Kröncke, K.: Stability and instability of Ricci solitons. *Calculus of Variations and Partial Differential Equations*, 53, 265–287 (2015).
- [55] Li, P., Yau, S. T.: On the parabolic kernel of the Schrödinger operator. *Acta Mathematica*, 156, 153–201 (1986).
- [56] Li, Y., Srivastava, S. K., Mofarreh, F., Kumar, A., Ali, A.: Ricci soliton of CR-warped product manifolds and their classifications, *Symmetry*, 15, Paper No. 976, 14 pp (2023).
- [57] Li, Y. L., Ganguly, D., Dey, S., Bhattacharyya, A.: Conformal η -Ricci solitons within the framework of indefinite Kenmotsu manifolds. *AIMS Math*, 7(4), 5[7]8–5430 (2022).
- [58] Miura, C., Ablowitz, M., Kaup, D., Newell, A., Segur, H.: Addendum: A classical perturbation theory [J. Math. Phys. 18, 110 (1977)]. *Stud. Appl. Math*, 53(249), 249 (1974).
- [59] Morgan, J. W., Tian, G.: Ricci flow and the Poincaré conjecture. *Clay Mathematics Monographs*, vol. 3. American Mathematical Society (2007).
- [60] Munteanu, O., Sung, C.-J. A., Wang, J.: Poisson equation on complete manifolds. *Advances in Mathematics*, 348, 81–145 (2019).
- [61] Munteanu, O., Wang, J.: Geometry of shrinking Ricci solitons. *Compositio Mathematica*, 151(12), 2273–2300 (2015).
- [62] O’Neill, B.: The fundamental equations of a submersion. *Michigan Mathematical Journal*, 13(4), 459–469 (1966).
- [63] O’Neill, B.: *Semi-Riemannian geometry with applications to relativity*. Academic Press, New York, NY (1983).
- [64] Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. *ArXiv Preprint Math/0211159* (2002).
- [65] Perelman, G.: Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, preprint, *ArXiv Preprint Math.DG/0307245*, 16 (2003).
- [66] Perelman, G.: Ricci flow with surgery on three-manifolds. *ArXiv Preprint Math/0303109* (2003).
- [67] Poincaré, H.: Sur la dynamique de l’électron. *Comptes rendus de l’Académie des Sciences*, 140, 1504–1508 (1905).
- [68] Rimoldi, M., Veronelli, G.: Extremals of log Sobolev inequality on non-compact manifolds and Ricci soliton structures. *Calculus of Variations and Partial Differential Equations*, 58(2), Paper No. 66, 26pp (2019).
- [69] Schoen, R. M., Yau, S. T.: *Lectures on differential geometry*. International Press, Cambridge, MA (1994).
- [70] Seshadri, H.: Weyl curvature and the Euler characteristic in dimension four. *Differential Geometry and Its Applications*, 24(2), 172–177 (2006).
- [71] Siddiqi, M. D.: Conformal η -Ricci solitons in δ -Lorentzian Trans Sasakian manifolds. *International Journal of Maps in Mathematics*, 1(1), 15–34 (2018).
- [72] Streets, J., Tian, G.: Symplectic curvature flow. *Journal für Die Reine Und Angewandte Mathematik*, 696, 143–185 (2014).
- [73] Tanno, S.: Variational problems on contact Riemannian manifolds. *Transactions of the American Mathematical Society*, 314(1), 349–379 (1989).
- [74] Thorpe, J. A.: Some remarks on the Gauss-Bonnet integral. *Journal of Mathematics and Mechanics*, 18(8), 779–786 (1969).
- [75] Wadati, M.: The modified Korteweg-de Vries equation. *Journal of the Physical Society of Japan*, 34(5), 1289–1296 (1973).
- [76] Wadati, M.: Introduction to solitons. *Pramana*, 57, 841–847 (2001).
- [77] Wadati, M., Toda, M.: The exact N-soliton solution of the Korteweg-de Vries equation. *Journal of the Physical Society of Japan*, 32(5), 1[7]3–1411 (1972).
- [78] Wang, X.-J., Zhu, X.: Kähler-Ricci solitons on toric manifolds with positive first Chern class. *Advances in Mathematics*, 188(1), 87–103 (2004).
- [79] Yano, K., Kon, M.: *Structures on manifolds*. World scientific, Singapore (1985).
- [80] Zabusky, N. J., Kruskal, M. D.: Interaction of “solitons” in a collisionless plasma and the recurrence of initial states. *Physical Review Letters*, 15(6), 2[7] (1965).
- [81] Zakharov, V. E., Faddeev, L. D.: Korteweg–de Vries equation: A completely integrable Hamiltonian system. *Funktsional’nyi Analiz i Ego Prilozheniya*, 5(4), 18–27 (1971).
- [82] Zakharov, V., Shabat, A.: Interaction between solutions in a stable medium. *Zhurnal Eksperimentalnoi I Teoreticheskoi Fiziki*, 64(5), 1627–1639 (1973).
- [83] Zhang, Q. S.: Extremal of Log Sobolev inequality and W entropy on noncompact manifolds. *Journal of Functional Analysis*, 263(7), 2051–2101 (2012).
- [84] Zhang, X.: Compactness theorems for gradient Ricci solitons. *Journal of Geometry and Physics*, 56(12), 2481–2499 (2006).
- [85] Zhang, Z.-H.: On the completeness of gradient Ricci solitons. *Proceedings of the American Mathematical Society*, 137(8), 2755–2759 (2009).

Affiliations

B.-Y. CHEN

ADDRESS: Department of Mathematics, Michigan State University, 619 Red Cedar Road, East Lansing, Michigan 48824–1027, U.S.A.

E-MAIL: chenb@msu.edu

ORCID ID: 0000-0002-1270-094X

M. A. CHODHARY

ADDRESS: Department of Mathematics, Maulana Azad National Urdu University, Hyderabad, India.

E-MAIL: majid_alichoudhary@yahoo.co.in

ORCID ID: 0000-0001-5920-1227

M. NISAR

ADDRESS: Department of Mathematics, Maulana Azad National Urdu University, Hyderabad, India.

E-MAIL: nisar.msc@gmail.com

ORCID ID: 0009-0006-0901-6123

M. D. SIDDIQI

ADDRESS: Department of Mathematics, Jazan University, P.O. Box 114, Jazan 45142, Saudi Arabia.

E-MAIL: msiddiqi@jazanu.edu.sa

ORCID ID: 0000-0002-1713-6831